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Asymptotically $\mathcal{J}_\sigma$-Equivalence of Sequences of Sets

Uğur Ulusu[*1], Esra Gülle[2]

Abstract

In this study, we introduce the concepts of Wijsman asymptotically $\mathcal{J}$-invariant equivalence ($W^L_{\mathcal{J}_\sigma}$), Wijsman asymptotically strongly p-invariant equivalence ($[W^L_{V_\sigma}]_p$) and Wijsman asymptotically $\mathcal{J}^*$-invariant equivalence ($W^L_{\mathcal{J}^*_\sigma}$). Also, we investigate the relationships among the concepts of Wijsman asymptotically invariant equivalence, Wijsman asymptotically invariant statistical equivalence, $W^L_{\mathcal{J}_\sigma}$, $[W^L_{V_\sigma}]_p$ and $W^L_{\mathcal{J}^*_\sigma}$.

Keywords: asymptotically equivalence, $\mathcal{J}$-convergence, invariant convergence, sequences of sets, Wijsman convergence

1. INTRODUCTION AND BACKGROUND

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ψ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

* $\psi(s) \geq 0$, when the sequence $s = (s_n)$ has $s_n \geq 0$ for all n,
* $\psi(e) = 1$, where $e = (1,1,1,\ldots)$ and
* $\psi(s_{\sigma(n)}) = \psi(s_n)$ for all $s \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, ψ extends the limit functional on c, the space of convergent sequences, in the sense that $\psi(s) = \lim s$ for all $s \in c$.

Mursaleen [4] defined the concept of strongly σ-convergent sequence. Then, using a positive real number p, Savaş [9] generalized the concept of strongly σ-convergent sequence. Following, Savaş and Nuray [10] defined the concept of lacunary σ-statistically convergent sequence.

The idea of $\mathcal{J}$-convergence which is based on the structure of the ideal $\mathcal{J}$ of subsets of the set $\mathbb{N}$ (natural numbers) was introduced by Kostyrko *et al.* [2].

$\mathcal{J} \subseteq 2^{\mathbb{N}}$ which is a family of subsets of $\mathbb{N}$ is called an ideal if it is satisfies following conditions:

* $\emptyset \in \mathcal{J}$,
* For each $X, Y \in \mathcal{J}$ we have $X \cup Y \in \mathcal{J}$,
* For each $X \in \mathcal{J}$ and each $Y \subseteq X$ we have $Y \in \mathcal{J}$.

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Let $\mathcal{J} \subseteq 2^{\mathbb{N}}$ be an ideal. If $\mathbb{N} \notin \mathcal{J}$, then $\mathcal{J}$ is called non-trivial and if $\{n\} \in \mathcal{J}$ for each $n \in \mathbb{N}$, then a non-trivial ideal $\mathcal{J}$ is called admissible.

All ideals considered in this study are assumed to be admissible.

An admissible ideal $\mathcal{J} \subset 2^{\mathbb{N}}$ is said to be satisfy the property (AP) if for every countable family of mutually disjoint sets $\{X_1, X_2, \dots\}$ belonging to $\mathcal{J}$ there exists a countable family of sets $\{Y_1, Y_2, \dots\}$ such that the symmetric difference $X_j \Delta Y_j$ is a finite set for $j \in \mathbb{N}$ and $Y = (\bigcup_{j=1}^{\infty} Y_j) \in \mathcal{J}$.

$\mathcal{F} \subseteq 2^{\mathbb{N}}$ which is a family of subsets of $\mathbb{N}$ is called a filter if it satisfies following conditions:

* $\emptyset \notin \mathcal{F}$,
* For each $X, Y \in \mathcal{F}$ we have $X \cap Y \in \mathcal{F}$,
* For each $X \in \mathcal{F}$ and each $Y \supseteq X$ we have $Y \in \mathcal{F}$.

For any ideal there is a filter $\mathcal{F}(\mathcal{J})$ corresponding with $\mathcal{J}$, given by

$$\mathcal{F}(\mathcal{J}) = \{M \subset \mathbb{N} : (\exists X \in \mathcal{J})(M = \mathbb{N} \backslash X)\}.$$

A sequence $s = (s_n)$ is said to be $\mathcal{J}$-convergent to L if for every $\gamma > 0$, the set

$$B_\gamma = \{n : |s_n - L| \geq \gamma\}$$

belongs to $\mathcal{J}$. It is denoted by $\mathcal{J} - \lim s_n = L$.

