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Note on the cross-section in the semi-tensor bundle

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Abstract: Using the fiber bundle M over a manifold B, we define a semi-tensor (pull-back) bundle tB of type (p,q). The complete and horizontal lift of projectable geometric objects on M to the semi-tensor (pull-back) bundle tB of type (p,q) are presented. The main purpose of this paper is to study the behaviour of complete lift of vector and affinor (tensor of type (1,1)) fields on cross-sections for pull-back (semi-tensor) bundle tB of type (p,q).

Keywords: Vector field, complete lift, horizontal lift, pull-back bundle, cross-section, semi-tensor bundle.

1 Introduction

Let M_n be an n-dimensional differentiable manifold of class C^{∞} and $\pi_1 : M_n \to B_m$ the differentiable bundle determined by a submersion π_1. Suppose that $(x^i) = (x^a, x^\alpha)$, $a, b, \ldots = 1, \ldots, n-m; \alpha, \beta, \ldots = n-m+1, \ldots, n; i, j, \ldots = 1, 2, \ldots, n$ is a system of local coordinates adapted to the bundle $\pi_1 : M_n \to B_m$, where x^α are coordinates in B_m, and x^a are fiber coordinates of the bundle $\pi_1 : M_n \to B_m$. If $(x^{i'}) = (x^{a'}, x^{\alpha'})$ is another system of local adapted coordinates in the bundle, then we have

$$\begin{cases} x^{a'} = x^{a'}\left(x^b, x^\beta\right), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right) \ . \end{cases} \tag{1}$$

The Jacobian of (1) has components

$$\left(A_j^{i'}\right) = \left(\frac{\partial x^{i'}}{\partial x^j}\right) = \begin{pmatrix} A_b^{a'} & A_\beta^{a'} \\ 0 & A_\beta^{\alpha'} \end{pmatrix},$$

where

$$A_\beta^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}.$$

Let $\left(T_q^p\right)_x (B_m)(x = \pi_1(\tilde{x}), \tilde{x} = (x^a, x^\alpha) \in M_n)$ be the tensor space at a point $x \in B_m$ with local coordinates $(x^1, \ldots, x^m)$, we have the holonomous frame field

$$\partial_{x^{i_1}} \otimes \partial_{x^{i_2}} \otimes \ldots \otimes \partial_{x^{i_p}} \otimes dx^{j_1} \otimes dx^{j_2} \otimes \ldots \otimes dx^{j_q},$$

for $i \in \{1, \ldots, m\}^p$, $j \in \{1, \ldots, m\}^q$, over $U \subset B_m$ of this tensor bundle, and for any (p,q)-tensor field t we have [[6], p.163]:

$$t|U = t_{j_1 \ldots j_q}^{i_1 \ldots i_p} \partial_{x^{i_1}} \otimes \partial_{x^{i_2}} \otimes \ldots \otimes \partial_{x^{i_p}} \otimes dx^{j_1} \otimes dx^{j_2} \otimes \ldots \otimes dx^{j_q},$$

then by definition the set of all points $(x^I) = (x^a, x^\alpha, x^{\overline{\alpha}})$, $x^{\overline{\alpha}} = t_{j_1 \ldots j_q}^{i_1 \ldots i_p}, \overline{\alpha} = \alpha + m^{p+q}, I, J, \ldots = 1, \ldots, n + m^{p+q}$ is a semi-tensor bundle $t_q^p(B_m)$ over the manifold M_n.

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The semi-tensor bundle $t_q^p(B_m)$ has the natural bundle structure over B_m, its bundle projection $\pi : t_q^p(B_m) \to B_m$ being defined by $\pi : (x^a, x^\alpha, x^{\overline{\alpha}}) \to (x^\alpha)$. If we introduce a mapping $\pi_2 : t_q^p(B_m) \to M_n$ by $\pi_2 : (x^a, x^\alpha, x^{\overline{\alpha}}) \to (x^a, x^\alpha)$, then $t_q^p(B_m)$ has a bundle structure over M_n. It is easily verified that $\pi = \pi_1 \circ \pi_2$.

On the other hand, let $\varepsilon = \pi : E \to B$ denote a fiber bundle with fiber F. Given a manifold B' and a map $f : B' \to B$, one can construct in a natural way a bundle over B' with the same fiber: Consider the subset

$$f^*E = \left\{ \left(b', e\right) \in B' \times E \middle| f\left(b'\right) = \pi\left(e\right) \right\}$$

together with the subspace topology from $B' \times E$, and denote by $\pi_1 : f^*E \to B'$, $\pi_2 : f^*E \to E$ the projections. $f^*\varepsilon = \pi_1 : f^*E \to B'$ is a fiber bundle with fiber F, called the pull-back bundle of ε via f [[5],[7],[11],[13]].

