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ON IRREGULAR COLORING OF WHEEL RELATED GRAPHS

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Abstract. An irregular coloring is a proper coloring in which distinct vertices have different color codes. In this paper, we obtain the irregular chromatic number for middle graph, total graph, central graph and line graph of wheel graph, helm graph, gear graph and closed helm graph.

1. Introduction

A proper coloring c is an *irregular coloring* [1] if no two like-colored vertices have the same color code. *i.e.*, for every pair of vertices u and w, $code(u) \neq code(w)$ whenever $c(u) = c(w)$. Thus, an irregular coloring distinguishes each vertex from each of other vertex by its color or by its color code.

In [5], Mary Radcliffe and Ping Zhang established sharp upper and lower bounds for the irregular chromatic number of a disconnected graph in terms of the irregular chromatic numbers of its components. Irregular chromatic number of some classes of disconnected graphs are determined.

It is shown that if G is a nontrivial graph of order n, then $\sqrt[2]{n} \leq \chi_{ir}(G)+\chi_{ir}(\bar{G}) \leq 2n, n \leq \chi_{ir}(G)\chi_{ir}(\bar{G}) \leq n^2$, and each bound in these inequalities is sharp.

In [4], Radcliffe and Zhang found a bound for the irregular chromatic number of a graph on n vertices and Let c be a (proper) coloring of the vertices of a nontrivial graph G and let u and v be two vertices of G then

(1) If $c(u) \neq c(v)$, *then* $code(u) \neq code(v)$
(2) If $d(u) \neq d(v)$, *then* $code(u) \neq code(v)$
(3) If c is irregular and $N(u) = N(v)$, *then* $c(u) \neq c(v)$

The *line graph* [3] of a graph G, denoted by $L(G)$, is the graph in which, all edges $e_i \in E(G)$ are represented by $e'_i \in V(L(G))$ and an edge $e'_ie'_j \in E(L(G))$ if and only if the edges e_i, e_j share a vertex (are incident) in G.

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The *middle graph* [7] of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds:
(i) x, y are in $E(G)$ and x, y are adjacent in G.
(ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *total graph* [2,7] of G, denoted by $T(G)$ is defined as follows: The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $T(G)$ are adjacent in $T(G)$ in case one of the following holds:
(i) x, y are in $V(G)$ and x is adjacent to y in G.
(ii) x, y are in $E(G), x, y$ are adjacent in G.
(iii) x is in $V(G)$ and y is in $E(G)$, and x, y are incident in G.

The *central graph* [6] of a graph G denoted by $C(G)$ is formed by adding an extra vertex on each edge of G, and joining each pair of vertices of the original graph which were previously non-adjacent.

The *wheel graph* W_n is defined to be the join $C_n + K_1$. The vertex corresponding to K_1 is known as apex vertex and the vertices corresponding to cycle are known as rim vertices. The edges corresponding to cycles are known as rim edges and the edges incident with the apex vertex are known as spoke edges.

The *helm graph* H_n is the graph obtained from wheel W_n by attaching a pendant edge to each of its rim vertices.

The *closed helm* CH_n is the graph obtained from a helm by joining each pendant vertex to form a cycle.

A *gear graph* G_n is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the n cycle of W_n.

2. Irregular Coloring of Wheel Graphs

Theorem 2.1. *For a wheel graph, W_n, $n \geq 4$ the irregular chromatic number of the total graph of wheel graph, $(T(W_n))$ is $\chi_{ir}(T(W_n)) = n + 1$.*

Proof. Let $V(H_n) = \{v, p_i, q_i : 1 \leq i \leq n\}$ and

$$V(M(H_n)) = \Big\{ v,\, p_i(1 \leq i \leq n),\, r_i(1 \leq i \leq n),\ s_i(1 \leq i \leq n) \Big\},$$

where r_i is the vertex of $T(W_n)$ corresponding to the edge vp_i $(1 \leq i \leq n)$ of W_n $(1 \leq i \leq n)$, s_i is the vertex of $T(W_n)$ corresponding to the edge p_ip_{i+1} $(1 \leq i \leq n-1)$ of W_n and s_n is the vertex corresponding to the edge p_np_1 of W_n, the degree of vertices are $d(v) = 2n$, $d(p_i) = 6$, $d(r_i) = n + 3$ and $d(s_i) = 6$ for $(1 \leq i \leq n)$.

Consider the following (n+1)-coloring of $T(W_n)$,

- Assign the color c_1 to v.
- Assign the color c_{i+1} to r_i for $(1 \leq i \leq n)$.
- Assign the color c_i to s_i for $(2 \leq i \leq n)$.
- Assign the color c_{i+2} to p_i for $(1 \leq i \leq n-1)$.

- Assign the color c_2 to p_n.
- Assign the color c_{n+1} to s_1.

