# SOME TAUBERIAN THEOREMS FOR WEIGHTED MEANS OF DOUBLE INTEGRALS

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ABSTRACT. Let $p(x)$ and $q(y)$ be nondecreasing continuous functions on $[0, \infty)$ such that $p(0) = q(0) = 0$ and $p(x), q(y) \to \infty$ as $x, y \to \infty$. For a locally integrable function $f(x, y)$ on $\mathbb{R}_+^2 = [0, \infty) \times [0, \infty)$, we denote its double integral by $F(x, y) = \int_0^x \int_0^y f(t, s) dt ds$ and its weighted mean of type $(\alpha, \beta)$ by

$$t_{\alpha,\beta}(x, y) = \int_0^x \int_0^y \left(1 - \frac{p(t)}{p(x)}\right)^\alpha \left(1 - \frac{q(s)}{q(y)}\right)^\beta f(t, s) dt ds$$

where $\alpha > -1$ and $\beta > -1$. We say that $\int_0^\infty \int_0^\infty f(t, s) dt ds$ is integrable to $L$ by the weighted mean method of type $(\alpha, \beta)$ determined by the functions $p(x)$ and $q(x)$ if $\lim_{x,y\to\infty} t_{\alpha,\beta}(x, y) = L$ exists. We prove that if $\lim_{x,y\to\infty} t_{\alpha,\beta}(x, y) = L$ exists and $t_{\alpha,\beta}(x, y)$ is bounded on $\mathbb{R}_+^2$ for some $\alpha > -1$ and $\beta > -1$, then $\lim_{x,y\to\infty} t_{\alpha+h,\beta+k}(x, y) = L$ exists for all $h > 0$ and $k > 0$. Finally, we prove that if $\int_0^\infty \int_0^\infty f(t, s) dt ds$ is integrable to $L$ by the weighted mean method of type $(1, 1)$ determined by the functions $p(x)$ and $q(x)$ and conditions $\dfrac{p(x)}{p'(x)} \int_0^y f(x, s) ds = O(1)$ and $\dfrac{q(y)}{q'(y)} \int_0^x f(t, y) dt = O(1)$ hold, then $\lim_{x,y\to\infty} F(x, y) = L$ exists.

## 1. INTRODUCTION

Let $p(x)$ and $q(y)$ be nondecreasing continuous functions on $[0, \infty)$ such that $p(0) = 0$, $q(0) = 0$ and $p(x), q(y) \to \infty$ as $x, y \to \infty$. For a locally integrable function $f(x, y)$ on $\mathbb{R}_+^2 = [0, \infty) \times [0, \infty)$, we denote its double integral on $\mathbb{R}_+^2$ by $F(x, y) = \int_0^x \int_0^y f(t, s) dt ds$ and its weighted mean of type $(\alpha, \beta)$ determined by the functions $p(x)$ and $q(y)$ by

$$t_{\alpha,\beta}(x, y) = \int_0^x \int_0^y \left(1 - \frac{p(t)}{p(x)}\right)^\alpha \left(1 - \frac{q(s)}{q(y)}\right)^\beta f(t, s) dt ds \qquad (1)$$

where $\alpha > -1$ and $\beta > -1$. An improper double integral

$$\int_0^\infty \int_0^\infty f(t,s)dtds \tag{2}$$

is said to be integrable to $L$ by the weighted mean method of type $(\alpha, \beta)$ determined by the functions $p(x)$ and $q(y)$ if

$$\lim_{x,y\to\infty} t_{\alpha,\beta}(x,y) = L. \tag{3}$$

We use the notion of convergence in Pringsheim's sense, that is, both $x$ and $y$ tend to $\infty$ independently of each other in (3).

If we take $p(x) = x$ and $q(y) = y$ in (1), we have the definition of $(C, \alpha, \beta)$ integrability of $f(x,y)$ on $[0,\infty) \times [0,\infty)$ given by [3]. The $(C, 0, 0)$ integrability of $f(x,y)$ is convergence of the improper double integral (2).