From the above definition it follows that the semi-tensor bundle $(t_q^p(B_m), \pi_2)$ is a pull-back bundle of the tensor bundle over B_m by π_1.

In other words, the semi-tensor bundle (induced or pull-back bundle) of the tensor bundle $\left(T_q^p(B_m), \widetilde{\pi}, B_m\right)$ is the bundle $\left(t_q^p(B_m), \pi_2, M_n\right)$ over M_n with a total space

$$t_q^p(B_m) = \left\{ \left((x^a, x^\alpha), x^{\overline{\alpha}} \right) \in M_n \times \left(T_q^p\right)_x (B_m) : \pi_1 (x^a, x^\alpha) = \widetilde{\pi}\left(x^\alpha, x^{\overline{\alpha}}\right) = (x^\alpha) \right\} \subset M_n \times \left(T_q^p\right)_x (B_m).$$

To a transformation (1) of local coordinates of M_n, there corresponds on $t_q^p(B_m)$ the coordinate transformation

$$\begin{cases} x^{a'} = x^{a'}\left(x^b, x^\beta\right), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right), \\ x^{\overline{\alpha}'} = t_{\alpha_1' \dots \alpha_q'}^{\beta_1' \dots \beta_p'} = A_{\alpha_1 \dots \alpha_p}^{\beta_1' \dots \beta_p'} A_{\alpha_1' \dots \alpha_q'}^{\beta_1 \dots \beta_q} t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} = A_{(\alpha)}^{(\beta')} A_{(\alpha')}^{(\beta)} x^{\overline{\beta}}. \end{cases} \tag{2}$$

The Jacobian of (2) is given by

$$\bar{A} = \left(A_J^{I'}\right) = \begin{pmatrix} A_b^{a'} & 0 & 0 \\ 0 & A_\beta^{\alpha'} & 0 \\ 0 & t_{(\sigma)}^{(\alpha)} \partial_\beta A_{(\alpha)}^{(\beta')} A_{(\alpha')}^{(\sigma)} & A_{(\alpha)}^{(\beta')} A_{(\alpha')}^{(\beta)} \end{pmatrix}, \tag{3}$$

where $I = (a, \alpha, \overline{\alpha})$, $J = (b, \beta, \overline{\beta})$, $I, J ... {=} 1, ..., n + m^{p+q}$, $t_{(\sigma)}^{(\alpha)} = t_{\sigma_1 \dots \sigma_q}^{\alpha_1 \dots \alpha_p}$, $A_\beta^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}$.

It is easily verified that the condition $Det\overline{A} \neq 0$ is equivalent to the condition:

$$Det(A_b^{a'}) \neq 0, Det(A_\beta^{\alpha'}) \neq 0, Det(A_{(\alpha)}^{(\beta')} A_{(\alpha')}^{(\beta)}) \neq 0.$$

Also, $\dim t_q^p(B_m) {=} n + m^{p+q}$. In the special case $n{=}m$, $t_q^p(B_m)$ is a tensor bundle $T_q^p(B_m)$ [[9], p.118]. In the special case, the semi-tensor bundles $t_0^1(B_m)$ $(p = 1,\ q = 0)$ and $t_1^0(B_m)$ $(p = 0,\ q = 1)$ are semi-tangent and semi-cotangent bundles, respectively.

We note that semi-tangent and semi-cotangent bundle were examined in [[1],[10],[12]] and [[14],[15]], respectively. Also, Fattaev studied the special class of semi-tensor bundle [3]. We denote by $\Im_q^p(t_q^p(B_m))$ and $\Im_q^p(B_m)$ the modules over $F\left(t_q^p(B_m)\right)$ and $F(B_m)$ of all tensor fields of type (p,q) on $t_q^p(B_m)$ and B_m respectively, where $F\left(t_q^p(B_m)\right)$ and $F(B_m)$ denote the rings of real-valued C^∞ $-$functions on $t_q^p(B_m)$ and B_m, respectively.

2 Vertical lifts of tensor fields and $\gamma-$operator

If $\psi \in t_q^p(B_m)$, it is regarded, in a natural way, by contraction on $t_q^p(B_m)$, which we denote by $\imath\psi$. If ψ has the local expression

$$\psi = \psi_{i_1\dots i_p}^{j_1\dots j_q}\partial_{j_1}\otimes\dots\otimes\partial_{j_q}\otimes dx^{i_1}\otimes\dots\otimes dx^{i_p}$$

in a coordinate neighborhood $U(x^\alpha)\subset B_m$, then $\imath\psi = \psi(t)$ has the local expression

$$\imath\psi = \psi_{i_1\dots i_p}^{j_1\dots j_q}t_{j_1\dots j_q}^{i_1\dots i_p}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ in $\pi^{-1}(U)$ [4].

