Universal Journal of Mathematics and Applications, 2 (1) (2019) 1-10
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: http://dx.doi.org/10.32323/ujma.520682

Special Graceful Labelings of Irregular Fences and Lobsters

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Article Info

Keywords: *Fence, Graceful labelling, Lobster.*
2010 AMS: *05C30, 05C78.*
Received: *2 February 2019*
Accepted: *6 March 2019*
Available online: *20 March 2019*

Abstract

Irregular fences are subgraphs of $P_m \times P_n$ formed with m copies of P_n in such a way that two consecutive copies of P_n are connected with one or two edges; if two edges are used, then they are located in levels separated an odd number of units. We prove here that any of these fences admits a special kind of graceful labeling, called α-labeling. We show that there is a huge variety of this type of fences presenting a closed formula to determine the number of them that can be built on the grid $[1,m] \times [1,n]$. If only one edge is used to connect any pair of consecutive copies of P_n, the resulting graph is a tree. We use the α-labelings of this type of fences to construct and label a subfamily of lobsters, partially answering the long standing conjecture of Bermond that states that all lobsters are graceful. The final labeling of the lobsters presented here is not only graceful, it is an α-labeling, therefore they can be used to produce new graceful trees.

1. Introduction

Suppose G is a graph of order n and size m. An injective function $f : V(G) \to \{0,1,\ldots,m\}$ is called a *graceful labeling* of G if every edge uv of G has assigned a *weight*, defined by $|f(u) - f(v)|$, and the set of all weights induced by f on the edges of G is $\{1,2,\ldots,m\}$. A graph that admits a graceful labeling is called *graceful*. This labeling, together with three other labelings, was introduced by Rosa [1] as a mean to study a problem in combinatorial design associated with the decomposition of the complete graph K_{2m+1} into copies of any tree of size m. Rosa proved that if there is a graceful labeling of a tree T of size m, then there exists a (cyclic) decomposition of K_{2m+1} into copies of T. Several applications of gracefully labeled graphs are known, we can mentioned here the ones presented by Bloom and Golomb [2] and [3], and the ones given by Brankovic and Wanless [4].

An α*-labeling* of G is a graceful labeling f for which there exists an integer λ, called the *boundary value* of f, such that for each edge uv of G, either $f(u) \le \lambda < f(v)$ or $f(v) \le \lambda < f(u)$. If G admits an α-labeling, then it is called an α*-graph*. This definition of an α-graph implies that G is bipartite and λ is the smaller of the two vertex labels that yield the weight 1. This type of labeling is the most restrictive one among the four labelings introduced by Rosa [1]. The existence of an α-labeling implies the existence of several other types of labelings; so, they are located at the center of this research area. Not all graphs are graceful or α, this fact motivates the search of new families of graphs admitting these types of labelings.

Let G be a graph of order n and size m. Suppose that f is a graceful labeling of G. The labeling $\bar{f} : V(G) \to \{0,1,\ldots,m\}$, defined as $\bar{f} = m - f(v)$ for every $v \in V(G)$, is called the *complementary* labeling of f; this is also a graceful labeling; thus, its existence can be used to prove that the number of graceful labelings of any graph is always even. Let g be a labeling of G defined as $g(v) = c + f(v)$ for every $v \in V(G)$; we say that g is a *c units shifting* of f. It is not difficult to see that both, f and g, induce the same weights. Suppose now that f is an α-labeling of G with boundary value λ; the labeling h, defined for every $v \in V(G)$ as

$$h(v) = \begin{cases} f(v) & \text{if } f(v) \le \lambda, \\ d - 1 + f(v) & \text{if } f(v) > \lambda, \end{cases}$$

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is called a *d-graceful labeling* of G. This type of labeling was introduced in 1982, independently, by Maheo and Thuillier [5] and Slater [6].

Suppose that $f(v)-f(u)=w>0$, then $h(v)-h(u)=d-1+f(v)-f(u)=d-1+w$. Since $1\leq w\leq m$, we get that $d\leq d-1+w\leq d-1+m$. In other terms, the weights induced by h on the edges of G are $d,d+1,\ldots,d-1+m$. This property of the α-labelings has been widely used to construct new graceful and α-graphs starting with smaller α-graphs. The *reverse* of f, denoted by f_r, is another α-labeling of G, it is defined as

$$f_r(v)=\begin{cases}\lambda-f(v) & \text{if } f(v)\leq\lambda,\\ m+\lambda+1-f(v) & \text{if } f(v)>\lambda.\end{cases}$$

Note that f and f_r have the same boundary value; in addition, if $f(v)-f(u)=w$, for any weight $w\in\{1,2,\ldots,m\}$, then $f_r(v)-f_r(u)=m+\lambda+1-f(v)-\lambda+f(u)=m+1-(f(v)-f(u))=m+1-w$.

In Section 2 we present an α-labeling for a large family of connected subgraphs of the grid $P_m\times P_n$. This family, denoted by $\mathscr{F}$, is formed by all the graphs built in the following way:

For every $i\in\{1,2,\ldots,m\}$, let P_n^i be the path of order n with vertex set $V(P_n^i)=\{v_{i,0},v_{i,1},\ldots,v_{i,n-1}\}$ and edge set $E(P_n^i)=\{v_{i,0}v_{i,1}, v_{i,1}v_{i,2},\ldots,v_{i,n-2}v_{i,n-1}\}$. Now, for every $i\in\{1,2,\ldots,m-1\}$, decide whether P_n^i is connected to P_n^{i+1} with one or two edges (also called *links*). If only one edge connects them, then choose any $j\in\{0,1,\ldots,n-1\}$ and connect with an edge the vertices $v_{i,j}$ and $v_{i+1,j}$. If two edges connect them, then choose $j_1,j_2\in\{0,1,\ldots,n-1\}$, where $|j_2-j_1|$ is odd, and introduce the edges $v_{i,j_1}v_{i+1,j_1}$ and $v_{i,j_2}v_{i+1,j_2}$. Given that the number of edges connecting two copies of P_n may vary, we refer to this type of graph as an *irregular fence*. In Figure 1.1 we show all the nonisomorphic fences in $\mathscr{F}$ built on $[1,3]\times[1,4]$. We claim that all the irregular fences are α-graphs.