It is clear that if $\lim_{x,y\to\infty} F(x,y) = L$ exists and $F(x,y)$ is bounded on $\mathbb{R}_+^2$, then the limit (3) also exists for $\alpha > -1$ and $\beta > -1$. The converse of this implication is not true in general. The converse of this implication may be true only by adding some suitable condition which is called a Tauberian condition. Any theorem which states that convergence of (2) follows from the integrability of $f(x,y)$ by the weighted mean method of type $(\alpha, \beta)$ determined by the function $p(x)$ and $q(y)$ and a Tauberian condition is said to be a Tauberian theorem.

In recent years, there has been an increasing interest on summability methods for functions of one and two variables. First, Laforgia [6] obtained a sufficient condition under which convergence of the improper integral follows from $(C, 1)$ integrability. Later, Çanak and Totur [1] extended the main results of Laforgia [6] to the $(C, \alpha)$ integrability of functions by weighted mean methods where $\alpha > -1$. Following these works, Totur and Çanak [10] obtained some Tauberian theorems in terms of the concept of the general control modulo of non-integer order for functions of one-variable. Recently, Özsaraç and Çanak [8] obtained Tauberian theorems for the iterations of weighted mean summable integrals. Totur et al. [9] introduced some new Tauberian conditions in terms of the weighted general control modulo for the weighted mean method of integrals. For some interesting Tauberian theorems for Cesàro and weighted integrability in quantum calculus, we refer the readers to Çanak et al. [2], Fitouhi and Brahim [5] and Totur et al. [11], etc. In [7], Móricz obtained one-sided Tauberian conditions which are necessary and sufficient in order that convergence follow from summability $(C, 1, 1)$ of (2). More generally, Çanak and Totur [3] obtained a sufficient condition under which convergence of (2) follows from $(C, \alpha, \beta)$ integrability of (2) where $\alpha > -1$ and $\beta > -1$.

In this paper we prove that if (3) exists and $t_{\alpha,\beta}(x,y)$ is bounded on $\mathbb{R}_+^2$ for some $\alpha > -1$ and $\beta > -1$ , then $\lim_{x,y\to\infty} t_{\alpha+h,\beta+k}(x,y) = L$ exists for all $h > 0$ and $k > 0$. As a corollary to this result, we show that if (2) is convergent to $L$ and the function $F(x,y)$ is bounded on $\mathbb{R}_+^2$, then $\lim_{x,y\to\infty} t_{1,1}(x,y) = L$. But, the converse of this implication may true under some conditions imposed on $p$, $q$ and $f$. Moreover, we give a Tauberian condition under which convergence of

improper double integrals follows from the existence of $\lim\limits_{x,y\to\infty} t_{11}(x,y) = L$. This paper has been presented at the 2nd International Conference of Mathematical Sciences (ICMS 2018), Maltepe University, İstanbul, Turkey [4].

## 2. Main Results

**Theorem 1.** *If* (3) *exists and* $t_{\alpha,\beta}(x,y)$ *is bounded on* $\mathbb{R}_+^2$ *for some* $\alpha > -1$ *and* $\beta > -1$, *then* $\lim_{x,y\to\infty} t_{\alpha+h,\beta+k}(x,y) = L$ *exists for all* $h > 0$ *and* $k > 0$.

*Proof.* Consider

$$\int_0^x \int_0^y \varphi(t,s;x,y)t_{\alpha,\beta}(x,y)dtds, \tag{4}$$

where

$$\varphi(t,s;x,y) = \frac{1}{B(\alpha+1,h)}\frac{p'(t)}{p(x)}\left(\frac{p(t)}{p(x)}\right)^\alpha\left(1 - \frac{p(t)}{p(x)}\right)^{h-1}$$
$$\frac{1}{B(\beta+1,k)}\frac{q'(s)}{q(y)}\left(\frac{q(s)}{q(y)}\right)^\beta\left(1 - \frac{q(s)}{q(y)}\right)^{k-1}$$

where $B$ denotes the Beta function defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x > 0, \quad y > 0.$$

