Celal Bayar University Journal of Science
Volume 15, Issue 1, 2019, p 1-7
Doi: 10.18466/cbayarfbe.416121

Taylor-Matrix Collocation Method to Solution of Differential Equations Characterizing Spherical Curves in Euclidean 4-Space

Tuba Ağırman Aydın[1*], **Mehmet Sezer**[2]

[1]Bayburt Universtiy Faculty of Education, No: 306 Bayburt, Turkey
[2]Manisa Celal Bayar University Faculty of Art And Science, Manisa, Turkey
[*]tubaaydin@bayburt.edu.tr

Received: 17 April 2018
Accepted: 13 February 2019
DOI: 10.18466/cbayarfbe.416121

Abstract

In this study we consider a third order linear differential equation with variable coefficients characterizing spherical curves according to Frenet frame in Euclidean 4-Space E^4. This equation whose coefficients are related to special function, curvature and torsion, is satisfied by the position vector of any regular unit velocity spherical curve. These type equations are generally impossible to solve analytically and so, for approximate solution we present a numerical method based on Taylor polynomials and collocations points by using initial conditions. Our method reduces the solution of problem to the solution of a system of algebraic equations and the approximate solution is obtained in terms of Taylor polynomials.

Keywords: Curves in Euclidean Space, Spherical curves; Taylor matrix method; Frenet frame; Linear differential equations; Matrix and collocation method.

1. Introduction

The concept of curve defined by Euler in the plane was moved by Fujiwara to the three-dimensional Euclidean space [1, 2]. Shortly after this work, the space curves of constant breadth was definited on the sphere [3]. Wong gave a global formulation of the condition that a general curve is lie on a sphere [4]. And this formula has taken place in books written on differential geometry as a necessary and sufficient condition for a curve to lie on a sphere. Reuleaux, in his work in the same years, showed kinematic and engineering applications of these curves [5]. The work by Gluck brought into the world of geometry the high-grade curvatures of the curves in Euclidean space [6]. The explicit solvability of the differential equation characterizing a spherical curve is shown and this solution is expressed in terms of the curvature radius and torsion of the curve [7]. Dannon worked on the integral characterization of curves [8]. Sezer gave integral characterizations of a system of differential equations like Frenet obtained for curves of constant breadth and spherical curves and he used these characterizations to determine a criterion for the closeness (periodicity) of a space curve [9].

At the same time, in the studies up to now, systems of differential equations related to spherical curves have been transformed into nonlinear differential equations and integral equations, but exact solutions have not been reached [10, 11].

Since spherical curves are used to operate various mechanisms, the results of this work can be used in field studies such as mechanical engineering, com design and kinematics. In addition, the solutions obtained for these curves in this study will fill an important gap in the literature.

2. Material and Methods

In this work, we firstly developed a Taylor matrix collocation method using Taylor polynomials to find approximate solutions of third order, linear differential equations with variable coefficients. We then have obtained differential equations characterizing unitary-speed spherical curves in 4-dimensional Euclidean space. We then reached approximate solutions of these equations using the Taylor matrix-collocation method developed by us.

3. Preliminaries

In this section, we give some basic concepts on differential geometry of space curves and spherical curves in Euclidean 3-space. A differentiable α function, defined as $\alpha: I \subseteq R \to E^n$ for $I = \{t: a < t < b\}$, is called a curve defined by coordinate neighborhood (I, α) in E^n. The variable $t \in I$ is called the parameter of the α curve. If the derivative $d\alpha(t)/dt$ of this curve differs from zero everywhere, this curve is called a regular curve [12].

Theorem 3.1. If α is a regular curve in E^n, then Frenet formulaes

$$V_1'(s) = k_1(s)V_2(s)$$
$$V_i'(s) = -k_{i-1}(s)V_{i-1}(s) + k_i(s)V_{i+1}(s), 1 < i < r$$
$$V_r'(s) = -k_{r-1}(s)V_{r-1}(s)$$

where k_i is the curvature function of the curve α, $s \in I$ is arc length parameter of the curve α, respectively [12].

The velocity vector of a regular curve $\alpha(t)$ at $t = t_0$ is the derivative $d\alpha(t)/dt$ evaluated at $t = t_0$. The velocity vector field is the vector valued function $d\alpha(t)/dt$. The speed of $\alpha(t)$ at $t = t_0$ is the lenght of the velocity vector at $t = t_0$, $|\alpha'(t_0)|$ [13].

4. Obtaining Differential Equation characterizing Spherical Curves in Euclidean 4-Space

In the simplest sense, the concept of spherical curve is defined as "the curve lying on a sphere" [14].