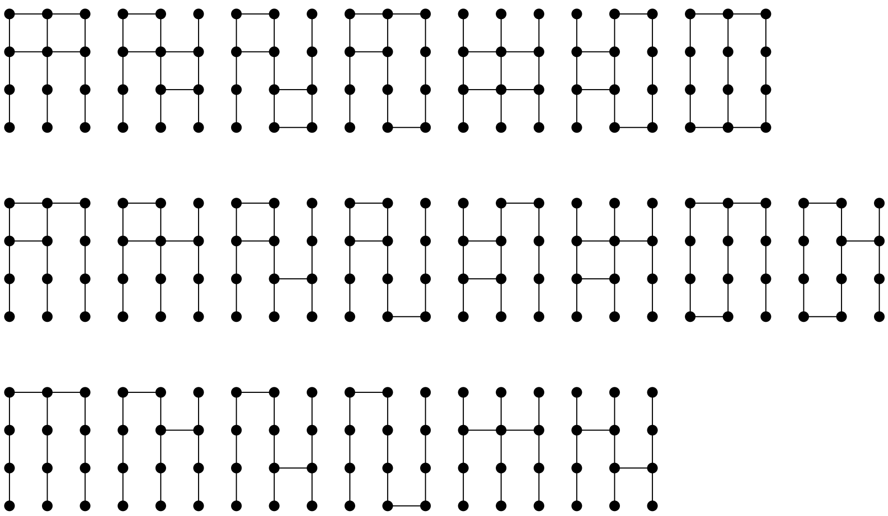


Figure 1.1: All nonisomorphic irregular fences built on $[1,3]\times[1,4]$

In Section 3 we study this type of irregular fences from an enumerative perspective. We present a closed formula for the number of nonisomorphic irregular fences built on $[1,m]\times[1,n]$. When every pair of consecutive copies of P_n is connected with only one edge, the resulting fence corresponds to a type of tree called *path-like tree*; it is known that they are α-trees [7]. In Section 4 we consider a subfamily of the path-like trees built on $[1,m]\times[1,5]$, with the extra property that they are lobsters. We characterize the lobsters that are irregular fences, therefore, α-trees; in addition we show that some other α-lobsters can be obtained from them by adding pendant vertices to some or all the vertices at distance one from the central path.

All graphs considered in this work are simple, i.e., no loops nor multiple edges are allowed. We mainly follow the notation and terminology used in [8] and [9].

2. α-labelings of irregular fences

As we mentioned before, α-labelings were introduced by Rosa [1]; he presented a labeling scheme for caterpillars that can be easily adapted for the case of paths. For the sake of completeness, we present here Rosa's α-labeling of the path P_n; we use this labeling in the construction of the α-labeled irregular fences.

Lemma 2.1. *For every $n\geq 1$, the path P_n is an α-graph.*

Assuming that $V(P_n)=\{v_0,v_1,\ldots,v_{n-1}\}$ and $E(P_n)=\{v_0v_1,v_1v_2,\ldots,v_{n-2}v_{n-1}\}$, the α-labeling $f:V(P_n)\to\{0,1,\ldots,n-1\}$ is defined as:

$$f(v_i)=\begin{cases}\frac{i}{2} & \text{if } i \text{ is even},\\ n-\frac{i+1}{2} & \text{if } i \text{ is odd}.\end{cases}$$

The labeling f has boundary value $\lambda = \frac{n-2}{2}$ when n is even and $\lambda = \frac{n-1}{2}$ when n is odd. Moreover, $f(v_0) = 0$ regardless the parity of n but $f(v_{n-1}) = \frac{n}{2} = \lambda + 1$ when n is even and $f(v_{n-1}) = \frac{n-1}{2} = \lambda$ when n is odd. We say that $v \in V(P_n)$ is a *black* vertex if $f(v) \leq \lambda$, otherwise v is a *white* vertex. In Figure 2.1 we show two examples of this labeling on P_{12} and P_{17}. Just for the examples, the boundary value is on a red vertex.

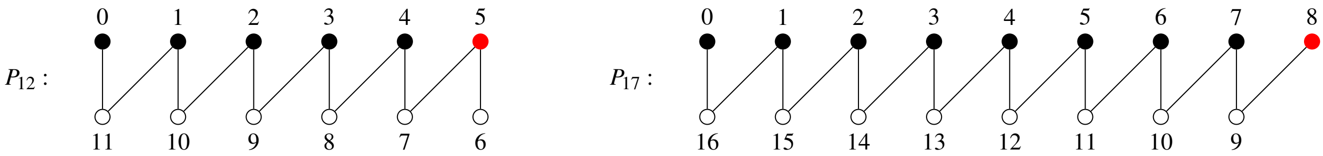


Figure 2.1: α-labelings of P_{12} and P_{17}

The construction of the α-labeled irregular fences, built on $P_m \times P_n$, is based on an embedding of the path P_{mn} on the grid $[1,m] \times [1,n]$. The division algorithm tell us that for each $i \in \{1,2,\ldots,mn\}$, there exist unique q and r such that $i = qn + r$, where $0 \leq r < n$. Using this fact we can define the embedding of P_{mn} on the grid $[1,m] \times [1,n]$ to be the bijective function $\phi : \{v_0, v_1, \ldots, v_{mn-1}\} \to [1,m] \times [1,n]$, where

$$\phi(v_i) = \begin{cases} (q+1, r+1) & \text{if } q \text{ is even,} \\ (q+1, n-r) & \text{if } q \text{ is odd.} \end{cases}$$

Once the embedding is done, we proceed to label the vertices of P_{mn} using the function f given in Lemma 2.1. In the first part of Figure 2.2 we show an embedding of P_{15} on the grid $[1,5] \times [1,3]$, on the second part we exhibit the α-labeling of this path at this embedding.