Letting $u = \dfrac{p(t)}{p(x)}$ and $v = \dfrac{q(s)}{q(y)}$, we have

$$\int_0^x \int_0^y \varphi(t,s;x,y)dtds = 1. \tag{5}$$

We first prove that

$$\lim_{x,y\to\infty}\int_0^x \int_0^y \varphi(t,s;x,y)t_{\alpha,\beta}(x,y)dtds = L. \tag{6}$$

Since

$$\lim_{x,y\to\infty} t_{\alpha,\beta}(x,y) = L \tag{7}$$

by the hypothesis, there exist numbers $x_\varepsilon$ and $y_\varepsilon$ for any given $\varepsilon > 0$ such that

$$|t_{\alpha,\beta}(x,y) - L| < \varepsilon, \quad x \geq x_\varepsilon, \quad y \geq y_\varepsilon. \tag{8}$$

It follows from (5) that

$$\int_0^x \int_0^y \varphi(t,s;x,y)t_{\alpha,\beta}(x,y)dtds - L = \int_0^x \int_0^y \varphi(t,s;x,y)[t_{\alpha,\beta}(x,y) - L]dtds. \tag{9}$$

To prove (6), we need to show that

$$\left|\int_0^x \int_0^y \varphi(t,s;x,y)t_{\alpha,\beta}(x,y)dtds - L\right| < 4\varepsilon, \tag{10}$$

provided that $x$ and $y$ are large enough. We realize that by the hypothesis, the function $t_{\alpha,\beta}(x,y)$ is bounded on $\mathbb{R}_+^2$. Therefore, there exists a constant $K$ such that

$$|t_{\alpha,\beta}(x,y) - L| < K, \quad \text{for} \quad 0 \le x, y < \infty.$$

Using (5) and (8), we obtain, by (9),

$$\left| \int_0^x \int_0^y \varphi(t,s;x,y) \left(t_{\alpha,\beta}(x,y) - L\right) dt ds \right|$$

$$\le \int_0^{x_\epsilon} \int_0^{y_\epsilon} \varphi(t,s;x,y) |t_{\alpha,\beta}(x,y) - L| dt ds$$

$$+ \int_0^{x_\epsilon} \int_{y_\epsilon}^y \varphi(t,s;x,y) |t_{\alpha,\beta}(x,y) - L| dt ds$$

$$+ \int_{x_\epsilon}^x \int_0^{y_\epsilon} \varphi(t,s;x,y) |t_{\alpha,\beta}(x,y) - L| dt ds$$

$$+ \epsilon \int_{x_\epsilon}^x \int_{y_\epsilon}^y \varphi(t,s;x,y) dt ds$$

$$\le K \int_0^{x_\epsilon} \int_0^{y_\epsilon} \varphi(t,s;x,y) dt ds + K \int_0^{x_\epsilon} \int_{y_\epsilon}^y \varphi(t,s;x,y) dt ds$$

$$+ K \int_{x_\epsilon}^x \int_0^{y_\epsilon} \varphi(t,s;x,y) dt ds + \epsilon \int_0^x \int_0^y \varphi(t,s;x,y) dt ds$$

$$= K \int_0^{x_\epsilon} \int_0^{y_\epsilon} \varphi(t,s;x,y) dt ds + K \int_0^{x_\epsilon} \int_{y_\epsilon}^y \varphi(t,s;x,y) dt ds$$

$$+ K \int_{x_\epsilon}^x \int_0^{y_\epsilon} \varphi(t,s;x,y) dt ds + \epsilon$$

By the substitution $u = \dfrac{p(t)}{p(x)}$, $v = \dfrac{q(s)}{q(y)}$, we have

$$\int_0^{x_\epsilon} \int_0^{y_\epsilon} \varphi(t,s;x,y) dt ds \;=\; \frac{1}{B(\alpha+1,h)} \int_0^{x_\epsilon} \frac{p'(t)}{p(x)} \left(\frac{p(t)}{p(x)}\right)^\alpha \left(1 - \frac{p(t)}{p(x)}\right)^{h-1} dt$$