To prove (n+1)-coloring is an irregular coloring of $T(W_n)$, since $deg(s_i) = deg(p_i)$ for $(1 \leq i \leq n)$ each p_i's are adjacent to v but s_i's are not adjacent to v, hence $code(s_i) \neq code(p_i)$ for $(1 \leq i \leq n)$. Hence $\chi_{ir}(T(W_n)) \leq (n+1)$. By the definition of total graph $\{v, r_i(1 \leq i \leq n)\}$ induces a clique of order (n+1) in $T(W_n)$. Therefore, $\chi_{ir}(T(W_n)) \geq \chi(T(W_n)) = n+1$. Hence $\chi_{ir}(T(W_n)) = n+1$. □

Theorem 2.2. *For a wheel graph, W_n, $n \geq 4$ the irregular chromatic number of the line graph of wheel graph, $(L(W_n))$ is $\chi_{ir}(L(W_n)) = n$.*

Proof. Let $V(W_n) = \{v, p_i : 1 \leq i \leq n\}$ and $V(L(W_n)) = \{r_i(1 \leq i \leq n), s_i(1 \leq i \leq n)\}$, where r_i is the vertex of $L(W_n)$ corresponding to the edge vp_i $(1 \leq i \leq n)$ of W_n, s_i is the vertex of $L(W_n)$ corresponding to the edge p_ip_{i+1} $(1 \leq i \leq n-1)$ of W_n and s_n is the vertex corresponding to the edge p_np_1 of W_n, the degree of vertices are $d(r_i) = n+1$ and $d(s_i) = 4$ for $(1 \leq i \leq n)$.
Consider the color function $c : V(L(W_n)) \rightarrow \{c_1, c_2, ...c_n\}$ defined by

$$\begin{aligned} c(r_i) &= c_i \quad \text{for } 1 \leq i \leq n \\ c(s_i) &= \begin{cases} c_{i-1} & \text{for } 2 \leq i \leq n \\ c_n & \text{for } i = 1 \end{cases} \end{aligned}$$

To prove c is an irregular coloring function of $L(W_n)$, since $deg(s_i) \neq deg(r_i)$ for $(1 \leq i \leq n)$ this shows that $code(s_i) \neq code(r_i)$ for $(1 \leq i \leq n)$. Hence $\chi_{ir}(L(W_n)) \leq n$. By the definition of line graph $\{r_i : (1 \leq i \leq n)\}$ induces a clique of order n in $L(W_n)$. Therefore, $\chi_{ir}(L(W_n)) \geq \chi(L(W_n)) = n$. Hence $\chi_{ir}(L(W_n)) = n$.

□

3. Irregular Coloring of Helm Graphs

Theorem 3.1. *For a helm graph, H_n, $n \geq 5$ the irregular chromatic number of $M(H_n)$ is $\chi_{ir}(M(H_n)) = n+1$.*

Proof. Let $V(H_n) = \{v, p_i, q_i : 1 \leq i \leq n\}$ and $V(M(H_n)) = \{v, p_i(1 \leq i \leq n),$ $q_i(1 \leq i \leq n), r_i(1 \leq i \leq n), s_i(1 \leq i \leq n), t_i(1 \leq i \leq n)\}$, where r_i is the vertex of $M(H_n)$ corresponding to the edge vp_i $(1 \leq i \leq n)$ of H_n, t_i is the vertex of $M(H_n)$ corresponding to the edge p_iq_i $(1 \leq i \leq n)$ of H_n, s_i is the vertex of $M(H_n)$ corresponding to the edge p_ip_{i+1} $(1 \leq i \leq n-1)$ of H_n and s_n is the vertex corresponding to the edge p_np_1 of H_n, the degree of vertices are $d(v) = n$, $d(p_i) = 4$, $d(q_i) = 1$, $d(r_i) = n+4$, $d(s_i) = 8$ and $d(t_i) = 5$ for $(1 \leq i \leq n)$.
Consider the following (n+1)-coloring of $M(H_n)$,

- Assign the color c_1 to v, p_i, q_i for $(1 \leq i \leq n)$.
- Assign the color c_{i+1} to r_i for $(1 \leq i \leq n)$.
- Assign the color c_i to s_i for $(2 \leq i \leq n)$.

- Assign the color c_{n+1} to s_1.
- Assign the color c_{2+i} to t_i for $(1 \leq i \leq n-1)$.
- Assign the color c_2 to t_n.