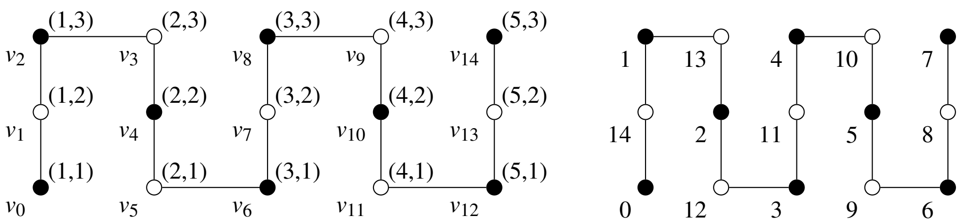
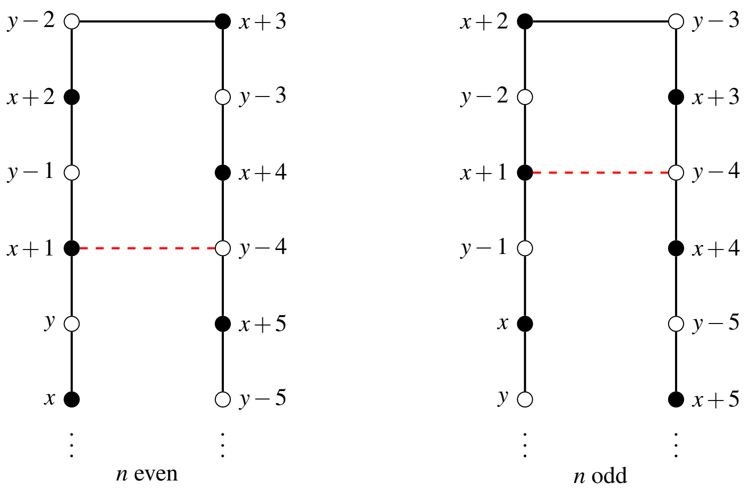


Figure 2.2: Embedding of P_{15} on $[1,5] \times [1,3]$ and its α-labeling

In the following lemmas we present the essential results that will allow us to prove that any irregular fence in $\mathscr{F}$ is an α-graph.

Lemma 2.2. *Any fence built on $[1,2] \times [1,n]$, with only one edge connecting the two copies of P_n, is an α-graph.*

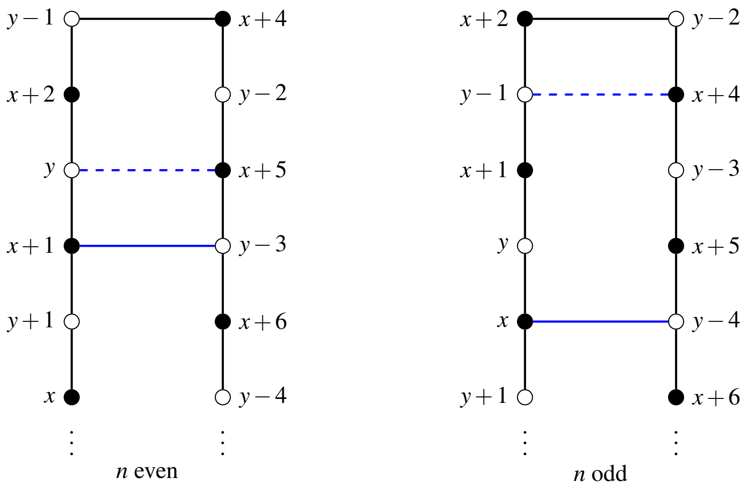
Proof. Suppose that P_{2n} has been embedded in the grid $[1,2] \times [1,n]$ in the way described before. In addition, assume that P_{2n} has been labeled using the function f given in Lemma 2.1. In the following diagram we show this labeling where the labels on the black vertices are at most λ, the boundary value of f, while the labels on the white vertices are at least $\lambda + 1$. Note that the edge connecting the vertices on $(1,n)$ and $(2,n)$ has weight $y - x - 5$, independently of the parity of n.



If for any feasible value of t, the vertices on $(1, n-t)$ and $(2, n-t)$ are connected, the new edge also has weight $y - x - 5$. This implies that all the horizontal "edges" on this embedding of P_{2n} have the same weight and any of them can be used to connect the two copies of P_n, being the final fence an α-graph. □

Lemma 2.3. *Any fence built on $[1,2]\times[1,n]$, with two edges connecting the two copies of P_n, is an α-graph.*

Proof. As we did in Lemma 2.2, suppose that P_{2n} has been embedded in the grid $[1,2]\times[1,n]$, in the way described before, and that it has been labeled using the α-labeling f in Lemma 2.1. In the following diagram, we show new labelings for the two copies of P_n.



These labelings are obtained from f by fixing the labels on the black vertices of the first copy of P_n and adding one unit to all other vertices. In this way, the edges on the first copy of P_n have the weights $n+2, n+3, \dots, 2n$; the weights on the edges of the second copy of P_n are $1,2,\dots,n-1$. We use all the labels in $\{0,1,\dots,2n\}$ except $\lceil \frac{n}{2} \rceil$. Since the white vertices on the second copy of P_n were augmented one unit while the black vertices on the first copy were fixed, any line connecting a black vertex with a white vertex will be an edge of weight $y-x-4$. Similarly, any line connecting a white vertex with a black vertex will be an edge of weight $y-x-5$ because the labels of both endvertices were augmented one unit. Hence, by connecting both copies of P_n with two edges, one of each kind, that is, one black-white and one white-black, we obtain an α-labeled irregular fence. This fence is in $\mathscr{F}$ because these types of edges are in alternated levels. This concludes the proof. □

In Figure 2.3 we show four examples of these labeled irregular fences, two for each lemma.