$$\times \; \frac{1}{B(\beta+1,k)} \int_0^{y_\epsilon} \frac{q'(s)}{q(y)} \left(\frac{q(s)}{q(y)}\right)^\beta \left(1 - \frac{q(s)}{q(y)}\right)^{k-1} ds$$

$$= \; \frac{1}{B(\alpha+1,h)} \int_0^{p(x_\epsilon)/p(x)} u^\alpha (1-u)^{h-1} du$$

$$\times \; \frac{1}{B(\beta+1,k)} \int_0^{q(y_\epsilon)/q(y)} v^\beta (1-v)^{k-1} dv$$

which tends to zero when $x, y \to \infty$ for any fixed $x_\varepsilon$ and $y_\varepsilon$. Thus, there exist some $\widehat{x_\varepsilon^1}$ and $\widehat{y_\varepsilon^1}$ such that

$$K \int_0^{x_\epsilon} \int_0^{y_\epsilon} \varphi(t,s;x,y)dtds < \varepsilon, \quad x \geq \widehat{x_\varepsilon^1}, \quad y \geq \widehat{y_\varepsilon^1}.$$

By the substitution $u = \dfrac{p(t)}{p(x)}$, $v = \dfrac{q(s)}{q(y)}$, we have

$$
\begin{aligned}
\int_0^{x_\epsilon} \int_{y_\epsilon}^y \varphi(t,s;x,y)dtds &= \frac{1}{B(\alpha+1,h)} \int_0^{x_\varepsilon} \frac{p'(t)}{p(x)} \left(\frac{p(t)}{p(x)}\right)^\alpha \left(1 - \frac{p(t)}{p(x)}\right)^{h-1} dt \\
&\times \frac{1}{B(\beta+1,k)} \int_{y_\epsilon}^y \frac{q'(s)}{q(y)} \left(\frac{q(s)}{q(y)}\right)^\beta \left(1 - \frac{q(s)}{q(y)}\right)^{k-1} ds \\
&= \frac{1}{B(\alpha+1,h)} \int_0^{p(x_\varepsilon)/p(x)} u^\alpha (1-u)^{h-1} du \\
&\times \frac{1}{B(\beta+1,k)} \int_{q(y_\varepsilon)/q(y)}^1 v^\beta (1-v)^{k-1} dv
\end{aligned}
$$

which tends to zero when $x, y \to \infty$ for any fixed $x_\varepsilon$ and $y_\varepsilon$ (Note that $\frac{1}{B(\beta+1,k)} \int_{q(y_\varepsilon)/q(y)}^1 v^\beta (1-v)^{k-1} dv$ tends to 1 as $y \to \infty$ ). Thus, there exist some $\widehat{x_\varepsilon^2}$ and $\widehat{y_\varepsilon^2}$ such that

$$K \int_0^{x_\epsilon} \int_{y_\epsilon}^y \varphi(t,s;x,y)dtds < \varepsilon, \quad x \geq \widehat{x_\varepsilon^2}, \quad y \geq \widehat{y_\varepsilon^2}.$$

Similarly, the integral

$$\int_{x_\epsilon}^x \int_0^{y_\epsilon} \varphi(t,s;x,y)dtds$$

tends to to zero when $x, y \to \infty$ for any fixed $x_\varepsilon$ and $y_\varepsilon$ (Note that $\frac{1}{B(\alpha+1,h)} \int_{p(x_\varepsilon)/p(x)}^1 u^\alpha (1-u)^{h-1} du$ tends to 1 as $x \to \infty$ ). Thus, there exist some $\widehat{x_\varepsilon^3}$ and $\widehat{y_\varepsilon^3}$ such that

$$K \int_{x_\epsilon}^x \int_0^{y_\epsilon} \varphi(t,s;x,y)dtds < \varepsilon, \quad x \geq \widehat{x_\varepsilon^3}, \quad y \geq \widehat{y_\varepsilon^3}.$$

