Universal Journal of Mathematics and Applications, 2 (1) (2019) 44-47
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: DOI: http://dx.doi.org/10.32323/ujma.509358

On the Quotients of Regular Operators

Erdal Bayram[1*] and Cansu Binnaz Binbaşıoglu[1]

[1]*Department of Mathematics, Faculty of Arts and Sciences, Tekirdağ Namık Kemal University, Tekirdağ, Turkey.*
[*]*Corresponding author*

Article Info

Keywords: *Banach lattice, L-weakly compact operator, M-weakly compact operator, Quotient space, Regular operators.*
2010 AMS: *46B42, 47B60, 47B65*
Received: *7 January 2019*
Accepted: *26 February 2019*
Available online: *20 March 2019*

Abstract

We give some results about quotients of regular operators on Banach lattices by the linear span of the positive M-weakly and positive L-weakly compact operators. We also present a representation of the quotient space created by the linear span of the positive L-weakly compact operators.

1. Introduction

A significant number of studies concerning L-weakly and M-weakly compact operators has been produced in the literature up to the present. The existing studies concern mostly the relationship between these operators and other operator classes or pathological properties of them. Recently, certain results regarding the order structure of these operator classes have been obtained [1]. Therefore, it is natural to consider other spaces with new ordered structure. The quotient spaces represent of the constructing new spaces from old ones. For this reason, we investigate certain order properties of quotients of the regular operators by our operators which are Banach lattice.
We refer to [2, 3, 4, 5] for unexplained concepts and properties about Banach lattices and positive operators. In the rest of this article, E and F are assumed as Banach lattices, X and Y are assumed Banach spaces unless otherwise stated, and neither of them is the zero space, as we will not indicate this fact in every result. $\mathscr{L}(E,F)$ $\left(\text{resp. } \mathscr{L}^+(E,F)\right)$ denotes all linear bounded (resp. positive) operators from E to F.
In general, the linear span of the positive operators $\mathscr{L}^r(E,F)$ which is called regular operators is neither a vector lattice (or Riesz space) nor a Banach space with respect to operator norm $\|.\|$. However, when another norm namely so-called regular norm $\|.\|_r$ is defined by

$$\|T\|_r = \inf\left\{\|S\| : S \in \mathscr{L}^+(E,F),\ |Tx| \leq S|x|,\ \forall x \in E\right\}$$

then $\mathscr{L}^r(E,F)$ turns into a Banach space. Also, the equality $\|T\|_r = \||T|\|$ is satisfied whenever $T \in \mathscr{L}^r(E,F)$ has a modulus. However, there are situations in which $\mathscr{L}^r(E,F)$ is a Riesz space. For example, $(\mathscr{L}^r(E,F), \|.\|_r)$ is a Banach lattice provided that F is Dedekind complete or E is atomic with an order continuous norm ([6],Theorem 3.3 and 3.4).
The operator $T \in \mathscr{L}(X,E)$ is called L-weakly compact if $T(B_X)$, where B_X is the closed unit ball of X, is an L-weakly compact in the sense that every disjoint sequence $(y_n)_{n\in\mathbb{N}}$ in the $sol(T(B_X))$ is norm null. The operator $T \in \mathscr{L}(E,X)$ is called M-weakly compact whenever the sequence $(Tx_n)_{n\in\mathbb{N}}$ is norm null for every bounded disjoint sequence (x_n) in E. From now on, we use the notations $\mathscr{W}_M(E,F)$ and $\mathscr{W}_L(E,F)$ for all operators which is M-weakly and L-weakly compact, respectively. There is very important duality property between our operators and is stated as follows: $T \in \mathscr{W}_M(E,F)$ (resp. $T \in \mathscr{W}_L(E,F)$) if and only if $T^* \in \mathscr{W}_L(F^*,E^*)$ (resp. $T^* \in \mathscr{W}_M(F^*,E^*)$) where T^* is adjoint operator for T. $\mathscr{W}_M(E,F)$ and $\mathscr{W}_L(E,F)$ are subclasses of weakly compact operators and are closed in $\mathscr{L}(E,F)$ with the operator norm.
E is said to have an order continuous norm whenever $\inf\{\|x_\alpha\|\} = 0$ for every downwards directed net (x_α) such that $\inf\{x_\alpha\} = 0$ in E. For example, c_0, ℓ_p and $L^p(\mu)$ $(1 \leq p < \infty)$ have order continuous norm whereas ℓ_∞ and c do not have respect their usual norms. The order continuous part of E is defined