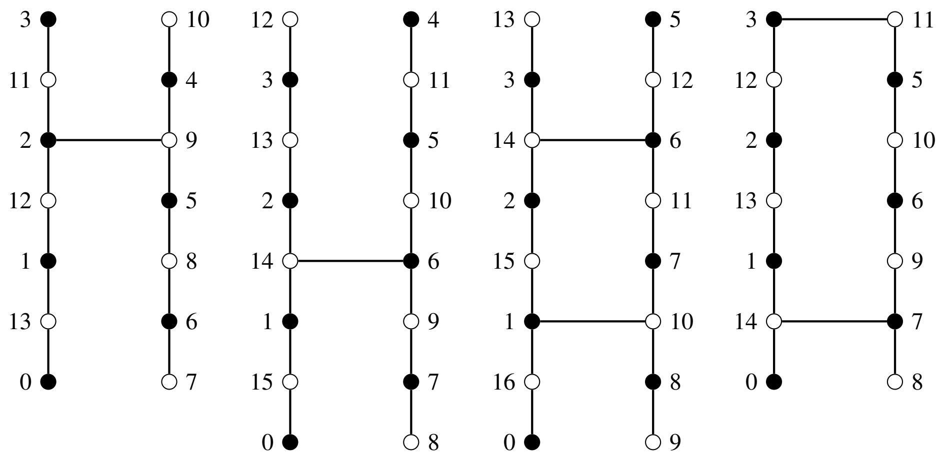


Figure 2.3: α-labelings of four irregular fences

Theorem 2.4. *If G is an irregular fence in $\mathscr{F}$, then G is an α-graph.*

Proof. Suppose that G is an irregular fence built on $P_m \times P_n$ such that it contains $1 \le k \le m-1$ pairs of consecutive copies of P_n connected by two edges. Thus, G has size $m(n-1)+(m-1)+k = mn-1+k$. Assume that the path P_{mn} has been labeled using the function f in Lemma 2.1 and is embedded in the grid $[1,m]\times[1,n]$. Thus, the weights induced on the edges of every copy of P_n are consecutive integers, and the horizontal edges, of this embedding of P_{mn}, have weights $(m-1)n, (m-2)n, \dots, 2n, n$.
Now we delete all the horizontal edges connecting consecutive copies of P_n in P_{mn}. Once this is done, we draw new horizontal edges following the pattern in G, In this way, we have a labeling of G; based on Lemma 2.2, this is an α-labeling when G is a tree, that is, when only one edge connects any pair of consecutive copies of P_n. If this is not the case, i.e., when there are $k>0$ pairs of consecutive copies of P_n connected with two edges, these two horizontal edges have the same weight. To eliminate this duplicity, we apply the procedure used in the proof of Lemma 2.3.
Suppose that $i_1, i_2, \dots, i_k$ are the indices for which there are two horizontal edges connecting $P_n^{i_j}$ and $P_n^{i_j+1}$. For every $i \le i_j$, the labels of the black vertices of all P_n^i are fixed and all the other labels are augmented in one unit. In this way, these horizontal edges have different

weights that are consecutive integers. Once this process has been applied to every pair of consecutive copies of P_n connected by two edges, the resulting labeling is indeed an α-labeling of G. In fact, since there are exactly k pairs of consecutive copies of P_n connected by two edges, the original labels of the white vertices have been shifted k units, avoiding the duplicity of vertex labels; the weights on each copy of P_n are consecutive integers, and the weights on the horizontal edges complement the ones on the vertical edges. Therefore, the final labeling of G is an α-labeling and G is an α-graph. □

In Figure 2.4 we show an example of this labeling where G is built on $P_{10} \times P_{10}$ and $k = 7$.

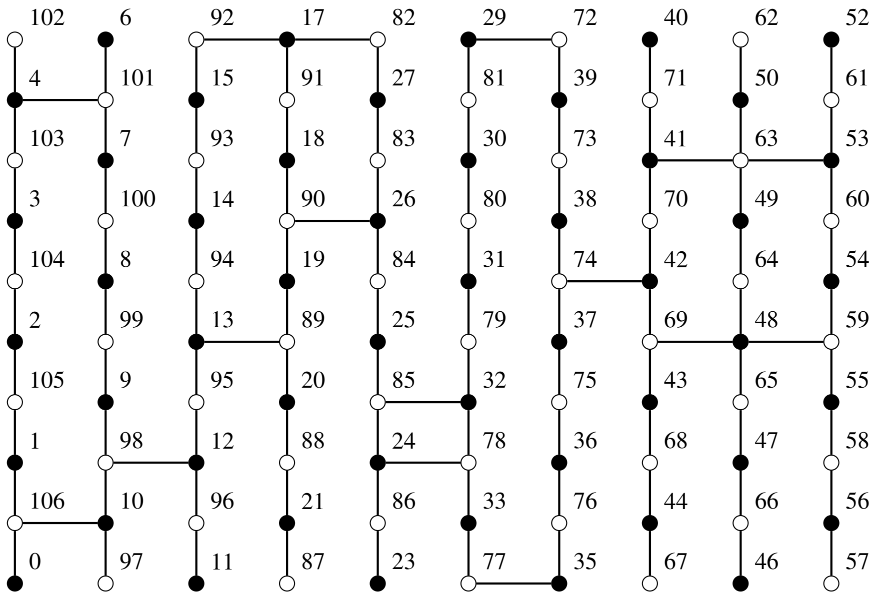


Figure 2.4: α-labelings of a fence of size 106 built on $[1,10] \times [1,10]$

3. Enumerating irregular fences

Motivated by the result in the previous section, we want to determine the number of this type of fences. In [10], we found the number of fences that can be built on the grid $[1,m] \times [1,n]$. Using that result, we present here a closed formula for the number of nonisomorphic irregular fences built on the grid.