Hence, we have (10) for $x \geq \max\{x_\epsilon, \widehat{x_\varepsilon}^1, \widehat{x_\varepsilon}^2, \widehat{x_\varepsilon}^3\}$, $y \geq \max\{y_\epsilon, \widehat{y_\varepsilon}^1, \widehat{y_\varepsilon}^2, \widehat{y_\varepsilon}^3\}$ and this proves (6). We obtain

$$
\begin{aligned}
&\int_0^x \int_0^y \varphi(t,s;x,y)t_{\alpha,\beta}(t,s)dtds \\
&= \int_0^x \int_0^y \varphi(t,s;x,y) \left(\int_0^t \int_0^s \left(1 - \frac{p(u)}{p(t)}\right)^\alpha \left(1 - \frac{q(v)}{q(s)}\right)^\beta f(u,v)dudv\right) dtds
\end{aligned}
$$

$$= \int_0^x \int_0^y f(u,v) \left( \int_u^x \int_v^y \varphi(t,s;x,y) \left( 1 - \frac{p(u)}{p(t)} \right)^\alpha \left( 1 - \frac{q(v)}{q(s)} \right)^\beta dtds \right) dudv$$

$$= \int_0^x \int_0^y f(u,v)I(u,v;x,y)dudv,$$

where

$$I(u,v;x,y) = \int_u^x \int_v^y \varphi(t,s;x,y) \left( 1 - \frac{p(u)}{p(t)} \right)^\alpha \left( 1 - \frac{q(v)}{q(s)} \right)^\beta dtds.$$

Here, we write $I(u,v;x,y)$ as

$$I(u,v;x,y) = \int_u^x \int_v^y \varphi(t,s;x,y) \left( 1 - \frac{p(u)}{p(t)} \right)^\alpha \left( 1 - \frac{q(v)}{q(s)} \right)^\beta dtds$$

$$= \left( \frac{1}{B(\alpha+1,h)} \int_u^x \frac{p'(x)}{p(x)} \left( \frac{p(t)}{p(x)} \right)^\alpha \left( 1 - \frac{p(t)}{p(x)} \right)^{h-1} \left( 1 - \frac{p(u)}{p(t)} \right)^\alpha dt \right)$$

$$\times \left( \frac{1}{B(\beta+1,k)} \int_v^y \frac{q'(y)}{q(y)} \left( \frac{q(s)}{q(y)} \right)^\beta \left( 1 - \frac{q(s)}{q(y)} \right)^{k-1} \left( 1 - \frac{q(v)}{q(s)} \right)^\beta ds \right)$$

$$= \left( \frac{1}{B(\alpha+1,h)} \left( \frac{1}{p(x)} \right)^{\alpha+1} \int_u^x p'(t) \left( 1 - \frac{p(t)}{p(x)} \right)^{h-1} (p(t)-p(u))^\alpha dt \right)$$

$$\times \left( \frac{1}{B(\beta+1,k)} \left( \frac{1}{q(y)} \right)^{\beta+1} \int_v^y q'(s) \left( 1 - \frac{q(s)}{q(y)} \right)^{k-1} (q(s)-q(v))^\beta ds \right)$$

$$= I_1(u,x)I_2(v,y),$$

where

$$I_1(u,x) = \frac{1}{B(\alpha+1,h)} \left( \frac{1}{p(x)} \right)^{\alpha+1} \int_u^x p'(t) \left( 1 - \frac{p(t)}{p(x)} \right)^{h-1} (p(t)-p(u))^\alpha dt,$$

and

$$I_2(v,y) = \frac{1}{B(\beta+1,k)} \left( \frac{1}{q(y)} \right)^{\beta+1} \int_v^y q'(s) \left( 1 - \frac{q(s)}{q(y)} \right)^{k-1} (q(s)-q(v))^\beta ds.$$