A sequence $s = (s_n)$ is said to be $\mathcal{J}^*$-convergent to L if there exists a set $M = \{m_1 < m_2 < \dots < m_n < \dots\} \in \mathcal{F}(\mathcal{J})$ such that

$$\lim_{n \to \infty} s_{m_n} = L.$$

It is denoted by $\mathcal{J}^* - \lim s_n = L$.

Recently, the concepts of σ-uniform density of a subset B of the set $\mathbb{N}$ and corresponding $\mathcal{J}_\sigma$-convergence for real sequences were introduced by Nuray *et al.* [5].

Let $B \subseteq \mathbb{N}$ and

$$s_m = \min_n |B \cap \{\sigma(n), \sigma^2(n), \dots, \sigma^m(n)\}|,$$

$$S_m = \max_n |B \cap \{\sigma(n), \sigma^2(n), \dots, \sigma^m(n)\}|.$$

If the following limits exists

$$\underline{\mathcal{V}}(B) = \lim_{m \to \infty} \frac{s_m}{m}, \qquad \overline{\mathcal{V}}(B) = \lim_{m \to \infty} \frac{S_m}{m},$$

then they are called a lower σ-uniform density and an upper σ-uniform density of the set B, respectively.

If $\underline{\mathcal{V}}(B) = \overline{\mathcal{V}}(B)$, then $\mathcal{V}(B) = \underline{\mathcal{V}}(B) = \overline{\mathcal{V}}(B)$ is called the σ-uniform density of B.

The class of all $B \subseteq \mathbb{N}$ with $\mathcal{V}(B) = 0$ is denoted by $\mathcal{J}_\sigma$.

A sequence $s = (s_n)$ is said to be $\mathcal{J}_\sigma$-convergent to L if for every $\gamma > 0$, the set

$$B_\gamma = \{n : |s_n - L| \geq \gamma\}$$

belongs to $\mathcal{J}_\sigma$, i.e., $\mathcal{V}(B_\gamma) = 0$. It is denoted by $\mathcal{J}_\sigma - \lim s_n = L$.

Let Z be any non-empty set. The function $g : \mathbb{N} \to P(Z)$ is defined by $g(i) = C_i \in P(Z)$ for each $i \in \mathbb{N}$, where $P(Z)$ is power set of Z. The sequence $\{C_i\} = (C_1, C_2, \dots)$, which is the range's elements of g, is said to be set sequences.

The concept of convergence for real sequences has been extended by many researchers to concepts of convergence for set sequences. The one of these such extensions considered in this study is the concept of Wijsman convergence (see, [6, 7]).

Let (Z, d) be a metric space. For any point $z \in Z$ and any non-empty subset C of Z, the distance from z to C is defined by

$$\rho(z, C) = \inf_{c \in C} d(z, c).$$

Throughout the study, we take (Z, d) be a metric space and C, C_i, D_i be any non-empty closed subsets of Z.

A sequence $\{C_i\}$ is said to be Wijsman convergent to C if for each $z \in Z$,

$$\lim_{i \to \infty} \rho(z, C_i) = \rho(z, C).$$

A sequence $\{C_i\}$ is said to be Wijsman invariant convergent to C if for each $z \in Z$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \rho(z, C_{\sigma^i(m)}) = \rho(z, C),$$

uniformly in $m = 1,2,\ldots$.

Let $0 < p < \infty$. A sequence $\{C_i\}$ is said to be Wijsman strongly p-invariant convergent to C if for each $z \in Z$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |\rho(z, C_{\sigma^i(m)}) - \rho(z, C)|^p = 0,$$

uniformly in m.

A sequence $\{C_i\}$ is said to be Wijsman invariant statistical convergent to C if for every $\gamma > 0$ and each $z \in Z$

$$\lim_{n\to\infty} \frac{1}{n} |\{i \le n: |\rho(z, C_{\sigma^i(m)}) - \rho(z, C)| \ge \gamma\}| = 0,$$

uniformly in m.

In [3], Marouf introduced the concept of asymptotically equivalence for real sequences. Then, this concept has been development by several researchers.

Two nonnegative sequences $s = (s_n)$ and $t = (t_n)$ are said to be asymptotically equivalent if

$$\lim_{n\to\infty} \frac{s_n}{t_n} = 1.$$

It is denoted by $s \sim t$.

The concept of asymptotically equivalence for real sequences has been firstly extended by Ulusu and Nuray [11] to concept of asymptotically equivalence (Wijsman sense) for set sequences. Similar concepts can be seen in [1, 8].