To prove (n+1)-coloring is an irregular coloring of $M(H_n)$, since $deg(v) \neq deg(q_i)$ for $(1 \leq i \leq n)$ it shows that $code(v) \neq code(q_i)$ for $(1 \leq i \leq n)$. Since $deg(r_i) = deg(s_i)$ for $(1 \leq i \leq n)$ each r_i's are adjacent to v but s_i's are not adjacent to v, hence $code(r_i) \neq code(s_i)$ for $(1 \leq i \leq n)$. Thus $\chi_{ir}(M(H_n)) \leq (n+1)$. By the definition of middle graph $\{v, r_i(1 \leq i \leq n)\}$ induces a clique of order (n+1) in $M(H_n)$. Therefore, $\chi_{ir}(M(H_n)) \geq \chi(M(H_n)) = n+1$. Hence $\chi_{ir}(M(H_n)) = n+1$. □

Theorem 3.2. *For a helm graph, H_n, $n \geq 5$ the irregular chromatic number of $T(H_n)$ is $\chi_{ir}(T(H_n)) = n+1$.*

Proof. Let $V(H_n) = \{v, p_i, q_i : 1 \leq i \leq n\}$ and $V(T(H_n)) = \{v, p_i(1 \leq i \leq n), q_i(1 \leq i \leq n), r_i(1 \leq i \leq n), s_i(1 \leq i \leq n), t_i(1 \leq i \leq n)\}$, where r_i is the vertex of $T(H_n)$ corresponding to the edge vp_i $(1 \leq i \leq n)$ of H_n, t_i is the vertex of $T(H_n)$ corresponding to the edge p_iq_i $(1 \leq i \leq n)$ of H_n, s_i is the vertex of $T(H_n)$ corresponding to the edge p_ip_{i+1} $(1 \leq i \leq n-1)$ of H_n and s_n is the vertex corresponding to the edge p_np_1 of H_n, the degree of vertices are $d(v) = 2n$, $d(p_i) = 8$, $d(q_i) = 2$, $d(r_i) = n+4$, $d(s_i) = 8$ and $d(t_i) = 5$ for $(1 \leq i \leq n)$.
Consider the following (n+1)-coloring of $T(H_n)$,

- Assign the color c_1 to v, q_i for $(1 \leq i \leq n)$.
- Assign the color c_{i+1} to r_i for $(1 \leq i \leq n)$.
- Assign the color c_{i+2} to p_i for $(1 \leq i \leq n-1)$.
- Assign the color c_2 to p_n.
- Assign the color c_i to t_i for $(2 \leq i \leq n)$.
- Assign the color c_{n+1} to t_1.
- case(i): if n is odd
  (1) Assign the color c_1 to s_{2i} for $(1 \leq i \leq \frac{n-1}{2})$.
  (2) Assign the color c_{2i} to s_{2i+1} for $(1 \leq i \leq (\frac{n-1}{2} - 1)$.
- case(ii): if n is even
  (1) Assign the color c_1 to s_{2i} for $(1 \leq i \leq \frac{n}{2})$.
  (2) Assign the color c_{2i} to s_{2i+1} for $(1 \leq i \leq \frac{n}{2} - 1)$
- Assign the color c_n to s_1.

To prove (n+1)-coloring is an irregular coloring of $T(H_n)$, since $deg(v) \neq deg(q_i)$ for $(1 \leq i \leq n)$ it shows that $code(v) \neq code(q_i)$ for $(1 \leq i \leq n)$. since $deg(s_i) = deg(p_i)$ for $(1 \leq i \leq n)$ each p_i's are adjacent to v but s_i's are not adjacent to v, hence $code(s_i) \neq code(p_i)$ for $(1 \leq i \leq n)$. Hence $\chi_{ir}(T(H_n)) \leq (n+1)$. By the definition of total graph $\{v, r_i(1 \leq i \leq n)\}$ induces a clique of order (n+1) in $T(H_n)$. Therefore, $\chi_{ir}(T(H_n)) \geq \chi(T(H_n)) = n+1$. Hence $\chi_{ir}(T(H_n)) = n+1$. □

Theorem 3.3. *For a helm graph, H_n, $n \geq 3$ the irregular chromatic number of $C(H_n)$ is $\chi_{ir}C(H_n)) = n+1$.*

Proof. Let $V(H_n) = \{v, p_i, q_i : 1 \leq i \leq n\}$ and $V(C(H_n)) = \{v, p_i(1 \leq i \leq n), q_i(1 \leq i \leq n), r_i(1 \leq i \leq n), s_i(1 \leq i \leq n), t_i(1 \leq i \leq n)\}$, where r_i is the vertex of $C(H_n)$ corresponding to the edge vp_i $(1 \leq i \leq n)$ of H_n, t_i is the vertex of $C(H_n)$ corresponding to the edge p_iq_i $(1 \leq i \leq n)$ of H_n, s_i is the vertex of $C(H_n)$ corresponding to the edge p_ip_{i+1} $(1 \leq i \leq n-1)$ of H_n and s_n is the vertex corresponding to the edge p_np_1 of H_n, the degree of vertices are $d(v) = 2n$, $d(p_i) = 2n$, $d(q_i) = 2n$, $d(r_i) = 2$, $d(s_i) = 2$ and $d(t_i) = 2$ for $(1 \leq i \leq n)$.
Consider the color function $c : V(C(H_n)) \to \{c_1, c_2, ...c_{n+1}\}$ defined by

$$\begin{aligned}
c\,(v) &= c_1 \\
c\,(t_i) &= c_1 \quad \text{for } 1 \leq i \leq n \\
c\,(r_i) &= c_{i+1} \quad \text{for } 1 \leq i \leq n \\
c\,(p_i) &= \begin{cases} c_i & \text{for } 2 \leq i \leq n \\ c_{n+1} & \text{for } i = 1 \end{cases} \\
c\,(q_i) &= \begin{cases} c_i & \text{for } 2 \leq i \leq n \\ c_{n+1} & \text{for } i = 1 \end{cases} \\
c\,(s_i) &= \begin{cases} c_{i+2} & \text{for } 1 \leq i \leq n-1 \\ c_2 & \text{for } i = n \end{cases}
\end{aligned}$$