Substituting $p(t) = p(x) - (p(x)-p(u))x$ in $I_1(u,x)$, we have

$$I_1(u,x) = \frac{1}{B(\alpha+1,h)} \left( 1 - \frac{p(u)}{p(x)} \right)^{h-1} \left( 1 - \frac{p(u)}{p(x)} \right)^{\alpha+1} \int_0^1 x^{h-1}(1-x)^\alpha dx$$

$$= \left( 1 - \frac{p(u)}{p(x)} \right)^{\alpha+h},$$

and similarly we have

$$I_2(v,y) = \left( 1 - \frac{q(v)}{q(y)} \right)^{\beta+k}.$$

These show that

$$\int_0^x \int_0^y \varphi(t,s;x,y) t_{\alpha,\beta}(t,s) dt ds$$

$$= \int_0^x \int_0^y \left(1 - \frac{p(u)}{p(x)}\right)^{\alpha+h} \left(1 - \frac{q(v)}{q(y)}\right)^{\beta+k} f(u,v) du dv$$

$$= t_{\alpha+h,\beta+k}(x,y).$$

This completes the proof of Theorem 1. □

**Corollary 2.** *If* $\lim_{x,y\to\infty} F(x,y) = L$ *exists and* $F(x,y)$ *is bounded on* $\mathbb{R}_+^2$, *then* $\lim_{x,y\to\infty} t_{1,1}(x,y) = L$ *exists.*

*Proof.* Take $\alpha = \beta = 0$ and $h = k = 1$ in Theorem 1. □

**Theorem 3.** *If* (2) *is integrable to* $L$ *by the weighted mean method of type* $(1,0)$ *determined by the function* $p(x)$ *and*

$$\frac{p(x)}{p'(x)} \int_0^y f(x,s) ds = O(1) \tag{11}$$

*holds, then* (2) *converges to* $L$.

**Theorem 4.** *If* (2) *is integrable to* $L$ *by the weighted mean method of type* $(0,1)$ *determined by the function* $q(y)$ *and*

$$\frac{q(y)}{q'(y)} \int_0^x f(t,y) dt = O(1) \tag{12}$$

*holds, then* (2) *converges to* $L$.

Since the proofs of Theorem 3 and Theorem 4 can be obtained by the similar techniques and steps as in the proof of Theorem 2.3 in [1], we omit them.

By the next theorem, we recover convergence of the improper double integral (2) from its weighted mean method of type $(1,1)$ determined by the functions $p(x)$ and $q(y)$ under conditions (13) and (14).

**Theorem 5.** *If* (2) *is integrable to* $L$ *by the weighted mean method of type* $(1,1)$ *determined by the functions* $p(x)$, $q(y)$ *and*

$$\frac{p(x)}{p'(x)} \int_0^y f(x,s) ds = O(1) \tag{13}$$

*and*

$$\frac{q(y)}{q'(y)} \int_0^x f(t,y) dt = O(1) \tag{14}$$

*then* (2) *converges to* $L$.

*Proof.* Assume that (2) is integrable to $L$ by the weighted mean method of type $(1,1)$ determined by the functions $p(x)$ and $q(y)$, that is,

$$G(x,y) := \int_0^x \int_0^y \left(1 - \frac{p(t)}{p(x)}\right) \left(1 - \frac{q(s)}{q(y)}\right) f(t,s) dt ds \to L, \quad x,y \to \infty \quad (15)$$

We rewrite $G(x,y)$ as

$$G(x,y) = \int_0^y \left(1 - \frac{q(s)}{q(y)}\right) \frac{\partial}{\partial s} G_1(x,s) ds, \quad (16)$$

where

$$G_1(x,y) := \int_0^x \int_0^y \left(1 - \frac{p(t)}{p(x)}\right) f(t,s) dt ds. \quad (17)$$

It follows from (15), (16) and (17) that $\frac{\partial}{\partial y} G_1(x,y)$ is integrable to $L$ by the weighted mean method of type $(0,1)$ determined by the functions $q(y)$.

By (15), we have

$$G(x,y) = G_1(x,y) - H_1(x,y), \quad (18)$$

where

$$H_1(x,y) = \frac{1}{q(y)} \int_0^y q(s) \frac{\partial}{\partial s} G_1(x,s) ds. \quad (19)$$

We have to show that $H_1(x,y) \to 0$ as $x,y \to \infty$.

