



Energy Conditions for Hamiltonian and Traceable Graphs

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Abstract

A graph is called Hamiltonian (resp. traceable) if the graph has a Hamiltonian cycle (resp. path), a cycle (resp. path) containing all the vertices of the graph. The energy of a graph is defined as the sum of the absolute values of the eigenvalues of the graph. In this note, we present new conditions based on energy for Hamiltonian and traceable graphs.

1. Introduction

All the graphs considered in this note are undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let G be a graph of order n with e edges. We use $\delta(G)$ and $\chi(G)$ to denote the minimum degree and the chromatic number of G , respectively. The independence number, denoted $\alpha = \alpha(G)$, is defined as the size of the largest independent set in G . The eigenvalues $\mu_1(G) \geq \mu_2(G) \geq \dots \geq \mu_n(G)$ of the adjacency matrix $A(G)$ of G are called the eigenvalues of G . We use $S^+(G)$ (resp. $S^-(G)$) to denote the sum of the squares of the positive (resp. negative) eigenvalues of G . Notice that $S^+(G) + S^-(G) = 2e(G)$ for a graph G . The energy, denoted $Eng(G)$, of G is defined as $\sum_{i=1}^n |\mu_i(G)|$ (see [2]). A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G . A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G . A graph G is called traceable if G has a Hamiltonian path. In this note, we will present the energy conditions for Hamiltonian and traceable graphs. The results are as follows.

Theorem 1.1. Let G be a k -connected ($k \geq 2$) graph with $n \geq 3$ vertices and e edges. If

$$Eng(G) \geq 2\sqrt{\frac{2e(\chi-1)(n-k-1)}{\chi}},$$

then G is Hamiltonian.

Theorem 1.2. Let G be a k -connected graph with $n \geq 3$ vertices and e edges. If

$$Eng(G) \geq 2\sqrt{\frac{2e(\chi-1)(n-k-2)}{\chi}},$$

then G is traceable or $K_{1,3}$.

2. Lemmas

In order to prove Theorem 1.1, we need the following results as our lemmas. Lemma 2.1 below is Theorem 2.3 on Pages 484 in [3].

Lemma 2.1. Let G be a graph. Then

$$\chi \geq 1 + \max \left\{ \frac{S^+}{S^-}, \frac{S^-}{S^+} \right\}.$$

Lemma 2.2 below is Theorem 3.14 on Pages 88 and 89 in [4].

Lemma 2.2. *Let G be a graph. If the number of eigenvalues of G which are greater than, less than, and equal to zero are p , q , and r , respectively, then*

$$\alpha \leq r + \min\{p, q\},$$

where α is the independence number of G .

3. Proofs

Next, we will present proofs for Theorems 1.1 and 1.2. Some ideas from [5] are used in our proofs.

Proof of Theorem 1.1. Let G be a graph satisfying the conditions in Theorem 1.1. Suppose, to the contrary, that G is not Hamiltonian. If $n = 3$, G must be Hamiltonian since G is k -connected ($k \geq 2$). From now on, we assume that $n \geq 4$. Since G is k -connected ($k \geq 2$), G has a cycle. Choose a longest cycle C in G and give an orientation on C . Since G is not Hamiltonian, there exists a vertex $u_0 \in V(G) - V(C)$. By Menger's theorem, we can find s ($s \geq k$) pairwise disjoint (except for u_0) paths P_1, P_2, \dots, P_s between u_0 and $V(C)$. Let v_i be the end vertex of P_i on C , where $1 \leq i \leq s$. Without loss of generality, we assume that the appearance of v_1, v_2, \dots, v_s agrees with the orientation of C . We use v_i^+ to denote the successor of v_i along the orientation of C , where $1 \leq i \leq s$. Since C is a longest cycle in G , we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq s$ and the index $s+1$ is regarded as 1. Moreover, $S := \{u_0, v_1^+, v_2^+, \dots, v_s^+\}$ is independent (otherwise G would have cycles which are longer than C). Then $\alpha \geq s+1 \geq k+1$.

Let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_p$ be the p positive eigenvalues of G and let $\mu_{n-q+1} \geq \mu_{n-q+2} \geq \dots \geq \mu_n$ be the q negative eigenvalues of G . Then $n - (p+q)$ is the number of eigenvalues of G which are equal to zero. Since $\sum_{i=1}^p \mu_i + \sum_{i=n-q+1}^n \mu_i = \text{trace of } A = 0$, $\sum_{i=1}^p |\mu_i| = \sum_{i=n-q+1}^n |\mu_i|$. Thus we have that

$$Eng(G) = 2 \sum_{i=1}^p |\mu_i| = 2 \sum_{i=n-q+1}^n |\mu_i|.$$

From Lemma 2.1, we have that

$$\chi \geq 1 + \frac{S^+}{S^-} = 1 + \frac{S^+}{2e - S^+} = \frac{2e}{2e - S^+}, \quad \chi \geq 1 + \frac{S^-}{S^+} = 1 + \frac{S^-}{2e - S^-} = \frac{2e}{2e - S^-}.$$

Therefore we further have that

$$S^+ \leq \frac{2e(\chi - 1)}{\chi}, \quad S^- \leq \frac{2e(\chi - 1)}{\chi}.$$

From Cauchy-Schwarz inequality, we have that

$$\frac{Eng(G)}{2} = \sum_{i=1}^p |\mu_i| \leq \sqrt{p \sum_{i=1}^p \mu_i^2} = \sqrt{p S^+} \leq \sqrt{\frac{2e(\chi - 1)p}{\chi}}.$$