Suppose that $A\in\Im_q^p(B_m)$. Then there is a unique vector fields ${}^{vv}A\in\Im_0^1(t_q^p(B_m))$ such that for $\psi\in t_q^p(B_m)$ [8]:

$$ {}^{vv}A(\imath\psi) = {}^{v}A\circ\pi_2 = \psi(A)\circ\pi_1\circ\pi_2 = \psi(A)\circ\pi = {}^{vv}(\psi(A)), \tag{4}$$

where ${}^{vv}(\psi(A))$ is the vertical lift of the function $\psi(A)\in F(B_m)$. We note that the vertical lift ${}^{vv}f = f\circ\pi$ of the arbitrary function $f\in F(B_m)$ is constant along each fiber of $\pi: t_q^p(B_m)\to B_m$. If ${}^{vv}A = {}^{vv}A^a\partial_a + {}^{vv}A^\alpha\partial_\alpha + {}^{vv}A^{\overline{\alpha}}\partial_{\overline{\alpha}}$, then we have from (4)

$$ {}^{vv}A = {}^{vv}A^a t_{\beta_1\dots\beta_q}^{\alpha_1\dots\alpha_p}\partial_a\psi_{\alpha_1\dots\alpha_p}^{\beta_1\dots\beta_q} + {}^{vv}A^\alpha t_{\beta_1\dots\beta_q}^{\alpha_1\dots\alpha_p}\partial_\alpha\psi_{\alpha_1\dots\alpha_p}^{\beta_1\dots\beta_q} + {}^{vv}A^{\overline{\alpha}}\psi_{\sigma_1\dots\sigma_p}^{\theta_1\dots\theta_q} = \psi_{\sigma_1\dots\sigma_p}^{\theta_1\dots\theta_q}A_{\theta_1\dots\theta_q}^{\sigma_1\dots\sigma_p}.$$

But $\psi_{\sigma_1\dots\sigma_p}^{\theta_1\dots\theta_q}$, $\partial_a\psi_{\alpha_1\dots\alpha_p}^{\beta_1\dots\beta_q}$ and $\partial_\alpha\psi_{\alpha_1\dots\alpha_p}^{\beta_1\dots\beta_q}$ can take any preassigned valued at each point. Thus we have

$$ {}^{vv}A^a t_{\beta_1\dots\beta_q}^{\alpha_1\dots\alpha_p} = 0, {}^{vv}A^\alpha t_{\beta_1\dots\beta_q}^{\alpha_1\dots\alpha_p} = 0, {}^{vv}A^{\overline{\alpha}} = A_{\theta_1\dots\theta_q}^{\sigma_1\dots\sigma_p}.$$

Hence ${}^{vv}A^\alpha = 0$ at all points of $t_q^p(B_m)$ except possibly those at which all the components $x^{\overline{\alpha}} = t_{\beta_1\dots\beta_q}^{\alpha_1\dots\alpha_p}$ are zero: that is, at points of the base space. Thus we see that the components ${}^{vv}A^\alpha$ are zero a point such that $x^{\overline{\alpha}}\neq 0$, that is, on $t_q^p(B_m)-B_m$. However, $t_q^p(B_m)-B_m$ is dense in $t_q^p(B_m)$ and the components of ${}^{vv}A$ are continuous at every point of $t_q^p(B_m)$. Hence, we have ${}^{vv}A^\alpha = 0$ at all points of $t_q^p(B_m)$. Consequently, the vertical lift ${}^{vv}A$ of A to $t_q^p(B_m)$ has components

$$ {}^{vv}A = \begin{pmatrix} {}^{vv}A^a \\ {}^{vv}A^\alpha \\ {}^{vv}A^{\overline{\alpha}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ A_{\beta_1\dots\beta_q}^{\alpha_1\dots\alpha_p} \end{pmatrix}, \tag{5}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t_q^p(B_m)$.

Let $\varphi\in\Im_1^1(B_m)$. We define a vector field $\gamma\varphi$ in $\pi^{-1}(U)$ by

$$\begin{cases} \gamma\varphi = \left(\sum_{\lambda=1}^p t_{\beta_1\dots\beta_q}^{\alpha_1\dots\varepsilon\dots\alpha_p}\varphi_\varepsilon^{\alpha_\lambda}\right)\frac{\partial}{\partial x^{\overline{\beta}}}, & (p\geq 1, q\geq 0) \\ \widetilde{\gamma}\varphi = \left(\sum_{\mu=1}^q t_{\beta_1\dots\varepsilon\dots\beta_q}^{\alpha_1\dots\alpha_p}\varphi_{\beta_\mu}^\varepsilon\right)\frac{\partial}{\partial x^{\overline{\beta}}}, & (p\geq 0, q\geq 1) \end{cases} \tag{6}$$

with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t_q^p(B_m)$. From (3) we easily see that the vector fields $\gamma\varphi$ and $\widetilde{\gamma}\varphi$ defined in each $\pi^{-1}(U)\subset t_q^p(B_m)$ determine respectively global vertical vector fields on $t_q^p(B_m)$. We call $\gamma\varphi$ (or $\widetilde{\gamma}\varphi$) the vertical-vector lift of the tensor field $\varphi\in\Im_1^1(B_m)$ to $t_q^p(B_m)$. For any $\varphi\in\Im_1^1(B_m)$, if we take account of (3) and (6), we can prove that $(\gamma\varphi)' = \bar{A}(\gamma\varphi)$ where $\gamma\varphi$ is a vector field defined by

$$\gamma\varphi = (\gamma\varphi)^I = \begin{pmatrix} 0 \\ 0 \\ \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\varphi_{\varepsilon}^{\alpha_\lambda} \end{pmatrix}. \tag{7}$$