By (16), we find

$$\frac{\partial}{\partial y} G(x,y) = \frac{q'(y)}{q(y)^2} \int_0^y q(s) \frac{\partial}{\partial s} G_1(x,s) ds = \frac{q'(y)}{q(y)} H_1(x,y). \quad (20)$$

We also have

$$\int_{y_1}^{y_2} \frac{\partial}{\partial y} G(x,y) dy = G(x,y_2) - G(x,y_1)$$

$$= \int_{y_1}^{y_2} \frac{q'(y)}{q(y)} H_1(x,y) dy$$

$$= \int_{\log q(y_1)}^{\log q(y_2)} H_1(x,q^{-1}(e^v)) dv$$

$$= \int_{\log q(y_1)}^{\log q(y_2)} R(x,v) dv.$$

Here, we used the substitution $q(y) = e^v$ and $R(x,v) = H_1(T,q^{-1}(e^v))$. We need to show that $\lim_{v \to \infty} R(x,v) = 0$. By the simple calculation, we have

$$\frac{\partial}{\partial v} R(x,v) = \frac{e^v}{q'(q^{-1}e^v))} \frac{\partial}{\partial v} H_1(x,q^{-1}(e^v)) = \frac{q(y)}{q'(y)} \frac{\partial}{\partial y} H_1(x,y). \quad (21)$$

By (19), we get

$$q(y) H_1(x,y) = \int_0^y q(s) \frac{\partial}{\partial s} G_1(x,s) ds. \quad (22)$$

Differentiating the both sides of (22) with respect to $y$ gives

$$H_1(x,y) + \frac{q(y)}{q'(y)} \frac{\partial}{\partial y} H_1(x,y) = \frac{q(y)}{q'(y)} \frac{\partial}{\partial y} G_1(x,y). \qquad (23)$$

Taking the weighted mean of type (1,0) of the both sides of (14) we have

$$\frac{q(y)}{q'(y)} \frac{\partial}{\partial y} G_1(x,y) = O(1), \qquad (24)$$

which implies that

$$\frac{q(y)}{q'(y)} \frac{\partial}{\partial y} H_1(x,y) = O(1) \qquad (25)$$

by (21). Thus by (21), we attain that $\frac{\partial}{\partial y} R(x,y)$ is bounded. Since $G(x,y)$ is convergent, given any $\varepsilon > 0$ there exists a $y_\varepsilon$ such that

$$|G(x,y_1) - G(x,y_2)| < \varepsilon$$

when $y_1, y_2 > y_\varepsilon$.

Suppose $\xi > \ln q(y_\varepsilon)$ and $R(x,\xi) > 0$. Then $R(x,y) > 0$ for $\xi - \psi < y < \xi$ and $\xi < y < \xi + \psi$ where $\psi = \dfrac{R(x,\xi)}{K}$. If we integrate $R(x,y)$ between $\xi - \psi$ and $\xi + \psi$, we have

$$\int_{\xi - \psi}^{\xi + \psi} R(x,y)dy > \frac{2R^2(x,\xi)}{K} \qquad (26)$$

Furthermore, we have, by (??),

$$\frac{2R^2(x,\xi)}{K} < \int_{\xi - \psi}^{\xi + \psi} R(x,y)dy = G(x, q^{-1}(e^{\xi + \psi})) - G(x, q^{-1}(e^{\xi - \psi})) < \varepsilon. \qquad (27)$$

Therefore,

$$R(x,\xi) < \sqrt{\frac{K\varepsilon}{2}} \qquad (28)$$

which shows that $H_1(x,y) \to 0$ as $x,y \to \infty$. It follows from (15) and (18) that $G_1(x,y) \to L$ as $x,y \to \infty$.

Since $G_1(x,y) \to L$ as $x,y \to \infty$ and the condition (13), we have $\lim_{x,y \to \infty} F(x,y) = L$ by Theorem 3. $\qquad \square$

## 3. Conclusion

In this paper we have extended Tauberian theorems given for $(C, \alpha, \beta)$ integrability method to the weighted mean method of type $(\alpha, \beta)$ determined by the functions $p(x)$ and $q(y)$. In a forthcoming work, we plan to obtain analogous results for the weighted mean method for functions of three or more variables.

## References

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