To prove c is an irregular coloring function of $C(H_n)$, since $deg(t_i) \neq deg(v)$ for $(1 \leq i \leq n)$, its shows that $code(t_i) \neq code(v)$ for $(1 \leq i \leq n)$ and $deg(p_i) = deg(q_i)$ for $(1 \leq i \leq n)$, each q_i's are adjacent to v but p_i's are not adjacent to v, hence $code(p_i) \neq code(q_i)$ for $(1 \leq i \leq n)$. Hence $\chi_{ir}(C(H_n)) \leq (n+1)$. By the definition of central graph $\{v, q_i(1 \leq i \leq n)\}$ induces a clique of order (n+1) in $C(H_n)$. Therefore, $\chi_{ir}(C(H_n)) \geq \chi(C(H_n)) = n+1$. Hence $\chi_{ir}(C(H_n)) = n+1$. □

Theorem 3.4. *For a helm graph, H_n, $n \geq 4$ the irregular chromatic number of $L(H_n)$ is $\chi_{ir}L(H_n)) = n$.*

Proof. Let $V(H_n) = \{v, p_i, q_i : 1 \leq i \leq n\}$ and $V(L(H_n)) = \{r_i(1 \leq i \leq n), s_i(1 \leq i \leq n), t_i(1 \leq i \leq n)\}$, where r_i is the vertex of $L(H_n)$ corresponding to the edge vp_i $(1 \leq i \leq n)$ of H_n, t_i is the vertex of $L(H_n)$ corresponding to the edge p_iq_i $(1 \leq i \leq n)$ of H_n, s_i is the vertex of $L(H_n)$ corresponding to the edge p_ip_{i+1} $(1 \leq i \leq n-1)$ of H_n and s_n is the vertex corresponding to the edge p_np_1 of H_n, the degree of vertices are $d(r_i) = n+2$, $d(s_i) = 6$ and $d(t_i) = 3$ for $(1 \leq i \leq n)$.
Consider the following n-coloring of $L(H_n)$,

- Assign the color c_i to r_i for $(1 \leq i \leq n)$.
- Assign the color c_{i-1} to t_i for $(2 \leq i \leq n)$.
- Assign the color c_n to t_1.
- Assign the color $c_{\lfloor \frac{n}{2} \rfloor + i}$ to s_i for $(1 \leq i \leq \lceil \frac{n}{2} \rceil)$.
- (1) If n is even, assign the color c_i to $s_{\frac{n}{2}+i}$ for $(1 \leq i \leq \frac{n}{2})$.

(2) If n is odd, assign the color c_i to $s_{\frac{n-1}{2}+1+i}$ for $(1 \leq i \leq \frac{n-1}{2})$.

To prove n-coloring is an irregular coloring of $L(H_n)$, since $deg(t_i) \neq deg(r_i)$ for $(1 \leq i \leq n)$ this shows that $code(t_i) \neq code(r_i)$ for $(1 \leq i \leq n)$. Hence $\chi_{ir}(L(H_n)) \leq n$. By the definition of line graph $\{r_i : (1 \leq i \leq n)\}$ induces a clique of order n in $L(H_n)$. Therefore, $\chi_{ir}(L(H_n)) \geq \chi(L(H_n)) = n$. Hence $\chi_{ir}(L(H_n)) = n$.

□

4. Irregular Coloring of closed helm Graphs

Theorem 4.1. *For a closed helm graph, CH_n, $n \geq 5$ the irregular chromatic number of the middle graph of closed helm, $M(CH_n)$ is $\chi_{ir}(M(CH_n)) = n+1$.*

Proof. Let $V(CH_n) = \{v, p_i, q_i : 1 \leq i \leq n\}$ and $V(M(CH_n)) = \{v, p_i(1 \leq i \leq n), q_i(1 \leq i \leq n), r_i(1 \leq i \leq n), s_i(1 \leq i \leq n), t_i(1 \leq i \leq n), u_i(1 \leq i \leq n)\}$, where r_i is the vertex of $M(CH_n)$ corresponding to the edge vp_i $(1 \leq i \leq n)$ of CH_n, t_i is the vertex of $M(CH_n)$ corresponding to the edge p_iq_i $(1 \leq i \leq n)$ of CH_n, s_i is the vertex of $M(CH_n)$ corresponding to the edge p_ip_{i+1} $(1 \leq i \leq n-1)$ of CH_n, s_n is the vertex corresponding to the edge p_np_1 of CH_n, u_i is the vertex of $M(CH_n)$ corresponding to the edge q_iq_{i+1} $(1 \leq i \leq n-1)$ of CH_n and u_n is the vertex corresponding to the edge q_nq_1 of CH_n, the degree of vertices are $d(v) = n$, $d(p_i) = 4$, $d(q_i) = 3$, $d(r_i) = n+4$, $d(s_i) = 8$, $d(t_i) = 7$ and $d(u_i) = 6$ for $(1 \leq i \leq n)$.
Consider the following (n+1)-coloring of $M(CH_n)$,