Let $\varphi \in \Im_1^1(B_m)$. On putting

$$\tilde{\gamma}\varphi = (\tilde{\gamma}\varphi)^I = \begin{pmatrix} 0 \\ 0 \\ \sum_{\mu=1}^{q} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\varphi_{\beta_\mu}^{\varepsilon} \end{pmatrix}, \tag{8}$$

we easily see that $(\tilde{\gamma}\varphi)' = \bar{A}(\tilde{\gamma}\varphi)$.

3 Complete lifts of vector fields

We now denote by $\Im_q^p(M_n)$ the module over $F(M_n)$ of all tensor fields of type (p,q) on M_n, where $F(M_n)$ denotes the ring of real-valued C^∞ −functions on M_n. Let $\widetilde{X} \in \Im_0^1(M_n)$ be a projectable vector field [9] with projection $X = X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{X} = \widetilde{X}^a(x^a,x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$. On putting

$$^{cc}\widetilde{X} = \begin{pmatrix} ^{cc}\widetilde{X}^b \\ ^{cc}\widetilde{X}^\beta \\ ^{cc}\widetilde{X}^{\overline{\beta}} \end{pmatrix} = \begin{pmatrix} \widetilde{X}^b \\ \widetilde{X}^\beta \\ \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_\varepsilon X^{\alpha_\lambda} - \sum_{\mu=1}^{q} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\beta_\mu}X^\varepsilon \end{pmatrix}, \tag{9}$$

we easily see that $^{cc}\widetilde{X}' = \bar{A}\left(^{cc}\widetilde{X}\right)$. The vector field $^{cc}\widetilde{X}$ is called the complete lift of $\widetilde{X}$ to the semi-tensor bundle $t_q^p(B_m)$.

4 Horizontal lifts of vector fields

Let $\widetilde{X} \in \Im_0^1(M_n)$ be a projectable vector field [9] with projection $X = X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{X} = \widetilde{X}^a(x^a,x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$. If we take account of (3), we can prove that $^{HH}\widetilde{X}' = \bar{A}\left(^{HH}\widetilde{X}\right)$, where $^{HH}\widetilde{X}$ is a vector field defined by

$$^{HH}\widetilde{X} = \begin{pmatrix} \widetilde{X}^b \\ \widetilde{X}^\beta \\ X^l\left(\sum_{\mu=1}^{q}\Gamma_{l\beta_\mu}^{\varepsilon} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p} - \sum_{\lambda=1}^{p}\Gamma_{l\varepsilon}^{\alpha_\lambda} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\right) \end{pmatrix}, \tag{10}$$

with respect to the coordinates $(x^b,x^\beta,x^{\overline{\beta}})$ on $t_q^p(B_m)$. We call $^{HH}\widetilde{X}$ the horizontal lift of the vector field $\widetilde{X}$ to $t_q^p(B_m)$.

Theorem 1. *If $\widetilde{X} \in \Im_0^1(M_n)$ then*

$$^{cc}\widetilde{X} - {}^{HH}\widetilde{X} = \gamma(\hat{\nabla}\widetilde{X}) - \tilde{\gamma}(\hat{\nabla}\widetilde{X}),$$

where the symmetric affine connection $\hat{\nabla}$ is the given by $\widehat{\Gamma}_{\beta\theta}^{\alpha} = \Gamma_{\theta\beta}^{\alpha}$.

