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Some new integral inequalities for n-times differentiable log-convex functions

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Abstract: In this work, by using an integral identity together with both the Hölder and the Power-Mean integral inequality we establish several new inequalities for n-time differentiable log-convex functions.

Keywords: Convex function, Log-Convex function, Hölder Integral inequality and Power-Mean Integral inequality.

1 Introduction

In this paper, we establish some new inequalities for functions whose nth derivatives in absolute value are log-convex functions. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. For some inequalities, generalizations and applications concerning convexity see [6,9,10,11]. Recently, in the literature there are so many papers about n-times differentiable functions on several kinds of convexities. In references [3,4,5,8,12,13,17,19], readers can find some results about this issue. Many papers have been written by a number of mathematicians concerning inequalities for different classes of log-convex functions see for instance the recent papers [1,2,7,14,15,16,18,20] and the references within these papers.

Definition 1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)$$

is valid for all $x,y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

Definition 2. *A positive function f is called log-convex on a real interval $I = [a,b]$, if for all $x,y \in [a,b]$ and $t \in [0,1]$,*

$$f(tx+(1-t)y) \le [f(x)]^t [f(y)]^{1-t}.$$

If f is a positive log-concave function, then the inequality is reversed. Equivalently, a function f is log-convex on I if f is positive and $\log f$ is convex on I. Also, if $f>0$ and f' exists on I, then f is log-convex if and only if $ff''-(f')^2 \ge 0$.

Let $0<a<b$, throughout this paper we will use

$$A(a,b)=\frac{a+b}{2},\ G(a,b)=\sqrt{ab},\ L(a,b)=\frac{b-a}{\ln b-\ln a},\quad L_p(a,b)=\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}},a\neq b,\ p\in R,\ p\neq -1,0$$

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for the arithmetic, geometric, logarithmic, generalized logarithmic mean for $a,b>0$ respectively.

2 Main results

We will use the following Lemma [13] for we obtain the main results.

Lemma 1. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be n-times differentiable mapping on I° for $n\in\mathbb{N}$ and $f^{(n)}\in L[a,b]$, where $a,b\in I^{\circ}$ with $a<b$, we have the identity*

$$\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx=\frac{(-1)^{n+1}}{n!}\int_a^b x^{(n)}f^{(n)}(x)dx.$$

where an empty sum is understood to be nil.

Theorem 1. *For $\forall n\in\mathbb{N}$; let $f:I\subseteq[0,\infty)\to(0,\infty)$ be n-times differentiable function on I° and $a,b\in I^{\circ}$ with $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q>1$ is log-convex on $[a,b]$, then the following inequality holds:*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\le\frac{1}{n!}(b-a)L_{np}^n(a,b)L^{\frac{1}{q}}\left(\left|f^{(n)}(b)\right|^q,\left|f^{(n)}(a)\right|^q\right)$$

where $\frac{1}{p}+\frac{1}{q}=1.$

Proof. Since $\left|f^{(n)}\right|^q$ for $q>1$ is log-convex on $[a,b]$, using Lemma 1, the Hölder integral inequality and

$$\left|f^{(n)}(x)\right|^q=\left|f^{(n)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right|^q\le\left[\left|f^{(n)}(b)\right|^q\right]^{\frac{x-a}{b-a}}\left[\left|f^{(n)}(a)\right|^q\right]^{\frac{b-x}{b-a}}$$

we have

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\le\frac{1}{n!}\int_a^b x^n f^{(n)}(x)dx$$

$$\le\frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}\le\frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b\left[\left|f^{(n)}(b)\right|^q\right]^{\frac{x-a}{b-a}}\left[\left|f^{(n)}(a)\right|^q\right]^{\frac{b-x}{b-a}}dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left|f^{(n)}(a)\right|\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b\left[\frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}(a)\right|^q}\right]^{\frac{x-a}{b-a}}dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left|f^{(n)}(a)\right|\left[\frac{b^{np+1}-a^{np+1}}{(np+1)}\right]^{\frac{1}{p}}\times\left\{\frac{b-a}{\ln\left|f^{(n)}(b)\right|^q-\ln\left|f^{(n)}(a)\right|^q}\left(\frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}(a)\right|^q}-1\right)\right\}^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left[\frac{b^{np+1}-a^{np+1}}{(np+1)(b-a)}\right]^{\frac{1}{p}}\left\{\frac{\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q}{\ln\left|f^{(n)}(b)\right|^q-\ln\left|f^{(n)}(a)\right|^q}\right\}^{\frac{1}{q}}=\frac{1}{n!}(b-a)L_{np}^n(a,b)L^{\frac{1}{q}}\left(\left|f^{(n)}(b)\right|^q,\left|f^{(n)}(a)\right|^q\right)$$

This completes the proof of theorem.