We start by counting the number of irregular fences that can be built on $[1,2] \times [1,n]$. Since the grid $[1,m] \times [1,n]$ can be seen as a linear amalgamation of $m-1$ copies of $[1,2] \times [1,n]$ we refer to the fences on $[1,2] \times [1,n]$ as *building blocks*, or just *blocks*, of $[1,m] \times [1,n]$. Thus, a block in an irregular fence consists of two copies of P_n and 1 or 2 (horizontal) links (edges), It is not difficult to see that the number of blocks with only one link is $C(n,1) = n$, i.e., the number of ways of selecting one element from $\{1,2,\ldots,n\}$. To determine the number of blocks with two links we may count the 2-element subsets of $\{1,2,\ldots,n\}$, such that the difference between the two elements is odd. Thus, for any subset $\{i,j\}$, with $i < j$, the possible values for j are determined by the value of i. When i is odd, there are $\lfloor \frac{n}{2} \rfloor - \frac{i-1}{2}$ possible values for j. When i is even, there are $\lceil \frac{n}{2} \rceil - \frac{i}{2}$ possible values for j.

Hence, when n is even, the number of 2-element subsets satisfying the conditions is given by

$$\sum_{i=1}^{\frac{n}{2}} i + \sum_{i=1}^{\frac{n}{2}-1} i = 2\sum_{i=1}^{\frac{n}{2}-1} i + \frac{n}{2} = \frac{2(\frac{n}{2}-1)\frac{n}{2}}{2} + \frac{n}{2} = \frac{n}{2}\left(\frac{n}{2} - 1 + 1\right) = \frac{n^2}{4}.$$

When n is odd, this number is

$$2\sum_{i=1}^{\frac{n-1}{2}} i = \frac{2(\frac{n-1}{2})(\frac{n+1}{2})}{2} = \frac{n^2-1}{4}.$$

Therefore, the number of blocks is $n + \frac{n^2}{4} = \frac{n^2+4n}{4}$ when n is even and $n + \frac{n^2-1}{4} = \frac{n^2+4n-1}{4}$ when n is odd. For $n \geq 1$, the sequence $a(n)$ formed by these values corresponds to the sequence A002620 in OEIS [11].

Another number needed in our counting process is the number of symmetric blocks. Once again, we start analyzing the case where the block has exactly one link. If n is even, there are no symmetric blocks. If n is odd, there is only one symmetric block. We have a similar situation when the block has two links. When n is odd there are no two numbers $i < j$ in $\{1,2,\ldots,n\}$ such that $j - i$ is odd and $i - 1 = n - j$. When n is even, for every $1 \leq i \leq \frac{n}{2}$, the number $j = n+1-i$ belongs to $\{\frac{n}{2}+1, \frac{n}{2}+2, \ldots, n\}$, $j - i = n+1-i-i = n+1-2i$ is odd and $i - 1 = n - j = n - (n+1-i) = i - 1$. Then, if $s(n)$ denotes the number of symmetric blocks, we get

$$s(n)=\begin{cases}\frac{n}{2} & \text{if } n \text{ is even,}\\ 1 & \text{if } n \text{ is odd.}\end{cases}$$

For $n \geq 1$, the sequence $s(n)$ is the sequence A152271 in OEIS [12].
Now we turn our attention to the general case. Given any irregular fence F built on $[1,m]\times[1,n]$, there are other three fences that are isomorphic to F: when F is rotated 180° around a central vertical axis, when F is rotated 180° around a central horizontal axis, and when F is rotated 180° around a central axis perpendicular to the plane containing F. Thus, there are three possible situations: F has four different representations, F has two different representations, or F has one representation. Let T be the set of all irregular fences on $[1,m]\times[1,n]$; we define V to be the subset of T containing the fences with a vertical symmetry, H to be the subset of T containing the fences with a horizontal symmetry, C to be the subset of T containing the fences with a central symmetry, and A to be the subset of T containing the fences with all these symmetries. In Figure 3.1 we show four examples, one for each of these subsets.

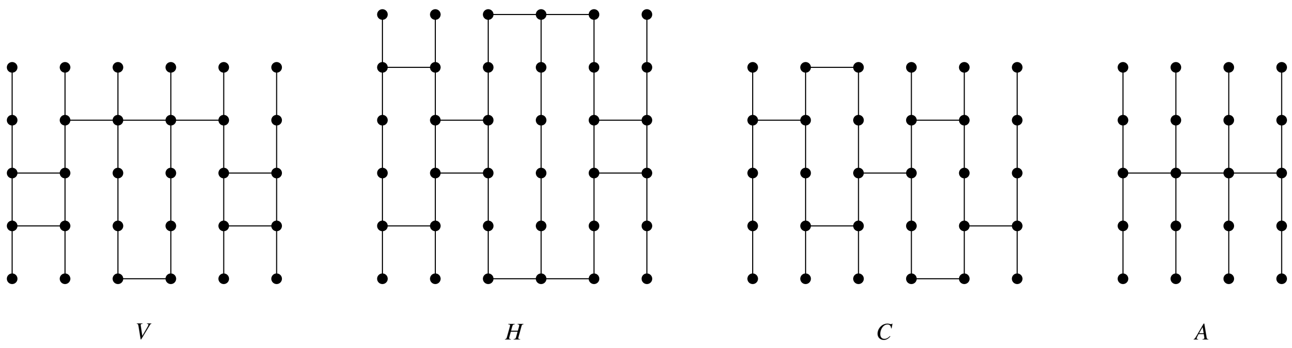


Figure 3.1: Different types of symmetric fences

Since the fences in A have all the described symmetries, each of them appears only once in the list of all possible fences built on $[1,m]\times[1,n]$. Every element of $V-A$, $H-A$, or $C-A$ appears twice in this list. Every nonsymmetric fence appears four times in the list. Thus, if we take the addition of cardinalities

$$|T|+|V|+|H|+|C|$$

every fence is counted four times. Therefore, the number of nonisomorphic irregular fences built on $[1,m]\times[1,n]$ is given by

$$f(m,n)=\tfrac{1}{4}(|T|+|V|+|H|+|C|).$$

In order to find a closed formula for $f(m,n)$ we just need to determine explicitely these four cardinalities.