Theorem 4.1. Let α be a frenet frame curve of class C^4 in E^3 with $\tau \neq 0$ everywhere. Then α lies on a sphere if and only if the following equation holds.

$$[\tau^{-1}(\varkappa^{-1})']' + \tau\varkappa^{-1} = 0 \qquad (1)$$

Using the differential equation(1), Dannon observed a system of differential equations like Frenet for the spherical curves of E^3. Then he proved the accuracy of observation by moving these curves to E^4 [8].

With this in mind, the differential equation characterizing spherical curves in E^3 is obtained as follows.

Thesis 4.2. Let $\alpha(s)$ be a unitary-speed Frenet curve that can be differentiated from class C^5 in E^4. And let $\varkappa(s), \tau(s), \mu(s)$ be the curvature functions of the curve α. In this case, there are $h(s)$ and $g(s)$ functions which can be differentiated from class C^2, which provides the following equations.

1) The $\alpha(s)$ curve lies on a sphere in E^4
2) For $\varkappa(s) \neq 0$
$\rho' = \tau h$
$h' = -\tau\rho + \mu g$
$g' = -\mu h$

where $\rho = 1/\varkappa(s)$ is defined as the radius of curvature of the curve [8].

Proof 4.2. Suppose that the α (s) curve lies on a sphere of radius "a" around $\overrightarrow{x_0}$ in E^4. Let's take the Frenet vector fields for the α (s) curve in E^4 as follows.

$$T_1'(s) = \varkappa(s)T_2(s)$$
$$T_2'(s) = -\varkappa(s)T_1(s) + \tau(s)T_3(s)$$
$$T_3'(s) = -\tau(s)T_2(s) + \mu(s)T_4(s)$$
$$T_4'(s) = -\mu(s)T_3(s).$$

Match the $\overrightarrow{x_0}$ point to the $\vec{0}$ point. In this case it is obvious that $a^2 = \langle\alpha(s), \alpha(s)\rangle$. To find the functions $h(s)$ and $g(s)$, let us use the repetitive differentiations of $a^2 = \langle\alpha(s), \alpha(s)\rangle$. So we get the following equations.

$$\langle\alpha(s), \alpha'(s)\rangle + \langle\alpha'(s), \alpha(s)\rangle = 0 \Rightarrow \langle\alpha'(s), \alpha(s)\rangle = 0$$
$$\langle T_1(s), \alpha(s)\rangle 0 \qquad (2)$$
$$\langle T_1'(s), \alpha(s)\rangle + \langle T_1(s), \alpha'(s)\rangle = 0 \Rightarrow$$
$$\langle\varkappa(s)T_2(s), \alpha(s)\rangle = -1$$
$$\langle T_2(s), \alpha(s)\rangle = -1/\varkappa(s) \qquad (3)$$
$$\langle T_2'(s), \alpha(s)\rangle + \langle T_2(s), \alpha'(s)\rangle = -(1/\varkappa(s))' = -\rho'(s)$$
$$\langle T_3(s), \alpha(s)\rangle = -1/\tau(s)\rho'(s) \qquad (4)$$
$$\langle T_3'(s), \alpha(s)\rangle + \langle T_3(s), \alpha'(s)\rangle = (-1/\tau(s)\rho'(s))'$$
$$\langle T_4(s), \alpha(s)\rangle =$$
$$-1/\mu(s)[\rho(s)\tau(s) - (-1/\tau(s)\rho'(s))] \qquad (5)$$

Then, there are functions $h_i = \langle -\vec{\alpha}, T_i\rangle$, $i = 2, 3, 4$ which can be differentiated from class C^2as $h' = Hh$. It is also clear that $h_1 = 0$. In contrast, if $h' = Hh$ is given by a system of equations like Frenet and $d/ds\{\alpha(s) - \sum_{i=2}^{4} h_i T_i\} = 0$, then $\alpha(s) = \sum_{i=2}^{4} h_i T_i + \overrightarrow{x_0}$ can be written. The equation of $|\alpha(s) - \overrightarrow{x_0}|^2 = |\sum_{i=2}^{4} h_i T_i|^2 = a$ (constant) can be clearly seen. Thus, the $\alpha(s)$ curve is located on the sphere with center $\overrightarrow{x_0}$ and radius a. Thus the proof is complete.