Proof. From (7), (8), (9) and (10), we have

$$
\begin{aligned}
{}^{cc}\widetilde{X} - {}^{HH}\widetilde{X} &= \begin{pmatrix} \widetilde{X}^{b} \\ \widetilde{X}^{\beta} \\ \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_\varepsilon X^{\alpha_\lambda} - \sum_{\mu=1}^{q} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\beta_\mu} X^{\varepsilon} \end{pmatrix} - \begin{pmatrix} \widetilde{X}^{b} \\ \widetilde{X}^{\beta} \\ X^{l}(\sum_{\mu=1}^{q}\Gamma_{l\beta_\mu}^{\varepsilon} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p} - \sum_{\lambda=1}^{p}\Gamma_{l\varepsilon}^{\alpha_\lambda} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}(\partial_\varepsilon X^{\alpha_\lambda} + \Gamma_{l\varepsilon}^{\alpha_\lambda} X^{l}) - \sum_{\mu=1}^{q} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}(\partial_{\beta_\mu} X^{\varepsilon} + \Gamma_{l\beta_\mu}^{\varepsilon} X^{l}) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\left(\partial_\varepsilon X^{\alpha_\lambda} + \Gamma_{l\varepsilon}^{\alpha_\lambda} X^{l}\right) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \sum_{\mu=1}^{q} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\left(\partial_{\beta_\mu} X^{\varepsilon} + \Gamma_{l\beta_\mu}^{\varepsilon} X^{l}\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\underbrace{(\partial_\varepsilon X^{\alpha_\lambda} + \widehat{\Gamma}_{\varepsilon l}^{\alpha_\lambda} X^{l})}_{\hat{\nabla}_\varepsilon \widetilde{X}^{\alpha_\lambda}} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \sum_{\mu=1}^{q} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\underbrace{(\partial_{\beta_\mu} X^{\varepsilon} + \widehat{\Gamma}_{\beta_\mu l}^{\varepsilon} X^{l})}_{\hat{\nabla}_{\beta_\mu} \widetilde{X}^{\varepsilon}} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\left(\hat{\nabla}_\varepsilon \widetilde{X}^{\alpha_\lambda}\right) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \sum_{\mu=1}^{q} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\left(\hat{\nabla}_{\beta_\mu} \widetilde{X}^{\varepsilon}\right) \end{pmatrix} \\
&= \gamma(\hat{\nabla}_\varepsilon \widetilde{X}^{\alpha_\lambda}) - \widetilde{\gamma}\left(\hat{\nabla}_{\beta_\mu} \widetilde{X}^{\varepsilon}\right) = \gamma(\hat{\nabla}\widetilde{X}) - \widetilde{\gamma}(\hat{\nabla}\widetilde{X}).
\end{aligned}
$$

Thus, we have Theorem 1.

5 Cross-sections in the semi-tensor bundle

Let $\xi \in \Im_q^p(B_m)$ be a tensor field on B_m. Then the correspondence $x \to \xi_x$, ξ_x being the value of ξ at $x \in M_n$, determines a cross-section β_ξ of semi-tensor bundle. Thus if $\sigma_\xi : B_m \to T_q^p(B_m)$ is a cross-section of $\left(T_q^p(B_m), \widetilde{\pi}, B_m\right)$, such that $\widetilde{\pi} \circ \sigma_\xi = I_{(B_m)}$, an associated cross-section $\beta_\xi : M_n \to t_q^p(B_m)$ of semi-tensor bundle $\left(t_q^p(B_m), \pi_2, M_n\right)$ defined by [[2], p. 217-218], [[9], p. 126-127]:

$$\beta_\xi\left(x^a, x^\alpha\right) = \left(x^a, x^\alpha, \sigma_\xi \circ \pi_1\left(x^a, x^\alpha\right)\right) = \left(x^a, x^\alpha, \sigma_\xi\left(x^\alpha\right)\right) = \left(x^a, x^\alpha, \xi_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\left(x^\beta\right)\right).$$

If the tensor field ξ has the local components $\xi_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\left(x^\beta\right)$, the cross-section $\beta_\xi\left(M_n\right)$ of $t_q^p(B_m)$ is locally expressed by

$$\begin{cases} x^b = x^b, \\ x^\beta = x^\beta, \\ x^{\overline{\beta}} = \xi_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\left(x^\beta\right), \end{cases} \tag{11}$$

with respect to the coordinates $x^B = (x^b, x^\beta, x^{\overline{\beta}})$ in $t_q^p(B_m)$. Differentiating (11) by x^c, we see that $n-m$ tangent vector fields $B_{(c)}$ $(c = 1, \ldots, n-m)$ to $\beta_\xi(M_n)$ have components

$$B_{(c)} = \frac{\partial x^B}{\partial x^c} = \partial_c x^B = \begin{pmatrix} \partial_c x^b \\ \partial_c x^\beta \\ \partial_c \xi_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p} \end{pmatrix},$$

which are tangent to the cross-section $\beta_\xi\,(M_n)$.

Thus $B_{(c)}$ have components

$$B_{(c)} : \left(B^B_{(c)}\right) = \begin{pmatrix} \delta^b_c \\ 0 \\ 0 \end{pmatrix},$$

with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ in $t^p_q(B_m)$. Where

$$\delta^b_c = A^b_c = \frac{\partial x^b}{\partial x^c}.$$

Let $\widetilde{X} \in \Im^1_0(M_n)$ be a projectable vector field [9] with projection $X = X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$. We denote by BX the vector field with local components

$$BX : \left(B^B_{(c)}\widetilde{X}^c\right) = \begin{pmatrix} \delta^b_c\widetilde{X}^c \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A^b_c\widetilde{X}^c \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \widetilde{X}^b \\ 0 \\ 0 \end{pmatrix}, \tag{12}$$

with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ in $t^p_q(B_m)$, which is defined globally along $\beta_\xi\,(M_n)$.