- Assign the color c_1 to v, p_i, q_i for $(1 \leq i \leq n)$.
- Assign the color c_{i+1} to r_i, u_i for $(1 \leq i \leq n)$.
- Assign the color c_{i-1} to t_i for $(3 \leq i \leq n)$.
- Assign the color c_n to t_1.
- Assign the color c_{n+1} to t_2.
- case(i): if n is odd
  (1) Assign the color $c_{\lceil \frac{n}{2} \rceil + i}$ to s_i for $(1 \leq i \leq \lceil \frac{n}{2} \rceil)$.
  (2) Assign the color c_{1+i} to $s_{\lceil \frac{n}{2} \rceil + i}$ for $(1 \leq i \leq \lfloor \frac{n}{2} \rfloor)$
- case(ii): if n is even
  (1) Assign the color $c_{\lceil \frac{n+1}{2} \rceil + i}$ to s_i for $(1 \leq i \leq \frac{n}{2})$.
  (2) Assign the color c_{1+i} to $s_{\frac{n}{2}+i}$ for $(1 \leq i \leq \frac{n}{2})$.

To prove (n+1)-coloring is an irregular coloring of $M(CH_n)$, since $deg(v) \neq deg(q_i)$ for $(1 \leq i \leq n)$ it shows that $code(v) \neq code(q_i)$ for $(1 \leq i \leq n)$, in the same sense $deg(r_i) \neq deg(u_i)$ for $(1 \leq i \leq n)$ it shows that $code(r_i) \neq code(u_i)$ for $(1 \leq i \leq n)$. Thus $\chi_{ir}(M(CH_n)) \leq (n+1)$. By the definition of middle graph $\{v, r_i(1 \leq i \leq n)\}$ induces a clique of order (n+1) in $M(CH_n)$. Therefore, $\chi_{ir}(M(CH_n)) \geq \chi(M(CH_n)) = n+1$. Hence $\chi_{ir}(M(CH_n)) = n+1$. □

Theorem 4.2. *For a closed helm graph, CH_n, $n \geq 5$ the irregular chromatic number of the total graph of closed helm, $T(CH_n)$ is $\chi_{ir}(T(CH_n)) = n + 1$.*

Proof. Let $V(CH_n) = \{v, p_i, q_i : 1 \leq i \leq n\}$ and $V(T(CH_n)) = \{v, p_i(1 \leq i \leq n), q_i(1 \leq i \leq n), r_i(1 \leq i \leq n), s_i(1 \leq i \leq n), t_i(1 \leq i \leq n)\}, u_i(1 \leq i \leq n)\}$, where r_i is the vertex of $T(CH_n)$ corresponding to the edge vp_i $(1 \leq i \leq n)$ of CH_n, t_i is the vertex of $T(CH_n)$ corresponding to the edge p_iq_i $(1 \leq i \leq n)$ of CH_n, s_i is the vertex of $T(CH_n)$ corresponding to the edge p_ip_{i+1} $(1 \leq i \leq n-1)$ of CH_n, s_n is the vertex corresponding to the edge p_np_1 of CH_n, u_i is the vertex of $T(CH_n)$ corresponding to the edge q_iq_{i+1} $(1 \leq i \leq n-1)$ of CH_n and u_n is the vertex corresponding to the edge q_nq_1 of CH_n, the degree of vertices are $d(v) = 2n$, $d(p_i) = 8$, $d(q_i) = 6$, $d(r_i) = n + 4$, $d(s_i) = 8$, $d(t_i) = 7$ and $d(u_i) = 6$ for $(1 \leq i \leq n)$.
Consider the following (n+1)-coloring of $T(CH_n)$,