Therefore, we can come to the conclusion that the differential equation characterizing spherical curves in E^4 can be obtained from this equation system like Frenet;

$$\rho' = \tau h$$
$$h' = -\tau\rho + \mu g \qquad (6)$$
$$g' = -\mu h$$

we first obtain the following equation using the equation system (6) given above

$$\rho^2 + h^2 + g^2 = a^2. \qquad (7)$$

We come to the conclusion that the $\alpha(s)$ curves determined by the solution set $\{\rho(s), h(s), g(s)\}$ of equation system (6) are on the sphere (7) with radius a. Let us now find the solution set $\{\rho(s), h(s), g(s)\}$ of this system of equations (6). By eliminating the functions h and g in this system of equations (6), we obtain the third order, variable coefficient, linear, homogeneous differential equation as follows.

$$(1/\tau\mu)\rho''' + [(1/\tau\mu)' + (1/\tau)'\,1/\mu]\rho'' +$$
$$\{[(1/\tau)'\,1/\mu]' + \tau/\mu + \mu/\tau\}\rho' + (\tau/\mu)'\rho = 0 \qquad (8)$$

5. Taylor Matrix Collocation Method

In this section, to obtain the Taylor polynomial solution of the differential equation defined by

$$\sum_{k=0}^{3} P_k(s)\,\lambda^{(k)}(s) = F(s), \qquad 0 \leq s \leq b \qquad (9)$$

about the point $s = 0$, under the initial conditions

$$\lambda(0) = \lambda_0$$
$$\lambda'(0) = \lambda_1 \qquad (10)$$
$$\lambda''(0) = \lambda_2$$

we develop the Taylor matrix method based on collocation points, which is given by Sezer [11]. Here λ_0, λ_1, λ_3 and b appropriate constants.

Firstly, let us assume that the desired solution $\lambda(s)$ can be expanded to Taylor series about $s = 0$ in the form, for $N \geq 3$

$$\lambda(s) = \lambda_N(s) = \sum_{n=0}^{N} a_n\, s^n, \qquad 0 \leq s \leq b \qquad (11)$$

where a_n $(n = 0,1,\dots,N)$ are the coefficients to be determined.

Now, we can convert the truncated Taylor series solution $\lambda(s)$defined by (11) and its derivatives $\lambda^{(k)}(s), k = 0,1,2,3$ to matrix forms;

$$\lambda(s) = S(s)A,$$
$$\lambda^{(k)}(s) = S^{(k)}(s)A, \quad k = 0,1,2,3,$$

where

$$S(s) = [\,1 \quad s \quad s^2 \quad \dots \quad s^N\,]_{1x(N+1)}$$
$$A = [a_0 \quad a_1 \quad a_2 \quad \dots \quad a_N]^t.$$

Also, it is clear that the relation between the matrix $S(s)$ and its derivative $S'(s)$ is $S'(s) = S(s)B$; where

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)X(N+1)}$$

$$B^0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{(N+1)X(N+1)}$$

By repeating this process, we get the matrix relation as follows:

$$S'(s) = S(s)B$$
$$S''(s) = S'(s)B = S(s)B^2$$
$$S'''(s) = S''(s)B = S(s)B^3$$
$$\vdots$$
$$S^{(k)}(s) = S(s)B^k, \qquad k = 0,1,2,3 \qquad (12)$$

From the matrix relations (11) and (12), it follows that

$$\lambda^{(k)}(s) \cong S^{(k)}(s)A = S(s)B^kA, \quad k = 0,1,2,3 \qquad (13)$$

We now ready to construct the fundamental corresponding to equation (9). For this purpose, by substituting the matrix relation (13) into equation (9) and by using the collocation points defined by

$$s_i = \frac{b}{N}i, \qquad i = 0,1,\dots,N$$

we get the system of the matrix equations

$$\{\sum_{k=0}^{3} P_k(s_i)S(s_i)B^k\}A = F(s_i)$$

or briefly the fundamental matrix equation.

$$\{\sum_{k=0}^{3} P_kSB^k\}A = F \qquad (14)$$

Where

$$P_k = \begin{bmatrix} P_k(s_0) & 0 & \cdots & 0 \\ 0 & P_k(s_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_k(s_N) \end{bmatrix}_{(N+1)x(N+1)},$$

$$S = \begin{bmatrix} S(s_0) \\ S(s_1) \\ \vdots \\ S(s_N) \end{bmatrix} = \begin{bmatrix} 1 & s_0 & {s_0}^2 & \dots & {s_0}^N \\ 1 & s_1 & {s_1}^2 & \dots & {s_1}^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_N & {s_N}^2 & \dots & {s_N}^N \end{bmatrix}_{(N+1)x(N+1)},$$

$$F = \begin{bmatrix} F(s_0) \\ F(s_1) \\ \vdots \\ F(s_N) \end{bmatrix}, \quad A = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}$$