Differentiating (11) by x^θ, we have vector fields $C_{(\theta)}$ $(\theta = n-m+1, \ldots, n)$ with components

$$C_{(\theta)} = \frac{\partial x^B}{\partial x^\theta} = \partial_\theta x^B = \begin{pmatrix} \partial_\theta x^b \\ \partial_\theta x^\beta \\ \partial_\theta \xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \end{pmatrix},$$

which are tangent to the cross-section $\beta_\xi\,(M_n)$.

Thus $C_{(\theta)}$ have components

$$C_{(\theta)} : \left(C^B_{(\theta)}\right) = \begin{pmatrix} A^b_\theta \\ \delta^\beta_\theta \\ \partial_\theta \xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \end{pmatrix},$$

with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ in $t^p_q(B_m)$. Where

$$A^b_\theta = \frac{\partial x^b}{\partial x^\theta}, \delta^\beta_\theta = A^\beta_\theta = \frac{\partial x^\beta}{\partial x^\theta}.$$

Let $\widetilde{X} \in \Im^1_0(M_n)$ be a projectable vector field [9] with projection $X = X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$. We denote by CX the vector field with local components

$$CX : \left(C^B_{(\theta)}X^\theta\right) = \begin{pmatrix} A^b_\theta X^\theta \\ X^\beta \\ X^\theta \partial_\theta \xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \end{pmatrix}, \tag{13}$$

with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ in $t^p_q(B_m)$, which is defined globally along $\beta_\xi\,(M_n)$.

On the other hand, the fibre is locally expressed by

$$\begin{cases} x^b = const., \\ x^\beta = const., \\ x^{\overline{\beta}} = t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}, \end{cases}$$

$t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$ being considered as parameters. Thus, on differentiating with respect to $x^{\overline{\beta}} = t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$, we easily see that the vector fields $E_{(\overline{\theta})}$ $(\overline{\theta} = n+1,\ldots,n+m^{p+q})$ with components

$$E_{(\overline{\theta})} : \left(E^B_{(\overline{\theta})}\right) = \partial_{\overline{\theta}} x^B = \begin{pmatrix} \partial_{\overline{\theta}} x^b \\ \partial_{\overline{\theta}} x^\beta \\ \partial_{\overline{\theta}} t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta^{\theta_1}_{\beta_1}\ldots\delta^{\theta_q}_{\beta_q}\delta^{\alpha_1}_{\gamma_1}\ldots\delta^{\alpha_p}_{\gamma_p} \end{pmatrix}$$

is tangent to the fibre, where δ is the Kronecker symbol.

Let ξ be a tensor field of type (p,q) with local components

$$\xi = \xi^{\gamma_1\ldots\gamma_p}_{\theta_1\ldots\theta_q} dx^{\theta_1} \otimes \ldots \otimes dx^{\theta_q} \otimes \partial_{\gamma_1} \otimes \ldots \otimes \partial_{\gamma_p}$$

on B_m.

We denote by $E\xi$ the vector field with local components

$$E\xi : \left(E^B_{(\overline{\theta})}\xi^{\gamma_1\ldots\gamma_p}_{\theta_1\ldots\theta_q}\right) = \begin{pmatrix} 0 \\ 0 \\ \xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \end{pmatrix}, \tag{14}$$

which is tangent to the fibre.

According to (12) and (13), we define new projectable vector field $H\widetilde{X}$ by

$$BX + CX = H\widetilde{X}$$

with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ in $t^p_q(B_m)$, where

$$H\widetilde{X} = \begin{pmatrix} A^b_c \widetilde{X}^c \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} A^b_\theta X^\theta \\ X^\beta \\ X^\theta \partial_\theta \xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \end{pmatrix} = \begin{pmatrix} \widetilde{X}^b \\ X^\beta \\ X^\theta \partial_\theta \xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \end{pmatrix}. \tag{15}$$

We consider in $\pi^{-1}(U) \subset t^p_q(B_m)$, $n+m^{p+q}$ local vector fields $B_{(c)}$, $C_{(\theta)}$ and $E_{(\overline{\theta})}$ along $\beta_\xi(M_n)$. They form a local family of frames $\left\{B_{(c)}, C_{(\theta)}, E_{(\overline{\theta})}\right\}$ along $\beta_\xi(M_n)$, which is called the adapted (B,C,E)-frame of $\beta_\xi(M_n)$ in $\pi^{-1}(U)$.