$$E^a = \{x \in E : |x| \geqslant x_\alpha \downarrow 0 \Rightarrow \|x_\alpha\| \to 0\}$$

For example, for an atomless measure μ, $(L^\infty(\mu))^a = \{0\}$ and $(\ell^\infty)^a = c_0$. From Proposition 2.4.10, Proposition 3.6.2 in [4], we see that E^a is a closed order ideal and all L-weakly compact subsets is included in E^a. Therefore, $E^a = \{0\}$ $\left(\text{resp. } (E')^a = \{0\}\right)$ if and only if $\mathscr{W}_L(E,F) = \{0\}$ (resp. $\mathscr{W}_M(E,F) = \{0\}$). For this reason, we assume that $E^a \neq \{0\}$ $\left(\text{resp. } (E')^a \neq \{0\}\right)$.

***Email addresses and ORCID numbers*: ebayram@nku.edu.tr, https://orcid.org/0000-0001-8488-359X (E. Bayram), cansubinnaz_@hotmail.com, https://orcid.org/0000-0001-9586-516X (C. Binnaz Binbaşıoğlu)**

2. Quotients

For the Riesz space E, we recall that the linear subspace $A \subset E$ is an ideal whenever $|u| \leq |v|$ and $v \in A$ implies $u \in A$. On the other hand, the quotient space $E \diagup A$ consists of all equivalence classes modulo A. We use the notation $[u]$ for the equivalence classes containing the element $u \in E$. The quotient $E \diagup A$ is a Riesz space according to the partial ordering: $[u] \leq [v]$ whenever there exist elements $u_1 \in [u]$ and $v_1 \in [v]$ such that $u_1 \leq v_1$ ([7], Sec. 18, also for equivalent ordering). The natural quotient map $\pi : E \to E \diagup A$, $\pi(u) = [u]$ is a linear, surjective Riesz (lattice) homomorphism in that $|\pi(u)| = \pi(u)$ for all $u \in E$ and its null space is A. Since a null space of a Riesz homomorfizm should be ideal, we consider quotients of Riesz spaces by the ideals. Moreover, the positive cone of $E \diagup A$ is $\pi(E^+) = \{[u] : u \in E^+\}$. Besides, for the closed subspace A of a normed space X, the function

$$\|.\| : X \diagup A \to \mathbb{R}, \|u\| = dist(u,A) = \inf\{\|u-y\| : y \in A\} = \inf\{\|u\| : u \in [x]\}$$

define the so-called quotient norm on the quotient space $X \diagup A$. Moreover, $X \diagup A$ with quotient norm is also a Banach space provided that X is a Banach space. Therefore, $E \diagup A$ with the quotient norm is a Banach lattice whenever E is a Banach lattice and A is a closed ideal of E ([2],Proposition II.5.4).
Theorem 2.2 of [8] show that a regular operator which is both L-weakly compact and M-weakly compact do not need to have a modulus. Also, Theorem 2.3 of [8] show that a modulus $|T|$ for the operator $T \in \mathscr{W}_L(E,F) \cap \mathscr{W}_M$ do not need to be L-weakly or M-weakly compact. These examples make it clear that $\mathscr{W}_M(E,F)$ (resp. $\mathscr{W}_L(E,F)$) and $\mathscr{W}_M(E,F) \cap \mathscr{L}^r(E,F)$ (resp. $\mathscr{W}_L(E,F) \cap \mathscr{L}^r(E,F)$) are not vector lattice generally. Nevertheless, considering smaller subclasses

$$\mathscr{W}_L^r(E,F) = \{T_1 - T_2 : T_1, T_2 \in \mathscr{W}_L^+(E,F)\}$$

and

$$\mathscr{W}_M^r(E,F) = \{T_1 - T_2 : T_1, T_2 \in \mathscr{W}_M^+(E,F)\}$$

we have nice order theoretic structures. Recently the following result have been proven:

Theorem 2.1 ([1] Theorem 2.2 and 2.3)**.** *$\mathscr{W}_L^r(E,F)$, equipped with the regular norm, is a Dedekind complete Banach lattice. Similarly, if F is a Dedekind complete Banach lattice, then $\mathscr{W}_M^r(E,F)$ equipped with the regular norm is a Dedekind complete Banach lattice.*

As we can also see in the proof of Theorems 2.2 and 2.3 in [1], $\mathscr{W}_L^r(E,F)$, equipped with the regular norm, is closed in $\mathscr{L}^r(E,F)$. Similarly, if F is a Dedekind complete Banach lattice, then $\mathscr{W}_M^r(E,F)$ equipped with the regular norm is closed in $\mathscr{L}^r(E,F)$. Note that our operator classes have also domination property, in other words, the inequality $0 \leq S \leq T$ implies that S is in the class of operators as T. Thus, the next two results are clear from Proposition II.5.4 of [2].

Corollary 2.2. *If $\mathscr{L}^r(E,F)$ with the regular norm is a Banach lattice, then $\mathscr{L}^r(E,F) \diagup \mathscr{W}_L^r(E,F)$ with the quotient norm is a Banach lattice.*

Corollary 2.3. *If F is a Dedekind complete, then $\mathscr{L}^r(E,F) \diagup \mathscr{W}_M^r(E,F)$ with the quotient norm is a Banach lattice.*

Dedekind completeness of the quotients depends on the quotient map $\pi : E \to E \diagup A$ to be order continuous since A is the kernel of the quotient map π. Note that our operator classes are not generally band in $\mathscr{L}^r(E,F)$. For example, this can be seen when $E = F = c_0$. The next proposition state a situation that our quotients are Dedekind complete. Recall that a Banach lattice E is an AM-space if $\|\sup\{x,y\}\| = maks\{\|x\|, \|y\|\}$ for all $x, y \in E^+$ and a strong order unit in Riesz space E is an element $e \in E^+$ whenever for every $x \in E$ there is $\lambda \in \mathbb{R}$ such that $-\lambda e \leq x \leq \lambda e$.

Proposition 2.4. *If E is an AM-space with strong order unit and the norm on F is order continuous, then $\mathscr{L}^r(E,F) \diagup \mathscr{W}_L^r(E,F)$ and $\mathscr{L}^r(E,F) \diagup \mathscr{W}_M^r(E,F)$ are Dedekind complete.*

Proof. It is easy to see that $\mathscr{W}_M^r(E,F)$ and $\mathscr{W}_L^r(E,F)$ are bands in $\mathscr{L}^r(E,F)$ under the assumptions. Therefore, the quotient map is order continuous ([3], Theorem 7.9). Since F is Dedekind complete $\mathscr{L}^r(E,F)$ is Dedekind complete so does our quotients. □