Hence the equation (14) can be written in the form

$$WA = F \quad \text{or} \quad [W\ ;\ F] \qquad (15)$$

$$W = [w_{pq}], \quad p,q = 0,1,\dots,N,$$

where

$$W = [w_{pq}] = \sum_{k=0}^{3} P_kSB^k$$

and B^0 unit matrix. On the other hand, we can obtain the following matrix forms for the initial conditions (10), by means of the relation (13);

$$\lambda(0) = \lambda_0 \Rightarrow S(0)B^0A = \lambda_0$$
$$\lambda'(0) = \lambda_1 \Rightarrow S(0)BA = \lambda_1$$
$$\lambda''(0) = \lambda_2 \Rightarrow S(0)B^2A = \lambda_2$$

or briefly

$$U_iA = [\lambda_i] \Rightarrow [U_i; \lambda_i], \qquad i = 0,1,2 \qquad (16)$$

where

$$U_i = [\,u_{i0} \quad u_{i1} \quad \cdots \quad u_{iN}\,] = S(0)B^i, \quad i = 0,1,2.$$

Celal Bayar University Journal of Science
Volume 15, Issue 1, 2019, p 1-7
Doi: 10.18466/cbayarfbe.416121

Finally, to obtain the solution of the equation (9) with the conditions (10), by replacing the 3 row matrics (16) by the last 3 rows (or appropriate 3 rows) of the augmented matrix (15) we have the required augmented matrix

$$[\tilde{W}\,;\ \tilde{F}] \qquad (17)$$

or clearly

$$\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & F(s_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & F(s_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{N-3,0} & w_{N-3,1} & \cdots & w_{N-3,N} & ; & F(s_{N-3}) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \\ u_{20} & u_{21} & \cdots & u_{2N} & ; & \lambda_2 \end{bmatrix}.$$

If rank $\tilde{W}$= rank$[\tilde{W}\,;\ \tilde{F}] = N+1$ then we can write $A = (\tilde{W})^{-1}\tilde{F}$. Thus the matrix A (thereby the coefficients $a_0, a_1, \dots, a_N$) is uniquely determined. Also the equation (9) with the initial conditions (10) has a unique solution. This solution is given by the truncated Taylor series (11). Thus we get the Taylor polynomial solution as

$$\lambda_N(s) = \sum_{n=0}^{N} a_n s^n.$$

6. The Solution of Differential Equation Characterizing Spherical Curves in E^4

We can arrange the equation (8) characterizing spherical curves in E^4 as fallows;

$$P_3(s)\lambda''' + P_2(s)\lambda'' + P_1(s)\lambda' + P_0(s)\lambda = 0 \qquad (18)$$

and

$$\sum_{k=0}^{3} P_k(s)\lambda^{(k)}(s) = F(s) \qquad (19)$$

so that

$$P_3(s) = 1/\tau\mu,$$
$$P_2(s) = (1/\tau\mu)' + (1/\tau)'\,1/\mu,$$
$$P_1(s) = [(1/\tau)'\,1/\mu]' + \tau/\mu + \mu/\tau,$$
$$P_0(s) = (\tau/\mu)' \text{ and } \rho = \lambda$$

It is clear that the expression (19) is equivalent to the differential equation (18) for F(s)=0. Suppose that an approximate solution of this the equation (18) for $0 \le s \le 2\pi$ under the conditions given. This solution has the form of the truncated Taylor series as fallows

$$\lambda(s) = \sum_{n=0}^{N} a_n\, s^n\,. \qquad (20)$$

Here we will take N=4 for simplicity. We show the expression (19) in the matrix form as follows;

$$\lambda(s) = S(s)A$$

where $S(s)$and A matrices are defined as

$$S(s) = [\,1 \quad s \quad s^2 \quad s^3 \quad s^4\,]$$
$$A = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4]^t$$

On the other hand, the matrices B, B^2 and B^3 alaong with the matrices the derivatives of $\lambda(s)$ are defined as follows:

$$\lambda'(s) = S(s)BA$$
$$\lambda''(s) = S(s)B^2A$$
$$\lambda'''(s) = S(s)B^3A$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$B^2 = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$B^3 = \begin{bmatrix} 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If we put all these expressions in the equation (18), we get following equation.

$$\{P_3(s)S(s)B^3 + P_2(s)S(s)B^2 + P_1(s)S(s)B + P_0(s)S(s)\}A = F(s). \qquad (21)$$