- Assign the color c_1 to v.
- Assign the color c_{i+1} to r_i, q_i for $(1 \leq i \leq n)$.
- Assign the color c_{i+2} to p_i for $(1 \leq i \leq n-1)$.
- Assign the color c_2 to p_n.
- case(i): if n is odd
  (1) Assign the color c_1 to s_{2i-1}, u_{2i-1} for $(1 \leq i \leq \lfloor \frac{n}{2} \rfloor)$.
  (2) Assign the color c_{i+1} to s_{2i}, u_{2i} for $(1 \leq i \leq \lfloor \frac{n}{2} \rfloor)$.
  (3) Assign the color $c_{\lceil \frac{n}{2} \rceil + 1}$ to s_n, u_n.
  (4) Assign the color $c_{(n+2)-i}$ to t_i for $(1 \leq i \leq \lfloor \frac{n}{2} \rfloor)$.
  (5) Assign the color $c_{\lfloor \frac{n}{2} \rfloor + i}$ to $t_{\lfloor \frac{n}{2} \rfloor + i}$ for $(1 \leq i \leq \lceil \frac{n}{2} \rceil)$.
- case(ii): if n is even
  (1) Assign the color c_1 to s_{2i-1}, u_{2i-1} for $(1 \leq i \leq \frac{n}{2})$.
  (2) Assign the color c_{i+1} to s_{2i}, u_{2i} for $(1 \leq i \leq \frac{n}{2})$.
  (3) Assign the color $c_{(n+2)-i}$ to t_i for $(1 \leq i \leq \frac{n}{2} - 1)$.
  (4) Assign the color $c_{(\frac{n}{2}-1)+i}$ to $t_{(\frac{n}{2}-1)+i}$ for $(1 \leq i \leq \frac{n}{2} + 1)$.

To prove (n+1)-coloring is an irregular coloring of $T(CH_n)$,since $deg(r_i) \neq deg(q_i)$ for $(1 \leq i \leq n)$ it shows that $code(r_i) \neq code(q_i)$ for $(1 \leq i \leq n)$ and $deg(s_i) \neq deg(u_i)$ for $(1 \leq i \leq n)$ it shows that $code(s_i) \neq code(u_i)$ for $(1 \leq i \leq n)$. Hence $\chi_{ir}(T(CH_n)) \leq (n{+}1)$. By the definition of total graph $\{v, r_i(1 \leq i \leq n)\}$ induces a clique of order (n+1) in $T(CH_n)$. Therefore, $\chi_{ir}(T(CH_n)) \geq \chi(T(CH_n)) = n + 1$. Hence $\chi_{ir}(T(CH_n)) = n + 1$. □

Theorem 4.3. *For a closed helm graph, CH_n, $n \geq 5$ the irregular chromatic number of the line grph of closed helm, $L(CH_n)$ is $\chi_{ir}L(CH_n) = n$.*

Proof. Let $V(CH_n) = \{v, p_i, q_i : 1 \leq i \leq n\}$ and $V(L(CH_n)) = \{r_i(1 \leq i \leq n), s_i(1 \leq i \leq n), t_i(1 \leq i \leq n), u_i(1 \leq i \leq n)\}$, where r_i is the vertex of $L(CH_n)$ corresponding to the edge vp_i $(1 \leq i \leq n)$ of CH_n, t_i is the vertex of $L(CH_n)$ corresponding to the edge p_iq_i $(1 \leq i \leq n)$ of CH_n, s_i is the vertex of $L(CH_n)$

corresponding to the edge p_ip_{i+1} $(1 \leq i \leq n-1)$ of CH_n, s_n is the vertex corresponding to the edge p_np_1 of CH_n, u_i is the vertex of $L(CH_n)$ corresponding to the edge q_iq_{i+1} $(1 \leq i \leq n-1)$ of CH_n and u_n is the vertex corresponding to the edge q_nq_1 of CH_n, the degree of vertices are $d(r_i) = n+2$, $d(s_i) = 6$, $d(t_i) = 5$ and $d(u_i) = 4$ for $(1 \leq i \leq n)$.
Consider the following n-coloring of $L(CH_n)$,

- Assign the color c_i to r_i, u_i for $(1 \leq i \leq n)$.
- Assign the color c_i to s_{1+i} for $(1 \leq i \leq n-1)$.
- Assign the color c_n to s_1.
- Assign the color c_{i+1} to t_i for $(1 \leq i \leq n-1)$.
- Assign the color c_1 to t_n.

To prove n-coloring is an irregular coloring of $L(CH_n)$, since $deg(u_i) \neq deg(r_i)$ for $(1 \leq i \leq n)$ this shows that $code(u_i) \neq code(r_i)$ for $(1 \leq i \leq n)$. Hence $\chi_{ir}(L(CH_n)) \leq n$. By the definition of line graph $\{r_i : (1 \leq i \leq n)\}$ induces a clique of order n in $L(CH_n)$. Therefore, $\chi_{ir}(L(CH_n)) \geq \chi(L(CH_n)) = n$. Hence $\chi_{ir}(L(CH_n)) = n$. □

5. Irregular Coloring of Gear Graphs

Theorem 5.1. *For a gear graph, G_n, $n \geq 4$ the irregular chromatic number of $M(G_n)$ is $\chi_{ir}(M(G_n)) = n+1$.*