From this point on, we assume that $\mathscr{L}^r(E,F)$ is a Banach lattice whenever we mention about quotients of it.
A Riesz space E is said to be a lattice ordered algebra whenever E is also an associative algebra such that the product of positive elements is positive. In addition, if E is a Banach lattice, then E is called a Banach lattice algebra provided that $\|xy\| \leq \|x\| \|y\|$ holds for all $x, y \in E^+$. It is well known that the composition of two positive operators is positive. Therefore, $\mathscr{L}^r(E,E)$ (briefly $\mathscr{L}^r(E)$) is closed under composition. This makes $\mathscr{L}^r(E)$ with the regular norm into a Banach lattice algebra whenever E is Dedekind complete. In this case,the identity of $\mathscr{L}^r(E)$ has the norm one. Moreover, if a linear subset $\mathscr{U}$ of the space $\mathscr{L}^r(E)$ is two sided ideal, in the sense that for every $S \in \mathscr{U}$ and for every $T \in \mathscr{L}^r(E)$, the compositions ST and TS belong to $\mathscr{U}$, then $\mathscr{L}^r(E) \diagup \mathscr{U}$ is also Banach lattice algebra.
On the contrary, regarding the regular weakly compact and regular compact operators, Example 1.2 in [9] shows that regular L-weakly and regular M-weakly compact operators do not need to be two sided ideals in $\mathscr{L}^r(E)$. In the same paper, it is proven that $\mathscr{W}_M(E) \cap \mathscr{L}^r(E)$ (resp. $\mathscr{W}_L(E) \cap \mathscr{L}^r(E)$) is a two sided ideal in $\mathscr{L}^r(E)$ if and only if E^* (resp. E)has an order continuous norm ([9], Theorem 3.3 and 3.4). Similar results can be given for $\mathscr{W}_M^r(E)$ and $\mathscr{W}_L^r(E)$.

Theorem 2.5. *$\mathscr{W}_M^r(E)$ (resp. $\mathscr{W}_L^r(E)$) is a two sided ideal in $\mathscr{L}^r(E)$ if and only if E^* (resp. E) has an order continuous norm.*

Proof. Proof is the same as with Theorem 3.3 and 3.4 in [9]. □

Since $\mathscr{W}_L^r(E)$ and $\mathscr{W}_M^r(E)$ are norm closed subspaces of $\mathscr{L}^r(E)$, the following result is obvious.

Corollary 2.6. *If E is Dedekind complete and E^* (resp. E) has an order continuous norm then $\mathscr{L}^r(E) \diagup \mathscr{W}_M^r(E)$ (resp. $\mathscr{L}^r(E) \diagup \mathscr{W}_L^r(E)$) with quotient norm is Banach lattice algebra.*

Order continuity properties of our operator classes regarding regular norm is given in [1] as it follows.

Theorem 2.7 ([1], Theorem 3.1 ve 3.2)**.** *The regular norm on* $\mathscr{W}_L^r(E,F)$ *(resp.* $\mathscr{W}_M^r(E,F)$*) is order continuous if and only if* E^* *(resp.* F*) has an order continuous norm.*

Order continuity is a hereditary property for the quotients by the closed ideals ([5], Example 1.5). But the regular norm is not order continuous in general. In the context of the order continuity of regular norm, some results were given by Z.Chen et all in [10].

Theorem 2.8 ([10], Proposition 1)**.** *If the regular norm on* $\mathscr{L}^r(E,F)$ *is order continuous, then the norms both on* E^* *and* F *are order continuous.*

Theorem 2.9 ([10], Theorem 2)**.** *The following statements are equivalent.*

1. $\mathscr{L}^r(E,F)$ is a vector lattice and the regular norm on $\mathscr{L}^r(E,F)$ is order continuous.
2. Every positive operator $T: E \to F$ is L- and M-weakly compact.

As a consequence, we obtain the following:

Corollary 2.10. *The regular norm on* $\mathscr{L}^r(E,F)$ *is order continuous if and only if* $\mathscr{L}^r(E,F)\diagup\mathscr{W}_M^r(E,F)=\{0\}$ *and* $\mathscr{L}^r(E,F)\diagup\mathscr{W}_L^r(E,F)=\{0\}$.

Note that E has order continuous norm if and only if E is σ-Dedekind complete and there does not exist any sublattice of E isomorphic to ℓ_∞ ([4], Corollary.2.4.3).