Now, we use collocation points $s = s_i$, $i = 0,1,\dots,4$, defined by
$s_0 = 0,\ s_1 = \pi/2,\ s_2 = \pi,\ s_3 = 3\pi/2,\ s_4 = 2\pi$ and $k = 0,1,2,3$ in the equation (21) and we determine the fundamental matrix equation as

$$\{P_3SB^3 + P_2SB^2 + P_1SB + P_0S\}A = F;$$

$$P_k = \begin{bmatrix} P_k(0) & 0 & 0 & 0 & 0 \\ 0 & P_k(\pi/2) & 0 & 0 & 0 \\ 0 & 0 & P_k(\pi) & 0 & 0 \\ 0 & 0 & 0 & P_k(3\pi/2) & 0 \\ 0 & 0 & 0 & 0 & P_k(2\pi) \end{bmatrix},$$

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & (\pi/2) & (\pi/2)^2 & (\pi/2)^3 & (\pi/2)^4 \\ 1 & (\pi) & (\pi)^2 & (\pi)^3 & (\pi)^4 \\ 1 & (3\pi/2) & (3\pi/2)^2 & (3\pi/2)^3 & (3\pi/2)^4 \\ 1 & (2\pi) & (2\pi)^2 & (2\pi)^3 & (2\pi)^4 \end{bmatrix},$$

Celal Bayar University Journal of Science
Volume 15, Issue 1, 2019, p 1-7
Doi: 10.18466/cbayarfbe.416121

$$F = [F(0) \quad F(\pi/2) \quad F(\pi) \quad F(3\pi/2) \quad F(2\pi)]^T .$$

If we get as

$$P_3SB^3 + P_2SB^2 + P_1SB + P_0S = W,$$

the equation (8) can be converted to

$$WA = F \rightarrow [W \; ; \; F]. \qquad (22)$$

We now consider the initial conditions defined by

$$\begin{aligned} \lambda(0) &= \lambda_0 \\ \lambda'(0) &= \lambda_1 \\ \lambda''(0) &= \lambda_2. \end{aligned} \qquad (23)$$

Then, using the procedure section 4, we obtain matrix forms of the conditions (23) as

$$\begin{aligned} \lambda(0) &= S(0)A = \lambda_0 \\ \lambda'(0) &= S(0)BA = \lambda_1 \\ \lambda''(0) &= S(0)B^2A = \lambda_2. \end{aligned} \qquad (24)$$

So, the augmented matrix forms (24) of the conditions (23) become fallows as;

$$\begin{aligned} [U_0; \lambda_0] &= [1 \quad 0 \quad 0 \quad 0 \quad 0 \; ; \; \lambda_0] \\ [U_1; \lambda_1] &= [0 \quad 1 \quad 0 \quad 0 \quad 0 \; ; \; \lambda_1] \\ [U_2; \lambda_2] &= [0 \quad 0 \quad 2 \quad 0 \quad 0 \; ; \; \lambda_2] \end{aligned}$$

or

$$[U; \lambda] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & ; & \lambda_0 \\ 0 & 1 & 0 & 0 & 0 & ; & \lambda_1 \\ 0 & 0 & 2 & 0 & 0 & ; & \lambda_2 \end{bmatrix}.$$

Here the following equality can be written, briefly,

$$UA = \lambda \quad \rightarrow [U \; ; \; \lambda]. \qquad (25)$$

Consequently, we get $W^*A = F^*$ from (22) and (25):

$$[W^* ; \; F^*] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & w_{03} & w_{04} & ; & F(0) \\ w_{10} & w_{11} & w_{12} & w_{13} & w_{14} & ; & F(\pi/2) \\ 1 & 0 & 0 & 0 & 0 & ; & \lambda_0 \\ 0 & 1 & 0 & 0 & 0 & ; & \lambda_1 \\ 0 & 0 & 2 & 0 & 0 & ; & \lambda_2 \end{bmatrix}$$

where, w_{ij} $(i = 0,1 \;\; j = 0,1, \dots ,4)$ are obtained as follows;

$$w_{00} = P_0(0), w_{01} = P_1(0), \; w_{02} = 2P_2(0),$$

$$w_{03} = 6P_3(0), w_{04} = 0, w_{10} = P_0\left(\frac{\pi}{2}\right),$$

$$w_{11} = \frac{\pi}{2}P_0\left(\frac{\pi}{2}\right) + P_1\left(\frac{\pi}{2}\right)$$

$$w_{12} = \left(\frac{\pi}{2}\right)^2 P_0\left(\frac{\pi}{2}\right) + \pi P_1\left(\frac{\pi}{2}\right) + 2P_2\left(\frac{\pi}{2}\right)$$