We can state following theorem:

Theorem 2. *Let $\widetilde{X}$ be a vector field on M_n with projection X on B_m. We have along $\beta_\xi(M_n)$ the formulas*

(i) ${}^{cc}\widetilde{X} = H\widetilde{X} + E(-L_X\xi)$, (16)
(ii) ${}^{vv}\xi = E\xi$

for any $t\in\Im_q^p(B_m)$, *where* $L_X\xi$ *denotes the Lie derivative of* ξ *with respect to* X.

Proof. (i) Using (9), (14) and (15), we have

$$H\widetilde{X}+E\left(-L_X\xi\right)=\begin{pmatrix}\widetilde{X}^b\\ X^\beta\\ X^\theta\partial_\theta\xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\end{pmatrix}+\begin{pmatrix}0\\0\\ -X^\theta\partial_\theta\xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}-\sum_{\mu=1}^q\partial_{\beta_\mu}X^\beta\xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta\ldots\beta_q}+\sum_{\lambda=1}^p\partial_\beta X^{\alpha_\lambda}\xi^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q}\end{pmatrix}$$
$$=\begin{pmatrix}\widetilde{X}^b\\ X^\beta\\ \sum_{\lambda=1}^p\partial_\beta X^{\alpha_\lambda}\xi^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q}-\sum_{\mu=1}^q\partial_{\beta_\mu}X^\beta\xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta\ldots\beta_q}\end{pmatrix}={}^{cc}\widetilde{X}.$$

Thus, we have (16).

(ii) This immediately follows from (5).

On the other hand, on putting $C_{(\overline{\beta})}=E_{(\overline{\beta})}$, we write the adapted frame of $\beta_\xi\left(M_n\right)$ as $\left\{B_{(b)},C_{(\beta)},C_{(\overline{\beta})}\right\}$. The adapted frame $\left\{B_{(b)},C_{(\beta)},C_{(\overline{\beta})}\right\}$ of $\beta_\xi\left(M_n\right)$ is given by the matrix

$$\widetilde{A}=\left(\widetilde{A}^A_B\right)=\begin{pmatrix}\delta^a_b & \partial_\beta x^a & 0\\ 0 & \delta^\alpha_\beta & 0\\ 0 & \partial_\beta\xi^{\sigma_1\ldots\sigma_p}_{\alpha_1\ldots\alpha_q} & \delta^{\beta_1}_{\alpha_1}\ldots\delta^{\beta_q}_{\alpha_q}\delta^{\sigma_1}_{\gamma_1}\ldots\delta^{\sigma_p}_{\gamma_p}\end{pmatrix}. \tag{16}$$

Since the matrix $\widetilde{A}$ in (16) is non-singular, it has the inverse. Denoting this inverse by $\left(\widetilde{A}\right)^{-1}$, we have

$$\left(\widetilde{A}\right)^{-1}=\left(\widetilde{A}^B_C\right)^{-1}=\begin{pmatrix}\delta^b_c & -\partial_\theta x^b & 0\\ 0 & \delta^\beta_\theta & 0\\ 0 & -\partial_\theta\xi^{\sigma_1\ldots\sigma_p}_{\beta_1\ldots\beta_q} & \delta^{\theta_1}_{\beta_1}\ldots\delta^{\theta_q}_{\beta_q}\delta^{\sigma_1}_{\gamma_1}\ldots\delta^{\sigma_p}_{\gamma_q}\end{pmatrix}, \tag{17}$$

where $\widetilde{A}\left(\widetilde{A}\right)^{-1}=(\widetilde{A}^A_B)\left(\widetilde{A}^B_C\right)^{-1}=\delta^A_C=\widetilde{I}$, where $A=(a,\alpha,\overline{\alpha})$, $B=\left(b,\beta,\overline{\beta}\right)$, $C=\left(c,\theta,\overline{\theta}\right)$.

Proof. In fact, from (16) and (17), we easily see that

$$\widetilde{A}\left(\widetilde{A}\right)^{-1}=(\widetilde{A}^A_B)\left(\widetilde{A}^B_C\right)^{-1}=\begin{pmatrix}\delta^a_b & \partial_\beta x^a & 0\\ 0 & \delta^\alpha_\beta & 0\\ 0 & \partial_\beta\xi^{\sigma_1\ldots\sigma_p}_{\alpha_1\ldots\alpha_q} & \delta^{\beta_1}_{\alpha_1}\ldots\delta^{\beta_q}_{\alpha_q}\delta^{\sigma_1}_{\gamma_1}\ldots\delta^{\sigma_p}_{\gamma_p}\end{pmatrix}\begin{pmatrix}\delta^b_c & -\partial_\theta x^b & 0\\ 0 & \delta^\beta_\theta & 0\\ 0 & -\partial_\theta\xi^{\sigma_1\ldots\sigma_p}_{\beta_1\ldots\beta_q} & \delta^{\theta_1}_{\beta_1}\ldots\delta^{\theta_q}_{\beta_q}\delta^{\sigma_1}_{\gamma_1}\ldots\delta^{\sigma_p}_{\gamma_q}\end{pmatrix}$$
$$=\begin{pmatrix}\delta^a_c & -\partial_\theta x^a+\partial_\theta x^a & 0\\ 0 & \delta^\alpha_\theta & 0\\ 0 & \partial_\theta\xi^{\sigma_1\ldots\sigma_p}_{\alpha_1\ldots\alpha_q}-\partial_\theta\xi^{\sigma_1\ldots\sigma_p}_{\alpha_1\ldots\alpha_q} & \delta^{\theta_1}_{\alpha_1}\ldots\delta^{\theta_q}_{\alpha_q}\end{pmatrix}=\begin{pmatrix}\delta^a_c & 0 & 0\\ 0 & \delta^\alpha_\theta & 0\\ 0 & 0 & \delta^\theta_\alpha\end{pmatrix}=\delta^A_C=\widetilde{I}.$$