Theorem 2.11 ([11], Theorem 2)**.** *Let* $E=(E,\tau)$ *be a Dedekind* $\sigma-$*complete Riesz space, let* τ *be locally convex-solid, and let* M *be a* τ*-closed ideal of* E*. If* E *contains a copy of* ℓ_∞*, then* $E\diagup M$ *or* M *contains a lattice copy of* ℓ_∞*.*

Combining Theorem 2.7 and Theorem 2.8 with Theorem 2.11, we obtain the following result:

Corollary 2.12. *If* F *is Dedekind complete, the following statements hold.*

1. If F has order continuous norm, but E^* does not have then $\mathscr{L}^r(E,F)\diagup\mathscr{W}_M^r(E,F)$ also does not have order continuous norm.
2. If E^* has order continuous norm, but F does not have then $\mathscr{L}^r(E,F)\diagup\mathscr{W}_L^r(E,F)$ also does not have order continuous norm.

Proof. If E^* (resp. F) does not have order continuous, then $\mathscr{L}^r(E,F)$ contains a lattice copy of ℓ_∞. Since F (resp. E^*) has order continuous norm, then $\mathscr{W}_M^r(E,F)$ (resp. $\mathscr{W}_L^r(E,F)$) has order continuous norm. Therefore, $\mathscr{L}^r(E,F)\diagup\mathscr{W}_M^r(E,F)$ (resp. $\mathscr{L}^r(E,F)\diagup\mathscr{W}_L^r(E,F)$) does not have order continuous norm. □

3. A Representation of $\mathscr{L}^r(E,F)\diagup\mathscr{W}_L^r(E,F)$

In [12], a representation of the weak Calkin algebra $\mathscr{L}(E)\diagup\mathscr{W}(E)$ where $\mathscr{W}(E)$ denotes the class of weakly compact operators on E is given. Similarly, in this section, we present a representation of the quotient $\mathscr{L}^r(E,F)\diagup\mathscr{W}_L^r(E,F)$.
We consider the operator $R(S): E^{**}\diagup E^a \to F^{**}\diagup F^a$ for every $S\in\mathscr{L}^r(E,F)$ as follows

$$R(S)([x^{**}]) = [S^{**}x^{**}].$$

Thus, we can define the induced map

$$\begin{array}{rcl} R:\mathscr{L}^r(E,F)\diagup\mathscr{W}_L^r(E,F) & \to & \mathscr{L}^r(E^{**}\diagup E^a, F^{**}\diagup F^a) \\ S+\mathscr{W}_L^r(E,F) & \to & R(S). \end{array}$$

In the following proposition, we present some properties of the above operator.

Proposition 3.1. *The following assumptions hold for the map* R*.*

1. *If* $S\in\mathscr{W}_L^r(E,F)$ *then* $R(S)=0$.
2. R *is a positive linear map.*
3. *For* $S\in\mathscr{L}^r(E,F)$ $\|R(S)\|_{F^{**}\diagup F^a}\le\|S^{**}\|_{E^{**}}$, *so* $\|R\|\le 1$.
4. *If* $E=F$ *then* $R([I_E])=I_{E^{**}\diagup E^a}$.
5. *Whenever* ST *is defined as* $R(ST)=R(S)R(T)$.

Proof. (1) Since L-weakly compact operators take values in F^a and are a subclass of weakly compact operators, then $S^{**}(E^{**})\subseteq F^a$ hold for all $S\in\mathscr{W}_L^r(E,F)$.
(2) Let choose $0\le S\in\mathscr{L}^r(E,F)$ and $[x^{**}]\in(E^{**}\diagup E^a)^+$.Then $x^{**}\in(E^{**})^+$ so $S^{**}x^{**}\in(F^{**})^+$. It means that $R([S])[x^{**}]\in(F^{**}\diagup F^a)^+$, i.e. R is positive. The linearity of R is a routine verification.
(3) For $T,S\in\mathscr{L}^r(E,F)$, $x^{**}\in E^{**}$