$$w_{13} = \left(\frac{\pi}{2}\right)^3 P_0\left(\frac{\pi}{2}\right) + 3\left(\frac{\pi}{2}\right)^2 P_1\left(\frac{\pi}{2}\right) + 3\pi P_2\left(\frac{\pi}{2}\right) + 6P_3\left(\frac{\pi}{2}\right)$$

$$w_{14} = \left(\frac{\pi}{2}\right)^4 P_0\left(\frac{\pi}{2}\right) + 4\left(\frac{\pi}{2}\right)^3 P_1\left(\frac{\pi}{2}\right) + 12\left(\frac{\pi}{2}\right)^2 P_2\left(\frac{\pi}{2}\right) + 12\pi P_3\left(\frac{\pi}{2}\right)$$

Thus the matrix of the unknown coefficients is obtained for $F(0) = 0$ and $F(\pi/2) = 0$, $A = W^{*^{-1}}F^*$

$$A = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2/2 \\ K \\ M \end{bmatrix}$$

Here we calculate $a_3 = K$ and $a_4 = M$ as follows:

$$K = \frac{(w_{00}w_{14} - w_{10}w_{04})p_0 + (w_{01}w_{14} - w_{11}w_{04})p_1 + (w_{02}w_{14} - w_{12}w_{04})p_2 /2}{w_{13}w_{04} - w_{03}w_{14}}$$

$$M = \frac{(w_{10}w_{03} - w_{00}w_{13})p_0 + (w_{11}w_{03} - w_{01}w_{13})p_1 + (w_{12}w_{03} - w_{02}w_{13})p_2/2}{w_{13}w_{04} - w_{03}w_{14}}$$

If we put the unknown coefficients a_n in (20), we get the Taylor polynomial solution

$$\rho(s) = \lambda(s) = \lambda_0 + \lambda_1 s + \lambda_2 s^2 + a_3 s^3 + a_4 s^4.$$

This expression is the radius of curvature of the E^4 spherical curve. Therefore, h(s) and g(s) functions are obtained through this function and its derivatives.

As another way, by using the same conditions and the same procedure as the similar method, the function h(s) (or g(s)) is found. Then substitute ρ(s) and h(s) (or g(s)) in equation (7) to obtain g(s) (or h(s)) as follows:

$$(\lambda_0 + \lambda_1 s + \lambda_2 s^2 + a_3 s^3 + a_4 s^4)^2 + (\lambda_0 + \lambda_1 s + \lambda_2 s^2 + m_3 s^3 + m_3 s^4)^2 + g^2 = a^2$$

As a result, the set of solution $\{\rho(s), h(s), g(s)\}$ of system of differential equations (6) like the Frenet characterizing E^4 spherical curves is reached in terms of Taylor polynomials.

On the other hand, the following equation is obtained by arranging the equation (18)

$$[\mu^{-1}(\tau^{-1}\rho')' + \tau\rho\mu^{-1}]' + \tau^{-1}\rho'\mu = 0. \qquad (26)$$

If the curvature functions $\varkappa(s), \tau(s), \mu(s)$ of a unit-speed curve in E^4 satisfy the differential equation (26), this curve lies on a sphere in E^4. This is true in the opposite direction.

7. Conclusions and Applications

We obtained the differential equation which characterizes spherical curves in E^4 space using the system of equations (6) like the Frenet. By taking $g = 0$ and $\mu = 0$ in this system of equations and eliminating h and its derivatives, the curve equation is reduced to E^3. In other words, if ρ and its derivatives are eliminated, the differential equation that characterizes spherical curves in E^3 space is obtained, depending on the function h that determines the curve, as follows:

$$1/\tau(s)\, h'' + (1/\tau(s))' h' + \tau(s) h = 0 \qquad (27)$$

From this equation, the function h is found by Taylor matrix collocation method as follows.

Example 7.1. The spherical indicator curves of the Frenet vector fields of a curve in space are spherical curves because they are formed on the sphere.

On the other hand, the curvature ($\varkappa_1$) and torsion (τ_1) of the spherical indenter of the t(s) unit tangent vector in E^3 space provides the following equations.

$$\varkappa_1 = \sqrt{\frac{\varkappa^2 + \tau^2}{\varkappa^2}}$$

$$\tau_1 = \frac{\varkappa\tau' - \varkappa'\tau}{\varkappa(\varkappa^2 + \tau^2)}$$

Torsion is the second curvature of a curve [15]. For

$$P_2(s) = 1/\tau_1(s)$$

$$P_1(s) = (1/\tau_1(s))'$$

$P_0(s) = \tau_1(s)$ and $h = \lambda$,

the following equation is obtained by using the differential equation (27)

$$P_2(s)\lambda'' + P_1(s)\lambda' + P_0(s)\lambda = 0 \qquad (28)$$

This is the differential equation of the spherical indicator curve of the t(s) unit tangent vector in E^3 space.