Let $A\in\Im_q^p(B_m)$. If we take account of (5) and (16), we can easily prove that ${}^{vv}A'=\widetilde{A}({}^{vv}A)$, where ${}^{vv}A\in\Im_0^1(t_q^p(B_m))$ is a vector field defined by

$${}^{vv}A=\begin{pmatrix}{}^{vv}A^a\\ {}^{vv}A^\alpha\\ {}^{vv}A^{\overline{\alpha}}\end{pmatrix}=\begin{pmatrix}0\\0\\ A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\end{pmatrix},$$

with respect to the adapted frame $\left\{B_{(b)},C_{(\beta)},C_{(\overline{\beta})}\right\}$ of $\beta_\xi\left(M_n\right)$.

Taking account of (9), (10) and (16), we see that the complete lift ${}^{cc}X$ and horizontal lift ${}^{HH}X$ have respectively components

$$ {}^{cc}\widetilde{X}:\begin{pmatrix}\widetilde{X}^{b}\\ X^{\beta}\\ -L_{X}\xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\end{pmatrix},\ {}^{HH}\widetilde{X}:\begin{pmatrix}\widetilde{X}^{b}\\ X^{\beta}\\ (\nabla_{X}\xi)^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\end{pmatrix} $$

with respect to the adapted frame $\left\{B_{(b)},C_{(\beta)},C_{(\overline{\beta})}\right\}$ of $\beta_{\xi}(M_n)$, where ${}^{cc}\widetilde{X}'=\widetilde{A}\left({}^{cc}\widetilde{X}\right)$ and ${}^{HH}\widetilde{X}'=\widetilde{A}\left({}^{HH}\widetilde{X}\right)$.

We now, from equations (7), (8) and (16) see that $\gamma\varphi$ and $\widetilde{\gamma}\varphi$ have respectively components

$$ \gamma\varphi=(\gamma\varphi)^{I}=\begin{pmatrix}0\\ 0\\ \sum_{\lambda=1}^{p}\xi^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q}\varphi^{\alpha_\lambda}_{\varepsilon}\end{pmatrix},\ \widetilde{\gamma}\varphi=(\widetilde{\gamma}\varphi)^{I}=\begin{pmatrix}0\\ 0\\ \sum_{\mu=1}^{q}\xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\varepsilon\ldots\beta_q}\varphi^{\varepsilon}_{\beta_\mu}\end{pmatrix} $$

with respect to the adapted frame $\left\{B_{(b)},C_{(\beta)},C_{(\overline{\beta})}\right\}$ of $\beta_{\xi}(M_n)$.

Let $S\in\Im^{1}_{2}(M_n)$ now. If we take account of (16), we see that $(\gamma S)'=\widetilde{A}(\gamma S)$. γS is given by

$$ \gamma S=(\gamma S)^{I}_{J}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & \sum_{\lambda=1}^{p}S^{\beta_\lambda}_{\beta_\varepsilon}\xi^{\beta_1\ldots\varepsilon\ldots\beta_p}_{\alpha_1\ldots\alpha_q} & 0\end{pmatrix}, $$

with respect to the adapted frame $\left\{B_{(b)},C_{(\beta)},C_{(\overline{\beta})}\right\}$, where $S^{\beta_\lambda}_{\beta_\varepsilon}$ are local componenets of S on B_m.

BX, CX and $E\xi$ also have components:

$$ BX=\begin{pmatrix}X^{\alpha}\\ 0\\ 0\end{pmatrix},CX=\begin{pmatrix}0\\ X^{\alpha}\\ 0\end{pmatrix},E\xi=\begin{pmatrix}0\\ 0\\ \xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\end{pmatrix} $$

respectively, with respect to the adapted frame $\left\{B_{(b)},C_{(\beta)},C_{(\overline{\beta})}\right\}$ of the cross-section $\beta_{\xi}(M_n)$ determined by a tensor field ξ of type (p,q) in M_n.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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