Based on the number of blocks and symmetric blocks, found above, and the fact that $[1,m]\times[1,n]$ can be formed with $m-1$ copies of $[1,2]\times[1,n]$, we can say that

$$|T|=\begin{cases}\left(\frac{n^2+4n}{4}\right)^{m-1} & \text{if } n \text{ is even,}\\ \left(\frac{n^2+4n-1}{4}\right)^{m-1} & \text{if } n \text{ is odd.}\end{cases}$$

If F is a fence in V, then its ith block is identical to its $(m-i)$th block. This implies that we need to determine the number of posibilities for the first $\lfloor\frac{m}{2}\rfloor$ blocks. Thus,

$$|V|=\begin{cases}\left(\frac{n^2+4n}{4}\right)^{\lfloor\frac{m}{2}\rfloor} & \text{if } n \text{ is even,}\\ \left(\frac{n^2+4n-1}{4}\right)^{\lfloor\frac{m}{2}\rfloor} & \text{if } n \text{ is odd.}\end{cases}$$

If $F \in H$, each block in F must be symmetric. So,

$$|H|=\begin{cases}\left(\frac{n}{2}\right)^{m-1} & \text{if } n \text{ is even,}\\ 1 & \text{if } n \text{ is odd.}\end{cases}$$

When $F \in C$, there are two cases that we need to analyze that depend on the parity of m. Recall that in this case the ith block of F is represented up side down in the $(m-i)$th block.

If m is even and $i=\frac{m}{2}$, then $i=m-i$. This implies that the ith block of F must be symmetric. So,

$$|C|=\begin{cases}\left(\frac{n^2+4n}{4}\right)^{\frac{m-2}{2}}\cdot\frac{n}{2} & \text{if } n \text{ is even,}\\ \left(\frac{n^2+4n-1}{4}\right)^{\frac{m-2}{2}}\cdot 1 & \text{if } n \text{ is odd.}\end{cases}$$

If m is odd

$$|C| = \begin{cases} \left(\frac{n^2+4n}{4}\right)^{\frac{m-1}{2}} & \text{if } n \text{ is even,} \\ \left(\frac{n^2+4n-1}{4}\right)^{\frac{m-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

Thus, we have found a closed formula for $F(m,n)$. We summarize these results in the following theorem.

Theorem 3.1. *The number $f(m,n)$ of nonisomorphic irregular fences built on $[1,m]\times[1,n]$ is:*

- *When both m and n are even.*
  $f(m,n) = \frac{1}{4}\left(\left(\frac{n^2+4n}{4}\right)^{m-1} + \left(\frac{n^2+4n}{4}\right)^{\frac{m}{2}} + \left(\frac{n}{2}\right)^{m-1} + \left(\frac{n^2+4n}{4}\right)^{\frac{m-2}{2}}\cdot\frac{n}{2}\right)$
- *When m is even and n is odd.*
  $f(m,n) = \frac{1}{4}\left(\left(\frac{n^2+4n-1}{4}\right)^{m-1} + \left(\frac{n^2+4n-1}{4}\right)^{\frac{m}{2}} + 1 + \left(\frac{n^2+4n-1}{4}\right)^{\frac{m-2}{2}}\right)$
- *When m is odd and n is even.*
  $f(m,n) = \frac{1}{4}\left(\left(\frac{n^2+4n}{4}\right)^{m-1} + \left(\frac{n^2+4n}{4}\right)^{\frac{m-1}{2}} + \left(\frac{n}{2}\right)^{m-1} + \left(\frac{n^2+4n}{4}\right)^{\frac{m-1}{2}}\right)$
- *When both m and n are odd*
  $f(m,n) = \frac{1}{4}\left(\left(\frac{n^2+4n-1}{4}\right)^{m-1} + \left(\frac{n^2+4n-1}{4}\right)^{\frac{m-1}{2}} + 1 + \left(\frac{n^2+4n-1}{4}\right)^{\frac{m-1}{2}}\right)$

In Table 1, read by rows, we show the first values of $f(m,n)$ for $2 \le m,n \le 10$. We have omitted the cases where $m=1$ or $n=1$ because $f(m,n)=1$.

	2	3	4	5	6	7	8	9	10
2	2	3	5	6	9	10	14	15	20
3	4	9	21	36	66	100	160	225	330
4	10	39	150	366	918	1810	3640	6315	11100
5	25	169	1060	3721	12789	32761	83296	177241	375925
6	70	819	8360	40626	190917	620830	1994944	5134095	13143500
7	196	3969	65808	443556	2849526	11764900	47783680	148718025	459591750
8	574	19719	525600	4875786	42730578	223502230	1146718720	4312651995	16085261781
9	1681	97969	4196416	53597041	640749609	4245955921	27519010816	125061956881	562969695625
10	5002	489219	33564800	589530846	9611072577	80672576050	660454273024	3626791798575	19703925162500

Table 1: Initial values for the numebr $f(m,n)$ of nonisomorphic irregular fences builon $[1,m]\times[1,n]$

4. Lobsters with an α-labeling

A *lobster* is a tree with the property that the removal of all its leaves results in a caterpillar, and a *caterpillar* is a tree with the property that the removal of all its leaves results in a path. We refer to this path as the *central path* of the lobster. An alternative definition was given in [13]. Let P be any of the longest paths in a tree T; T is called a *k-distance* tree if every vertex is at distance at most k from P. Thus, paths are 0-distance trees, caterpillars are 1-distance trees, and lobsters are 2-distance trees.

It was conjectured by Bermond [14] that all lobsters are graceful. Several families of graceful lobsters are known. Using the construction of Stanton and Zarnke [15] it is possible to obtain a graceful labeling of any lobster constructed by attaching, to every vertex of a path, a leaf of the star $K_{1,n}$. Burzio and Ferrarese [16] proved that any tree obtained from a graceful tree by replacing each edge with a path of fixed length is graceful. Thus, if the starting tree is a caterpillar and every edge is replaced with a path of length 2, the resulting graph is a lobster. This is one of the strongest results in this area, the weakest part is that the distance between any two leaves, at distance two, is always even. This problem is solved in the work of Wang et al. [17], as well as in the series of articles of Mishra and Panagrahi [18], [19], [20], and [21]. In all these papers, the lobsters considered share the property that all the vertices in the central path have degree larger than two and the subtrees attached to them must satisfy some structural conditions. Morgan [13] proved that all lobsters with a perfect matching are graceful. In a similar line, Krop [22] showed the same for lobsters with an almost perfect matching.