It is clear that for F(s)=0 the differential equation (28) is equal to the following expression

$$\sum_{k=0}^{2} P_k(s)\lambda^{(k)}(s) = F(s). \qquad (29)$$

We suppose that the equation (29) has an approximate solution, for $0 \le s \le 2\pi$, under the initial conditions

$$\lambda(0) = \lambda_0$$

$$\lambda'(0) = \lambda_1$$

This solution has the form of truncated Taylor series in the form

$$\lambda(s) = \sum_{m=0}^{N} a_m s^m. \qquad (30)$$

Here we will take N=3 for simplicity. We show this expression in the matrix form as follows;

$$\lambda(s) = S(s)A$$

where S(s) and A matrices are defined as

$$S(s) = [\,1 \quad s \quad s^2 \quad s^3\,]$$

$$A = [a_0 \quad a_1 \quad a_2 \quad a_3\,]^T.$$

On the other hand, B and B^2 matrices are defined as follows

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ B^2 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for derivatives of $\lambda'(s) = S(s)BA$ and $\lambda''(s) = S(s)B^2A$. If we put all these expressions in the equation (28), we get following equation

$$\{P_2(s)S(s)B^2 + P_1(s)S(s)B + P_0(s)S(s)\}A = F(s) \quad (31)$$

Now, we use the collocation points $s = s_i$, $i = 0,1,2,3$, $s_0 = 0$, $s_1 = 2\pi/3$, $s_2 = 4\pi/3$, $s_3 = 2\pi$ and then for $k = 0, 1, 2$ we obtain the fundamental matrix equation of (28):

$$\{P_2SB^2 + P_1SB + P_0S\}A = F\,;$$

$$P_k = \begin{bmatrix} P_k(0) & 0 & 0 & 0 \\ 0 & P_k(2\pi/3) & 0 & 0 \\ 0 & 0 & P_k(4\pi/3) & 0 \\ 0 & 0 & 0 & P_k(2\pi) \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & (2\pi/3) & (2\pi/3)^2 & (2\pi/3)^3 \\ 1 & (4\pi/3) & (4\pi/3)^2 & (4\pi/3)^3 \\ 1 & (2\pi) & (2\pi)^2 & (2\pi)^3 \end{bmatrix}.$$

Therefore, briefly, this equation becomes

$$WA = F \rightarrow [W\,;\ F] \qquad (32)$$

so that

$$W = P_2SB^2 + P_1SB + P_0S,$$

$$F = [F(0)\ F(2\pi/3) \quad F(4\pi/3)\ ) \quad F(2\pi)]^T.$$

On the other hand, the matrix equation of the conditions is obtained as follows:

$$\lambda(0) = S(0)A = \lambda_0$$

$\lambda'(0) = S(0)BA = \lambda_1$.

So, the expression in the augmented matrix form of conditions cab be written as

$$[U_{0;}\lambda_0] = [1 \quad 0 \quad 0 \quad 0 \;;\; \lambda_0]$$
$$[U_1;\lambda_1] = [0 \quad 1 \quad 0 \quad 0 \;;\; \lambda_1]$$

or

$$[U;\lambda] = \begin{bmatrix} 1 & 0 & 0 & 0; & \lambda_0 \\ 0 & 1 & 0 & 0; & \lambda_1 \end{bmatrix}.$$

Here the following equality can be written, briefly,

$$UA = F \quad \to [U \;;\; F]. \qquad (33)$$

We get $W^*A = F^*$, from (32) and (33), as

$$[W^*;\ F^*] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & w_{03} & ; & 0 \\ w_{10} & w_{11} & w_{12} & w_{13} & ; & 0 \\ 1 & 0 & 0 & 0 & ; & \lambda_0 \\ 0 & 1 & 0 & 0 & ; & \lambda_1 \end{bmatrix}$$

where, w_{ij} $(i = 0,1 \;\; j = 0,1,\dots,3)$ are obtained as follows: for $F(0) = 0$ and $F(2\pi/3) = 0$

$$w_{00} = P_0(0),\;\; w_{01} = P_1(0),\; w_{02} = 2P_2(0),\; w_{03} = 0,$$

$$w_{10} = P_0(2\pi/3),$$

$$w_{11} = (2\pi/3)P_0(2\pi/3) + P_1(2\pi/3),$$

$$w_{12} = (2\pi/3)^2P_0(2\pi/3) + 2(2\pi/3)P_1(2\pi/3) + 2\,P_2(2\pi/3)$$

$$w_{13} = (2\pi/3)^3P_0(2\pi/3) + 3(2\pi/3)^2P_1(2\pi/3) + 6(2\pi/3)P_2(2\pi/3)$$