Proof. Let $V(G_n) = \{v, v_i, w_i : 1 \leq i \leq n\}$ and $V(M((G_n)) = \{v, v_i(1 \leq i \leq n),$ $w_i(1 \leq i \leq n), r_i(1 \leq i \leq n), s_i(1 \leq i \leq n), t_i(1 \leq i \leq n)\}$, where r_i is the vertex of $M(G_n)$ corresponding to the edge vv_i of G_n $(1 \leq i \leq n)$, t_i is the vertex of $M(G_n)$ corresponding to the edge v_iw_i of G_n $(1 \leq i \leq n)$, s_i is the vertex of $M(G_n)$ corresponding to the edge w_iv_{i+1} of G_n $(1 \leq i \leq n-1)$ and s_n is the vertex corresponding to the edge w_nv_1 of G_n, the degree of vertices are $d(v) = n$, $d(v_i) = 3$, $d(w_i) = 2$, $d(r_i) = n+3$, $d(s_i) = 5$ and $d(t_i) = 5$ for $(1 \leq i \leq n)$.
Consider the color function $c : V(M(G_n)) \rightarrow \{c_1, c_2, ...c_{n+1}\}$ defined by

$$\begin{aligned} c(v) &= c_1 \\ c(s_i) &= c_1 \text{ for } 1 \leq i \leq n \\ c(r_i) &= c_{i+1} \text{ for } 1 \leq i \leq n \\ c(w_i) &= c_{i+1} \text{ for } 1 \leq i \leq n \\ c(v_i) &= \begin{cases} c_{i+2} & \text{for } 1 \leq i \leq n-1 \\ c_2 & \text{for } i = n \end{cases} \\ c(t_i) &= \begin{cases} c_i & \text{for } 2 \leq i \leq n \\ c_{n+1} & \text{for } i = 1 \end{cases} \end{aligned}$$

To prove c is an irregular coloring function of $M(G_n)$, Since $deg(r_i) \neq deg(w_i)$ for $(1 \leq i \leq n)$ this shows that $code(r_i) \neq code(w_i)$ for $(1 \leq i \leq n)$. Except at $n = 5$ for all other n, $deg(v) \neq deg(s_i)$ for $(1 \leq i \leq n)$ this shows that $code(v) \neq code(s_i)$

for $(1 \leq i \leq n)$. At $n = 5$, $deg(v) = deg(s_i)$ for $(1 \leq i \leq 5)$ each s_i's are adjacent to v_i but v is not adjacent to v_i, hence $code(v) \neq code(s_i)$ for $(1 \leq i \leq 5)$. Thus $\chi_{ir}(M(G_n)) \leq (n+1)$. By the definition of middle graph $\{v, r_i(1 \leq i \leq n)\}$ induces a clique of order (n+1) in $M(G_n)$. Therefore, $\chi_{ir}(M(G_n)) \geq \chi(M(G_n)) = n + 1$. Hence $\chi_{ir}(M(G_n)) = n + 1$. □

Theorem 5.2. *For a gear graph, G_n, $n \geq 4$ the irregular chromatic number of $T(G_n)$ is $\chi_{ir}(T(G_n)) = n + 1$.*

Proof. Let $V(G_n) = \{v, v_i, w_i : 1 \leq i \leq n\}$ and $V(T(G_n)) = \{v, v_i(1 \leq i \leq n), w_i(1 \leq i \leq n), r_i(1 \leq i \leq n), s_i(1 \leq i \leq n), t_i(1 \leq i \leq n)\}$, where r_i is the vertex of $T(G_n)$ corresponding to the edge vv_i $(1 \leq i \leq n)$ of G_n, t_i is the vertex of $T(G_n)$ corresponding to the edge v_iw_i $(1 \leq i \leq n)$ of G_n, s_i is the vertex of $T(G_n)$ corresponding to the edge w_iv_{i+1} $(1 \leq i \leq n-1)$ of G_n and s_n is the vertex corresponding to the edge w_nv_1 of G_n, the degree of vertices are $d(v) = 2n$, $d(v_i) = 6$, $d(w_i) = 4$, $d(r_i) = n + 3$, $d(s_i) = 5$ and $d(t_i) = 5$ for $(1 \leq i \leq n)$.
Consider the following (n+1)-coloring of $T(G_n)$,

- Assign the color c_1 to v, s_i for $(1 \leq i \leq n)$.
- Assign the color c_{1+i} to r_i, w_i for $(1 \leq i \leq n)$.
- Assign the color c_{2+i} to v_i for $(1 \leq i \leq n-1)$.
- Assign the color c_2 to v_n.
- Assign the color c_i to t_i for $(2 \leq i \leq n)$.
- Assign the color c_{n+1} to t_1.