Similarly, we have that

$$\frac{Eng(G)}{2} = \sum_{i=n-q+1}^n |\mu_i| \leq \sqrt{q \sum_{i=n-q+1}^n \mu_i^2} = \sqrt{q S^-} \leq \sqrt{\frac{2e(\chi - 1)q}{\chi}}.$$

Therefore we get that

$$\begin{aligned} Eng(G) &= \frac{Eng(G)}{2} + \frac{Eng(G)}{2} \\ &\leq \sqrt{\frac{2e(\chi - 1)p}{\chi}} + \sqrt{\frac{2e(\chi - 1)q}{\chi}} = \sqrt{\frac{2e(\chi - 1)}{\chi}} (\sqrt{p} + \sqrt{q}). \end{aligned}$$

From Lemma 2.2, we have that $\alpha \leq n - (p+q) + \min\{p, q\} \leq n - p - q + p = n - q$ and $\alpha \leq n - (p+q) + \min\{p, q\} \leq n - p - q + q = n - p$. Thus $p \leq n - \alpha$ and $q \leq n - \alpha$. Therefore we have that

$$\begin{aligned} 2\sqrt{\frac{2e(\chi - 1)(n - k - 1)}{\chi}} &\leq Eng(G) \leq 2\sqrt{\frac{2e(\chi - 1)(n - \alpha)}{\chi}} \\ &\leq 2\sqrt{\frac{2e(\chi - 1)(n - s - 1)}{\chi}} \leq 2\sqrt{\frac{2e(\chi - 1)(n - k - 1)}{\chi}}. \end{aligned}$$

From the above proofs, we have that

$$S^+ = S^- = \frac{2e(\chi - 1)}{\chi},$$

$$\mu_1 = \mu_2 = \dots = \mu_p, \mu_{n-q+1} = \mu_{n-q+2} = \dots = \mu_n,$$

$$p = q = n - \alpha, \alpha = s + 1 = k + 1.$$

Thus $p\mu_1^2 = S^+ = S^- = q\mu_n^2$. Since $p = q$, $\mu_1^2 = \mu_n^2$. Hence $\mu_1 = -\mu_n$. Since G is connected and $\mu_1 = -\mu_n$, G is a bipartite graph. From Perron-Frobenius theorem, we have that $\mu_1 > \mu_2$. Since $\mu_1 = \mu_2 = \dots = \mu_p$, we must have $p = 1$. Now $\alpha = n - p = n - 1$, which implies that G cannot be 2-connected, a contradiction.

Therefore the proof of Theorem 1 is complete. □

Proof of Theorem 1.2. Let G be a graph satisfying the conditions in Theorem 1.2. Suppose, to the contrary, that G is not traceable. If $n = 3$, G must be traceable since G is k -connected ($k \geq 1$). From now on, we assume that $n \geq 4$. Choose a longest path P in G and give an orientation on P . Let x and y be the two end vertices of P . Since G is not traceable, there exists a vertex $u_0 \in V(G) - V(P)$. By Menger's theorem, we can find s ($s \geq k$) pairwise disjoint (except for u_0) paths P_1, P_2, \dots, P_s between u_0 and $V(P)$. Let v_i be the end vertex of P_i on P , where $1 \leq i \leq s$. Without loss of generality, we assume that the appearance of v_1, v_2, \dots, v_s agrees with the orientation of P . Since P is a longest path in G , $x \neq v_i$ and $y \neq v_i$, for each i with $1 \leq i \leq s$, otherwise G would have paths which are longer than P . We use v_i^+ to denote the successor of v_i along the orientation of P , where $1 \leq i \leq s$. Since P is a longest path in G , we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq s - 1$. Moreover, $S := \{u_0, v_1^+, v_2^+, \dots, v_s^+, x\}$ is independent (otherwise G would have paths which are longer than P). Then $\alpha \geq s + 2 \geq k + 2$.

Using the arguments similar to the ones in Proof of Theorem 1.1, we have that

$$2\sqrt{\frac{2e(\chi - 1)(n - k - 2)}{\chi}} \leq Eng(G) \leq 2\sqrt{\frac{2e(\chi - 1)(n - \alpha)}{\chi}}$$

$$\leq 2\sqrt{\frac{2e(\chi - 1)(n - s - 2)}{\chi}} \leq 2\sqrt{\frac{2e(\chi - 1)(n - k - 2)}{\chi}}.$$

Therefore we have that

$$S^+ = S^- = \frac{2e(\chi - 1)}{\chi},$$

$$\mu_1 = \mu_2 = \dots = \mu_p, \mu_{n-q+1} = \mu_{n-q+2} = \dots = \mu_n, p = q = n - \alpha, \alpha = s + 2 = k + 2.$$

Thus $p\mu_1^2 = S^+ = S^- = q\mu_n^2$. Since $p = q$, $\mu_1^2 = \mu_n^2$. Hence $\mu_1 = -\mu_n$. Since G is connected and $\mu_1 = -\mu_n$, G is a bipartite graph. From Perron-Frobenius theorem, we have that $\mu_1 > \mu_2$. Since $\mu_1 = \mu_2 = \dots = \mu_p$, we must have $p = 1$. Now $\alpha = n - p = n - 1$. So G is $K_{1, n-1}$ with $n \geq 4$. Since now $k = 1$ and $n - 1 = \alpha = k + 2$, we have that G is $K_{1,3}$.

Therefore the proof of Theorem 1.2 is complete.

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