$$\begin{array}{rcl} \|R(S)[x^{**}]\|_{F^{**}\diagup F^a} & = & dist\left(R(S)[x^{**}],F^a\right)=\inf\{\|R(S)[x^{**}]-y\|: y\in F^a\} \\ & = & \inf\{\|S^{**}x^{**}+z-y\|: z,y\in F^a\} \\ & = & \inf\{\|S^{**}x^{**}-(y-z)\|: z,y\in F^a\} \\ & = & \inf\{\|S^{**}x^{**}-u\|: u\in F^a\} \\ & \le & \inf\{\|S^{**}x^{**}-S^{**}w\|: w\in E^a\} \\ & \le & \|S^{**}\|\inf\{\|x^{**}-w\|: w\in E^a\}=\|S^{**}\|\,\|[x^{**}]\| \end{array}$$

show that $\|R(S)\| \leq \|S^{**}\| = \|S\|$ so $\|R\| \leq 1$.
(4) It follows that for every $x^{**} \in E^{**}$

$$R([I_E])[x^{**}] = \left[(I_E)^{**} x^{**}\right] = [x^{**}] = I_{E^{**}/E^a}[x^{**}].$$

(5) Let E, F, G be Banach lattices and $S \in \mathscr{L}^r(E,F)$, $T \in \mathscr{L}^r(G,E)$. For every $x^{**} \in E^{**}$ we obtain that

$$\begin{aligned} R([ST])[x^{**}] &= \left[(ST)^{**} x^{**}\right] = [S^{**}(T^{**}x^{**})] = [S^{**}(R([T])[x^{**}])] \\ &= R([S])R([T])[x^{**}]. \end{aligned}$$

□

Definition 3.2. *The pair of Banach lattices (E,F) has the invariant modulus property if the equality $|T^*| = |T|^*$ holds for every $T \in \mathscr{L}^r(E,F)$ for which $|T|$ exists in $\mathscr{L}^r(E,F)$.*

Theorem 3.3. *If (E,F) and (F^*,E^*) have invariant modulus property, then R is a Riesz homomorphism.*

Proof. For $S \in \mathscr{L}^r(E,F)$ and $x^{**} \in E^{**}$ we obtain

$$R(|[S]|)[x^{**}] = R([|S|])[x^{**}] = \left[|S|^{**} x^{**}\right] = [|S^{**}|x^{**}].$$

On the other hand, by the help Riesz-Kantorovich formulae we get

$$\begin{aligned} R(|[S]|)[x^{**}] &= [\sup\{|S^{**}y^{**}| : |y^{**}| \leq x^{**}\}] = [\sup\{|R([S])[y^{**}]| : |y^{**}| \leq x^{**}\}] \\ &= \sup\{|R([S])[y^{**}]| : [|y^{**}|] \leq [x^{**}]\} = \sup\{|R([S])[y^{**}]| : |[y^{**}]| \leq [x^{**}]\} \\ &= |R([S])|[x^{**}]. \end{aligned}$$

□

Analogously to Gantmacher's theorem (see [3], Theorem 17.2) can be modified for L-weakly compact operators as follows:

Lemma 3.4. *If E^* has Schur property and $T \in \mathscr{L}(E,F)$, then the following statements are equivalent.*

1. $T \in \mathscr{W}_L(E,F)$,
2. $T^{**}(E^{**}) \subseteq F^a$,
3. $T^* : \left((F^a)^*, \sigma\left((F^a)^*, F^a\right)\right) \to (E^*, \sigma(E^*, E^{**}))$,
4. $T^* \in \mathscr{W}_M(E,F)$.