In this section we explore lobsters that are path-like trees and how to use the α-labeling, given in Section 2, to produce new α-labeled lobsters.

Suppose that the path P_{5m} has been labeled using the labeling in Lemma 2.1, and embedded in the grid $[1,m]\times[1,5]$, as we did in Section 2. Thus, every column in this embedding is a copy of P_5; moreover, the labeling of the ith copy of P_5 is a d_i-graceful labeling shifted c_i units, where $d_i = n(m-i)+1$ and

$$c_i = \begin{cases} \frac{n(i-1)}{2} & \text{if } i \text{ is odd,} \\ \frac{n(i-1)+1}{2} & \text{if } i \text{ is even.} \end{cases}$$

We claim that when every copy of P_5 is replaced by a copy of any caterpillar of diameter four, the result still holds; that is, we can concatenate the central vertices of these caterpillars to obtain a lobster with an α-labeling. In Figure 4.1 we show the labeling scheme given by Rosa [1] to get an α-labeling of a caterpillar of size $n-1$.

Let G be a caterpillar of diameter 4 and order n. If all the leaves of G are deleted, we get the path P_3; thus, we can use the notation $C(n_1,n_2,n_3)$ to denote the caterpillar of order $n = n_1+n_2+n_3+3$, obtained from P_3 by attaching n_i pendant vertices to the vertex v_i of P_3. In Figure 4.2 we show an α-labeling f of $C(n_1,n_2,n_3)$ together with the reverse of its complementary labeling.

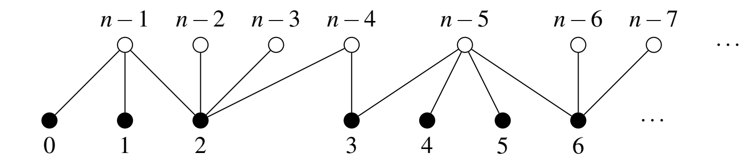


Figure 4.1: α-labeling scheme of a caterpillar of size $n-1$

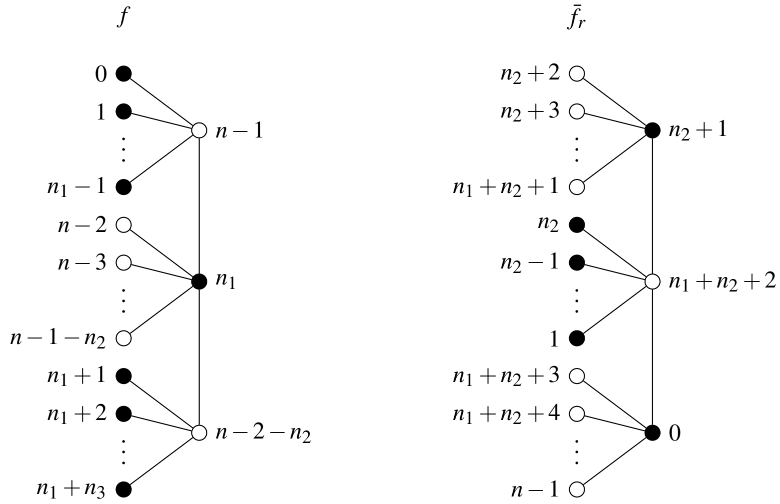


Figure 4.2: α-labelings of $C(n_1,n_2,n_3)$

Lemma 4.1. *The lobster L, obtained by connecting with an edge the central vertices of two copies of the caterpillar $C(n_1,n_2,n_3)$, is an α-tree.*

Proof. The caterpillar $C(n_1,n_2,n_3)$ has size n_1,n_2,n_3+2; the α-labeling f of it has boundary value $\lambda = n_1+n_3$. Then, we label the first copy of this caterpillar using the labeling f, which is transformed into a $(n+1)$-graceful labeling. In this way, its central vertex has label n_1. The second copy of the caterpillar is originally labeled using $\overline{f}_r$, this labeling is shifted n_1+n_3+1 units, thus there is no repetition of labels between both copies. The new label of the central vertex of the second copy is $(n_1+n_2+2)+(n_1+n_3+1) = n+n_1$. Hence, if we connect with an edge the central vertices, this edge will have weight n. Therefore, the lobster L is an α-tree. □

This process can be applied to any number of copies of $C(n_1,n_2,n_3)$, in the same way that it was applied to any number of copies of P_n in Section 2. Thus, we get the following theorem.

Theorem 4.2. *For each $1 \le i \le k$, let G_i be a copy of the caterpillar $C(n_1,n_2,n_3)$. If for every $1 \le i \le k-1$, the central vertex of G_i is connected with an edge to the central vertex of G_{i+1}, then the resulting graph is a lobster that admits an α-labeling.*

In Figure 4.3 we show an example of this construction using the caterpillar $C(2,4,3)$ four times. We must observe that the lobsters obtained using these caterpillars do not have a perfect (or almost perfect) matching.