Corollary 1. *Under the conditions Theorem 1 for $n = 1$ we have the following inequality:*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le L_p(a,b)L^{\frac{1}{q}}\left(\left|f'(b)\right|^q,\left|f'(a)\right|^q\right).$$

Proposition 1. *Let $a,b\in(0,\infty)$ with $a<b$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and we have*

$$L_{1-\frac{1}{q}}^{1-\frac{1}{q}}(a,b) \le L_p(a,b)\left[\frac{L(a,b)}{G^2(a,b)}\right]^{\frac{1}{q}}$$

Proof. Under the assumption of the Proposition, let $f(x)=\frac{q}{q-1}x^{1-\frac{1}{q}}, x\in(0,\infty)$. Then

$$\left|f'(x)\right| = x^{-\frac{1}{q}}$$

is log-convex on $(0,\infty)$ and the result follows directly from Corollary 1.

Theorem 2. *For $\forall n\in\mathbb{N}$; let $f: I\subseteq[0,\infty)\to(0,\infty)$ be n-times differentiable function on I° and $a,b\in I^\circ$ with $a<b$. If $\left|f^{(n)}\right|^q\in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q\ge 1$ is log-convex on $[a,b]$, then the following inequality holds:*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right| \le \frac{1}{n!}\left|f^{(n)}(a)\right|(b-a)^{1-\frac{1}{q}}L_n^{n\left(1-\frac{1}{q}\right)}(a,b)M^{\frac{1}{q}}(a,b,n,q)$$

where $M(a,b,n,q)=\int_a^b x^n\left[\frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}(a)\right|^q}\right]^{\frac{x-a}{b-a}}dx$.

Proof. From Lemma 1 and Power-mean integral inequality, we obtain

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right| \le \frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx$$

$$\le \frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}}\left(\int_a^b x^n\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \le \frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}}\left(\int_a^b x^n\left[\left|f^{(n)}(b)\right|^q\right]^{\frac{x-a}{b-a}}\left[\left|f^{(n)}(a)\right|^q\right]^{\frac{b-x}{b-a}}dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left|f^{(n)}(a)\right|\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}}\left(\int_a^b x^n\left[\frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}(a)\right|^q}\right]^{\frac{x-a}{b-a}}dx\right)^{\frac{1}{q}} = \frac{1}{n!}\left|f^{(n)}(a)\right|\left[\frac{b^{n+1}-a^{n+1}}{b-a}\right]^{1-\frac{1}{q}}M^{\frac{1}{q}}(a,b,n,q)$$

$$=\frac{1}{n!}\left|f^{(n)}(a)\right|(b-a)^{1-\frac{1}{q}}\left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{1-\frac{1}{q}}M^{\frac{1}{q}}(a,b,n,q) = \frac{1}{n!}\left|f^{(n)}(a)\right|(b-a)^{1-\frac{1}{q}}L_n^{n\left(1-\frac{1}{q}\right)}M^{\frac{1}{q}}(a,b,n,q).$$

This completes the proof of theorem.

Corollary 2. *Under the conditions Theorem 2 for $n = 1$ we have the following inequality.*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le A^{1-\frac{1}{q}}(a,b)\left\{\frac{b\left|f'(b)\right|^q-a\left|f'(a)\right|^q}{\ln\left|f'(b)\right|^q-\ln\left|f'(a)\right|^q}-\frac{(b-a)L\left(\left|f'(b)\right|^q,\left|f'(a)\right|^q\right)}{\ln\left|f'(b)\right|^q-\ln\left|f'(a)\right|^q}\right\}^{\frac{1}{q}}.$$

Proposition 2. *Let $a,b\in(0,\infty)$ with $a<b$, $q\geq 1$ and, we have*

$$L_{1-\frac{1}{q}}^{1-\frac{1}{q}}(a,b)\leq A^{1-\frac{1}{q}}(a,b)G^{-\frac{2}{q}}(a,b)L^{\frac{2}{q}}(a,b)$$

Proof. The result follows directly from Corollary 2 for the function $f(x)=\frac{q}{q-1}x^{1-\frac{1}{q}},\ x\in(0,\infty)$.

Corollary 3. *Under the conditions Theorem 2 for $q=1$ we have the following inequality:*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\leq\frac{1}{n!}\left|f^{(n)}(a)\right|M(a,b,n,1)$$

Theorem 3. *For $\forall n\in\mathbb{N}$; let $f:I\subseteq[0,\infty)\to(0,\infty)$ be n-times differentiable function on I° and $a,b\in I^\circ$ with $a<b$. If $\left|f^{(n)}\right|^q\in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q>1$ is log-convex on $[a,b]$, then the following inequality holds.*

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\\&\leq\frac{1}{n!}(b-a)L_{p(n-1)+1}^{n-1+1/p}(a,b)\times\left\{\frac{b\left|f^{(n)}(b)\right|^q-a\left|f^{(n)}(a)\right|^q}{\ln\left|f^{(n)}(b)\right|^q-\ln\left|f^{(n)}(a)\right|^q}-\frac{(b-a)L\left(\left|f^{(n)}(b)\right|^q,\left|f^{(n)}(a)\right|^q\right)}{\ln\left|f^{(n)}(b)\right|^q-\ln\left|f^{(n)}(a)\right|^q}\right\}^{\frac{1}{q}}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $\left|f^{(n)}\right|^q$ for $q>1$ is log-convex on $[a,b]$, using Lemma 1 and the Hölder integral inequality, we have the following inequality:

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\leq\frac{1}{n!}\int_a^b x^{n-\frac{1}{q}}x^{\frac{1}{q}}\left|f^{(n)}(x)\right|dx\\&\leq\frac{1}{n!}\left[\int_a^b\left(x^{n-\frac{1}{q}}\right)^p dx\right]^{\frac{1}{p}}\left[\int_a^b\left(x^{\frac{1}{q}}\right)^q\left|f^{(n)}(x)\right|^q dx\right]^{\frac{1}{q}}\leq\frac{1}{n!}\left[\int_a^b x^{p\frac{qn-1}{q}}dx\right]^{\frac{1}{p}}\left[\int_a^b x\left[\left|f^{(n)}(b)\right|^q\right]^{\frac{x-a}{b-a}}\left[\left|f^{(n)}(a)\right|^q\right]^{\frac{b-x}{b-a}}dx\right]^{\frac{1}{q}}\\&=\frac{1}{n!}\left|f^{(n)}(a)\right|\left[\int_a^b x^{p(n-1)+1}dx\right]^{\frac{1}{p}}\left[\int_a^b x\left[\frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}(a)\right|^q}\right]^{\frac{x-a}{b-a}}dx\right]^{\frac{1}{q}}=\frac{1}{n!}(b-a)^{\frac{1}{p}}\left[\frac{b^{p(n-1)+2}-a^{p(n-1)+2}}{(p(n-1)+2)(b-a)}\right]^{\frac{1}{p}}\\&\times\left\{\frac{(b-a)\left[b\left|f^{(n)}(b)\right|^q-a\left|f^{(n)}(a)\right|^q\right]}{\ln\left|f^{(n)}(b)\right|^q-\ln\left|f^{(n)}(a)\right|^q}-\frac{(b-a)^2L\left(\left|f^{(n)}(b)\right|^q,\left|f^{(n)}(a)\right|^q\right)}{\ln\left|f^{(n)}(b)\right|^q-\ln\left|f^{(n)}(a)\right|^q}\right\}^{\frac{1}{q}}=\frac{1}{n!}(b-a)\left[\frac{b^{p(n-1)+2}-a^{p(n-1)+2}}{(p(n-1)+2)(b-a)}\right]^{\frac{1}{p}}\\&\times\left\{\frac{b\left[b\left|f^{(n)}(b)\right|^q-a\left|f^{(n)}(a)\right|^q\right]}{\ln\left|f^{(n)}(b)\right|^q-\ln\left|f^{(n)}(a)\right|^q}-\frac{(b-a)L\left(\left|f^{(n)}(b)\right|^q,\left|f^{(n)}(a)\right|^q\right)}{\ln\left|f^{(n)}(b)\right|^q-\ln\left|f^{(n)}(a)\right|^q}\right\}^{\frac{1}{q}}\\&=\frac{1}{n!}(b-a)L_{p(n-1)+1}^{n-1+1/p}(a,b)\times\left\{\frac{b\left[b\left|f^{(n)}(b)\right|^q-a\left|f^{(n)}(a)\right|^q\right]}{\ln\left|f^{(n)}(b)\right|^q-\ln\left|f^{(n)}(a)\right|^q}-\frac{(b-a)L\left(\left|f^{(n)}(b)\right|^q,\left|f^{(n)}(a)\right|^q\right)}{\ln\left|f^{(n)}(b)\right|^q-\ln\left|f^{(n)}(a)\right|^q}\right\}^{\frac{1}{q}}.\end{aligned}$$

Corollary 4. *Under the conditions Theorem 3 for $n = 1$ we have the following inequality.*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq A^{\frac{1}{p}}(a,b)\left\{\frac{b\left[b\left|f'(b)\right|^q-a\left|f'(a)\right|^q\right]}{\ln\left|f'(b)\right|^q-\ln\left|f'(a)\right|^q}-\frac{(b-a)L\left(\left|f'(b)\right|^q,\left|f'(a)\right|^q\right)}{\ln\left|f'(b)\right|^q-\ln\left|f'(a)\right|^q}\right\}^{\frac{1}{q}}$$

Proposition 3. *Let $a,b \in (0,\infty)$ with $a < b, p,q > 1\frac{1}{p}+\frac{1}{q} = 1$, we have*

$$L_{1-\frac{1}{q}}^{1-\frac{1}{q}}(a,b) \leq A^{\frac{1}{p}}(a,b)G^{-\frac{2}{q}}(a,b)L^{\frac{2}{q}}(a,b)$$

Proof. The result follows directly from Corollary 4 for the function $f(x) = \frac{q}{q-1}x^{1-\frac{1}{q}}, x \in (0,\infty)$.

3 Conclusions

In this paper, by using an integral identity we obtain some new type inequalities for n-time differentiable log-convex functions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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