Proof. $(1 \Rightarrow 2 \Rightarrow 3)$ is clear from Theorem 17.2 in [3].
$(3 \Rightarrow 4)$ If $(f_n)_{n\in\mathbb{N}}$ is a norm bounded disjoint sequence in F^* so in $(F^a)^*$, then we have $f_n \to 0$ in the topology $\sigma\left((F^a)^*, F^a\right)$ since F^a has order continuous norm (see [4], Corollary 2.4.3). Thus, $(T^* f_n)$ is $\sigma(E^*, E^{**})$-null sequence by the hypothesis. Hence $\|T^* f_n\| \to 0$ since E^* has Schur property. This show that T^* is an M-weakly compact operator.
$(4 \Rightarrow 1)$ is clear from Theorem 18.13 in [3]. □

Lemma 3.5. *If F^a has Schur property and $T \in \mathscr{L}(E,F)$ then the following statements are equivalent.*

1. $T \in \mathscr{W}_L(E,F)$,
2. $T^{**}(E^{**}) \subseteq F^a$,
3. $T^* \in \mathscr{W}_M(E,F)$.

Proof. $(1 \Rightarrow 2)$ and $(3 \Rightarrow 1)$ are obvious. $(2 \Rightarrow 3)$ If $T^{**}(E^{**}) \subseteq F^a$ hold then the operator T is weakly compact (see [3], Theorem 17.2). Hence $T^{**}(B_{E^{**}})$, where $B_{E^{**}}$ is unit ball of E^{**}, is relatively weakly compact subset of F^a, so is L-weakly compact from Corollary 3.6.8 in [4] since F^a has Schur property. □

Theorem 3.6. *Suppose that E^* or F has Schur property. Then, $S \in \mathscr{W}_L^r(E,F)$ if and only if $R(S) = 0$.*

Proof. Necessity has been proved in Proposition 3.1. Suppose that $R(S) = 0$. The equality $R(S) = 0$ implies $S^{**}(E^{**}) \subseteq F^a$. Hence, $S \in \mathscr{W}_L^r(E,F)$ from Lemma 3.4 and Lemma 3.5 which we are looking for. □

Hence, if E^* and F has Schur property with order continuous norm, then $S + \mathscr{W}_L^r(E,F) \to R(S)$ provides a representation of the quotient $\mathscr{L}^r(E,F) / \mathscr{W}_L^r(E,F)$, and its image $\{R(S) : S \in \mathscr{L}^r(E)\}$ whenever $E = F$ is a subalgebra of $\mathscr{L}^r(E^{**}/E^a)$ containing the identity.

References

[1] E. Bayram, A. W. Wickstead, *Banach lattices of L-weakly and M-weakly compact operators*, Arch. Math. (Basel) **108**(2017), 293–299.
[2] H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer-Verlag, Berlin-Heidelberg-New York, 1974.
[3] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Pure and Applied Mathematics, vol. 119, Academic Press, Inc., Orlando, FL, 1985.
[4] P. Meyer-Nieberg, *Banach Lattices*, Universitext, Springer-Verlag, Berlin, 1991.
[5] W. Wnuk, *Banach Lattices with Order Continuous Norms*, Advenced Topics in Mathematics, Polish Scientific Publishers PWN, Warsaw, 1999.
[6] A. W. Wickstead, *Regular operators between Banach lattices*, Positivity, TrendsMath., Birkhäuser, Basel, 2007.
[7] W. A. J. Luxemburg, A.C. Zaanen, *Riesz Spaces I*, North-Holland Publ., Amsterdam, 1971.
[8] Z. L. Chen and A. W. Wickstead, *L-weakly and M-weakly compact operators*, Indag. Math. (N.S.), **10**(3) (1999), 321-336.
[9] E. Bayram , W. Wnuk, *Some Algebra Ideals Of Regular Operators*, Comment. Math. **532** (2013), 127-133.
[10] Z. L. Chen, Y. Feng, J.X. Chen, *The Order Continuity of the Regular Norm on Regular Operator Spaces*, Abstr. Appl. Anal., (2013), Article ID 183786, 7 pages.
[11] M. Wojtowicz, *Copies of ℓ_∞ in quotients of locally solid Riesz spaces*, Arch. Math. **80**(2003), 294–301.
[12] M. Gonzalez, E. Saksman, H.O. Tylli, *Representing non-weakly compact operators*, Studia Math., **113** (1995), 265-282.