To prove (n+1)-coloring is an irregular coloring of $T(G_n)$, since $deg(v) \neq deg(s_i)$ for $(1 \leq i \leq n)$ shows that $code(v) \neq code(s_i)$ for $(1 \leq i \leq n)$. Since $deg(r_i) \neq deg(w_i)$ for $(1 \leq i \leq n)$ this shows that $code(r_i) \neq code(w_i)$ for $(1 \leq i \leq n)$. Thus $\chi_{ir}(T(G_n)) \leq (n+1)$. By the definition of middle graph $\{v, r_i(1 \leq i \leq n)\}$ induces a clique of order (n+1) in $T(G_n)$. Therefore, $\chi_{ir}(T(G_n)) \geq \chi(T(G_n)) = n + 1$. Hence $\chi_{ir}(T(G_n)) = n + 1$. □

Theorem 5.3. *For a gear graph, G_n, $n \geq 3$ the irregular chromatic number of $C(G_n)$ is $\chi_{ir}(C(G_n)) = n + 1$.*

Proof. Let $V(G_n) = \{v, v_i, w_i : 1 \leq i \leq n\}$ and $V(C(G_n)) = \{v, v_i(1 \leq i \leq n), w_i(1 \leq i \leq n), r_i(1 \leq i \leq n), s_i(1 \leq i \leq n), t_i(1 \leq i \leq n)\}$, where r_i is the vertex of $C(G_n)$ corresponding to the edge vv_i $(1 \leq i \leq n)$ of G_n, t_i is the vertex of $C(G_n)$ corresponding to the edge v_iw_i $(1 \leq i \leq n)$ of G_n, s_i is the vertex of $C(G_n)$ corresponding to the edge w_iv_{i+1} $(1 \leq i \leq n-1)$ of G_n and s_n is the vertex corresponding to the edge w_nv_1 of G_n, the degree of vertices are $d(v) = 2n$, $d(v_i) = 2n$, $d(w_i) = 2n$, $d(r_i) = 2$, $d(s_i) = 2$ and $d(t_i) = 2$ for $(1 \leq i \leq n)$.
Consider the following (n+1)-coloring of $C(G_n)$,

- Assign the color c_1 to v, s_i, t_i for $(1 \leq i \leq n)$.
- Assign the color c_{1+i} to r_i for $(1 \leq i \leq n)$.
- Assign the color c_i to v_i, w_i for $(2 \leq i \leq n)$.

- Assign the color c_{n+1} to v_1, w_1.

To prove (n+1)-coloring is an irregular coloring of $C(G_n)$, since $deg(v) \neq deg(s_i)$ for $(1 \leq i \leq n)$ this shows that $code(v) \neq code(s_i)$ for $(1 \leq i \leq n)$. Since $deg(v_i) \neq deg(w_i)$ for $(1 \leq i \leq n)$ each w_i's are adjacent to v but $v_i^{'}$s are not adjacent to v, hence $code(v_i) \neq code(w_i)$ for $(1 \leq i \leq n)$. Thus $\chi_{ir}(C(G_n)) \leq (n+1)$. By the definition of central graph the vertices $\{v, w_i : 1 \leq i \leq n\}$ induce a clique of order $n+1$. Therefore, $\chi_{ir}(C(G_n)) \geq \chi(C(G_n)) = n+1$. Hence $\chi_{ir}(C(G_n)) = n+1$. □

Theorem 5.4. *For a gear graph, G_n, $n \geq 3$ the irregular chromatic number of $L(G_n)$ is $\chi_{ir}(L(G_n)) = n$.*

Proof. Let $V(G_n) = \{v, v_i, w_i : 1 \leq i \leq n\}$ and $V(L(G_n)) = \{r_i(1 \leq i \leq n), s_i(1 \leq i \leq n),$
$t_i(1 \leq i \leq n)\}$, where r_i is the vertex of $L(G_n)$ corresponding to the edge vv_i $(1 \leq i \leq n)$ of G_n, t_i is the vertex of $L(G_n)$ corresponding to the edge v_iw_i $(1 \leq i \leq n)$ of G_n, s_i is the vertex of $L(G_n)$ corresponding to the edge w_iv_{i+1} $(1 \leq i \leq n-1)$ of G_n and s_n is the vertex corresponding to the edge w_nv_1 of G_n, the degree of vertices are $d(r_i) = n+1$, $d(s_i) = 3$ and $d(t_i) = 3$ for $(1 \leq i \leq n)$.
Consider the following n-coloring of $L(G_n)$,

- Assign the color c_i to r_i, s_i for $(1 \leq i \leq n)$.
- Assign the color c_{1+i} to t_i for $(1 \leq i \leq n-1)$.
- Assign the color c_1 to t_n.

To prove n-coloring is an irregular coloring of $L(G_n)$, since $deg(s_i) \neq deg(r_i)$ for $(1 \leq i \leq n)$ this shows that $code(s_i) \neq code(r_i)$ for $(1 \leq i \leq n)$. Hence $\chi_{ir}(L(G_n)) \leq n$. By the definition of line graph $\{r_i : (1 \leq i \leq n)\}$ induces a clique of order n in $L(G_n)$. Therefore, $\chi_{ir}(L(G_n)) \geq \chi(L(G_n)) = n$. Hence $\chi_{ir}(L(G_n)) = n$. □

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