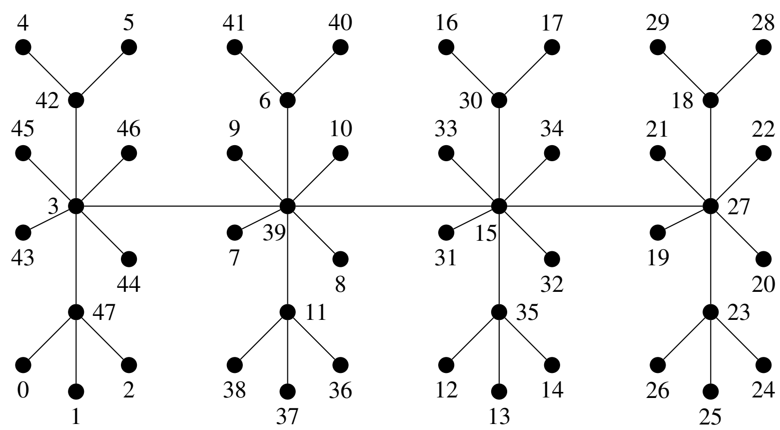


Figure 4.3: α-labeling of a lobster in $\mathscr{G}$

For each $1 \le i \le k$, let G_i be a copy of the caterpillar $C(n_1, n_2, n_3)$. The family $\mathscr{G}_k$ consists of all lobsters formed connecting with an edge the central vertices of G_i and G_{i+1} where $1 \le i \le k-1$. Thus we can say that all members of $\mathscr{G}_k$ are α-trees. Furthermore, for any $G \in \mathscr{G}_k$, the α-labeling of G, obtained using Theorem 4.2, assigns the label 0 to a leaf of G and the label λ (when k is odd) or $\lambda + 1$ (when k is even) to another leaf, and the distance between these leaves is $k+3$, that is, the diameter of G. In [23] we proved that if $B_1, B_2, \ldots, B_k$ is a collection of α-labeled blocks, with boundary value λ_i, then the graph obtained amalgamating the vertex labeled 0 in B_i with the vertex labeled λ_{i-1} in B_{i-1}, for every $2 \le i \le k$, is an α-graph. We refer to this process as the $(0, \lambda)$-*amalgamation*. As we showed before, if G is a caterpillar, there exists an α-labeling of G that assigns the labels 0 and λ (when the diameter is even) or 0 and $\lambda + 1$ (when the diameter is odd) on the leaves of a path of maximum length in G. These two properties allow us to prove the following theorem.

Theorem 4.3. *Let $G_1, G_2, \ldots, G_t$ be a collection of α-graphs, such that $G_i \in \mathscr{G}_{k_i}$ or G_i is a caterpillar. Then, the lobster L, obtained via $(0-\lambda)$-amalgamation of these graphs, is an α-tree.*

Proof. Suppose that f_i is an α-labeling of G_i with boundary value λ_i. If G_i is a caterpillar, we assume that f_i is the labeling f in Figure 4.1. If G_i is a lobster in $\mathscr{G}_{k_i}$, we assume that f_i is the labeling obtained in Theorem 4.2. In both cases, the vertex of G_i labeled 0 belongs to a path of maximum length in G_i. If the vertex of G_i labeled λ_i is on a leaf, then we can identify the vertex labeled 0 in G_{i+1} with the vertex labeled λ_i in G_i. The α-labeling of the new graph, denoted by Γ_{i+1}, is obtained by shifting λ_i units the labeling f_{i+1} and transforming f_i into a d_i-graceful labeling where $d_i - 1$ is the size of G_{i+1}. If the boundary value of this labeling of Γ_{i+1} is on a leaf, we concatenate Γ_{i+1} with G_{i+2}, to obtain an α-graph Γ_{i+2}, and so on until all the amalgamations are done. If the boundary value of this labeling of Γ_{i+1} is not on a leaf, then we use the complementary labeling, which puts its boundary value on a leaf, and connect Γ_{i+1} with G_{i+2}, and continue in this way until all the amalgamations are done. Given the position of the vertices labeled 0 and λ_i, the final graph is a lobster with an α-labeling. $\square$

In Figure 4.4 we show an example of this construction where $G_1 \in \mathscr{G}_2$, G_2 is a caterpillar of size 10, and $G_3 \in \mathscr{G}_3$.

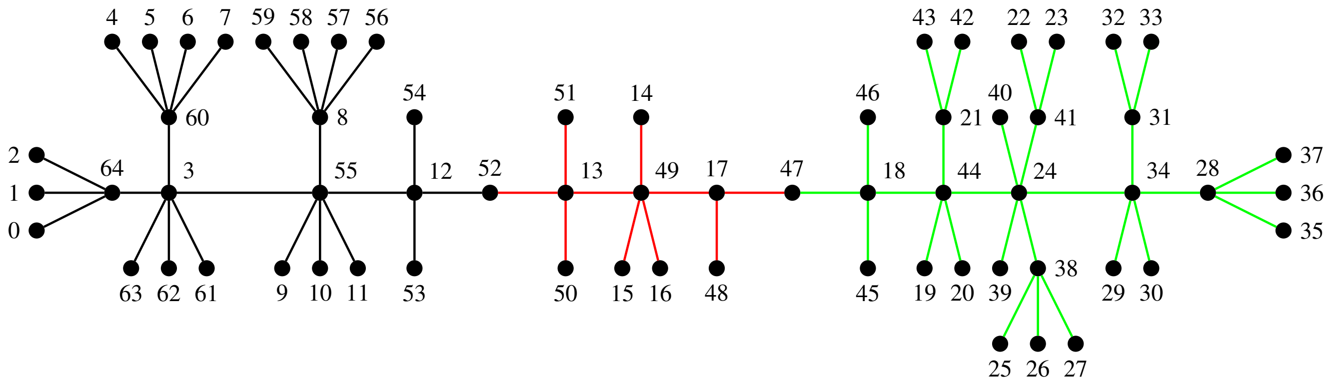


Figure 4.4: α-labeling of a lobster

5. Conclusions

There is a wide variety of fences, we explored here one of these varieties where two consecutive copies of P_n are connected by one or two links, if two links are used, the distance between them is odd. These constraints can be modified to explore the existence of α-labelings of general fences, where the number of links is not restricted to 1 or 2. We think that all fences admit an α-labeling, except when the fence is isomorphic to the cycle C_n with $n \equiv 2 (\text{mod } 4)$, that is not a graceful graph.

The construction of α-lobsters presented in Theorem 4.3 can be use in a more general case, where a lobster could be decomposed into sublobsters, each of them with an α-labeling that assigns the labels 0 and λ to leaves u and v such that the distance between them equals the diameter of the sublobster. We think that this technique should be explored with more details in future works.

6. Acknowledgment

We want to thank the referees for their comments, suggestions, and fine job, that certainly helped us to improve the quality of this work.

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