Thus the matrix of unknown coefficients is obtained as

$$A = {W^*}^{-1}F^*;$$

where, $a_0 = \lambda_0$, $a_1 = \lambda_1$,

$$a_2 = (w_{00}w_{13} - w_{03}w_{10}/w_{12}w_{03} - w_{13}w_{02})\lambda_0 + (w_{11}w_{00} - w_{01}w_{13}/w_{13}w_{02} - w_{12}w_{03})\lambda_1$$

$$a_3 = (w_{00}w_{12} - w_{10}w_{02}/w_{13}w_{02} - w_{12}w_{03})\lambda_0 + (w_{01}w_{12} - w_{11}w_{02}/w_{13}w_{02} - w_{12}w_{03})\lambda_1.$$

If we put this a_n unknowns in equation

$$h(s) = \lambda(s) = \sum_{m=0}^{3} a_m\, s^m.$$

we get following equation

$$h(s) = \lambda_0 + \lambda_1 s + [(w_{00}w_{13} - w_{03}w_{10}/w_{12}w_{03} - w_{13}w_{02})\lambda_0 + (w_{11}w_{00} - w_{01}w_{13}/w_{13}w_{02} - w_{12}w_{03})\lambda_1]s^2 + [(w_{00}w_{12} - w_{10}w_{02}/w_{13}w_{02} - w_{12}w_{03})\lambda_0 + (w_{01}w_{12} - w_{11}w_{02}/w_{13}w_{02} - w_{12}w_{03})\lambda_1]s^3.$$

This is the function h which determines the spherical indicator curve of the $t(s)$ unit tangent vector in E^3 space.

Author's Contributions

Tuba Ağırman Aydın: Drafted and wrote the manuscript, performed the experiment and result analysis.
Mehmet Sezer: Assisted in analytical analysis on the structure, supervised the experiment's progress, result interpretation and helped in manuscript preparation.

Ethics

There are no ethical issues after the publication of this manuscript.

References

1. Euler, L. 1778. De curvis trangularibis, *Acta Academica Petropol*; 1780: 3-30.
2. Fujivara, M. 1914. On space curves of constant breadth, *Thoku Mathematical Journal*; 5: 179-184.
3. Blaschke, W, Leipziger Berichte; 1917, 67, pp 290.
4. Wong, Y-C. 1963. A global formulation of the condition for a curve to Lie in a sphere, *Monatshefte für Mathematik,* 67(4), 363-365.
5. Reuleaux, F, The Kinematics of Machinery; Trans. By Kennedy A.B.W. Dover Publishers: New York, 1963.
6. Gluck. H. 1966. Higher curvatures of curves in Euclidean space, *The American Mathematical Montly*, 73, 699-704.
7. Bruer, S, Gottlieb, D. 1971. Explicit characterization of spherical curves, *Proceedings of the American Mathematical Society*, 27(1), 126-127.
8. Dannon, V. 1981. Integral characterizations and the theory of curves, *Proceedings of the American Mathematical Society*, 81(4), 600-602.
9. Sezer, M, Integral properties and applications of a Frenet-like differential equation system, II. National Mathematical Symposium, Ege University, İzmir, 1989, 1, pp 435-444.
10. Akdoğan, Z, Mağden, A. 2001. Some characterization of curves of constant breadth in E^n space, *Turkish Journal of Mathematics;* 25, 433-444.
11. Sezer, M. 1996. A method for the approximate solution of the second order linear differential equations in terms of Taylor Polynomials, *International Journal of Mathematical Education in Science and Technology*, 27(6), 821- 834.
12. Hacısalihoğlu, H.H, Differential Geometry; Ankara Univ. Press: Ankara, Turkey, 1993.
13. O'Neill, B, Elemantary Differential Geometry; Academic Press Inc: 1966.
14. Karger, A, Novak, J, Space Kinematics and Lie Groups; Gordon And Breach Science Publishers: 1985.
15. Millman, R.S, Parker, G.D, Elements of Differential Geometry; Prentice-Hall, Inc., Englewood Cliffs Press: New Jersey, 1977.