For any non-empty closed subsets $C_i, D_i \subseteq Z$ such that $\rho(z, C_i) > 0$ and $\rho(z, D_i) > 0$ for each $z \in Z$, the sequences $\{C_i\}$ and $\{D_i\}$ are said to be asymptotically equivalent (Wijsman sense) if for each $z \in Z$,

$$\lim_{i\to\infty} \frac{\rho(z, C_i)}{\rho(z, D_i)} = 1.$$

It is denoted by $C_i \sim D_i$.

As an example, consider the following sequences:

$$C_i = \{(x, y): x^2 + y^2 + 2ix = 0\},$$
$$D_i = \{(x, y): x^2 + y^2 - 2ix = 0\}.$$

Since

$$\lim_{i\to\infty} \frac{\rho(z, C_i)}{\rho(z, D_i)} = 1,$$

the sequences $\{C_i\}$ and $\{D_i\}$ are asymptotically equivalent (Wijsman sense), i.e., $C_i \sim D_i$.

The term $\rho(z; C_i, D_i)$ is defined as follows:

$$\rho(z; C_i, D_i) = \begin{cases} \dfrac{\rho(z, C_i)}{\rho(z, D_i)} &, \quad z \notin C_i \cup D_i \\ L &, \quad z \in C_i \cup D_i. \end{cases}$$

Two sequences $\{C_i\}$ and $\{D_i\}$ are said to be asymptotically invariant equivalent (Wijsman sense) of multiple L if for each $z \in Z$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \rho(z; C_{\sigma^i(m)}, D_{\sigma^i(m)}) = L,$$

uniformly in m. It is denoted by $C_i \overset{WV_\sigma^L}{\sim} D_i$.

Two sequences $\{C_i\}$ and $\{D_i\}$ are said to be asymptotically invariant statistical equivalent (Wijsman sense) of multiple L if for every $\gamma > 0$ and each $z \in Z$

$$\lim_{n\to\infty} \frac{1}{n} |\{i \le n: |\rho(z; C_{\sigma^i(m)}, D_{\sigma^i(m)}) - L| \ge \gamma\}| = 0,$$

uniformly in m. It is denoted by $C_i \overset{WS_\sigma^L}{\sim} D_i$.

The set of all asymptotically invariant statistical equivalent (Wijsman sense) sequences is denoted by WS_σ^L.

From now on, for short, we use $\rho_z(C)$, $\rho_z(C_i)$ and $\rho_z(C_i, D_i)$ instead of $\rho(z, C)$, $\rho(z, C_i)$ and $\rho(z; C_i, D_i)$, respectively.

2. MAIN RESULTS

Definition 2.1 Two sequences $\{C_i\}$ and $\{D_i\}$ are said to be Wijsman asymptotically $\mathcal{J}$-invariant equivalent or Wijsman asymptotically $\mathcal{J}_\sigma$-

equivalent of multiple L if for every $\gamma > 0$ and each $z \in Z$, the set

$$B_{\gamma,z}^{\sim} := \{i : |\rho_z(C_i, D_i) - L| \geq \gamma\}$$

belongs to $\mathcal{J}_\sigma$, that is, $\mathcal{V}(B_{\gamma,z}^{\sim}) = 0$. In this case, we write $C_i \overset{W_{\mathcal{J}_\sigma}^L}{\sim} D_i$ and if $L = 1$, simply Wijsman asymptotically $\mathcal{J}$-invariant equivalent.

The set of all Wijsman asymptotically $\mathcal{J}_\sigma$-equivalent sequences will be denoted by $W_{\mathcal{J}_\sigma}^L$.

Theorem 2.2 Let $\rho_z(C_i) = \mathcal{O}(\rho_z(D_i))$. If $C_i \overset{W_{\mathcal{J}_\sigma}^L}{\sim} D_i$, then $C_i \overset{WV_\sigma^L}{\sim} D_i$.

Proof. Let $m, n \in \mathbb{N}$ are arbitrary and $\gamma > 0$ is given. Now, we calculate

$$T_z(m,n) := \left| \frac{1}{m} \sum_{i=1}^{m} \rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L \right|.$$

Then, for each $z \in Z$ we have

$$T_z(m,n) \leq T_z^1(m,n) + T_z^2(m,n)$$

where

$$T_z^1(m,n) := \frac{1}{m} \sum_{\substack{i=1 \\ |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L| \geq \gamma}}^{m} |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L|$$

and

$$T_z^2(m,n) := \frac{1}{m} \sum_{\substack{i=1 \\ |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L| < \gamma}}^{m} |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L|.$$

For each $z \in Z$ and every $n = 1,2,\ldots$, it is obvious that $T_z^2(m,n) < \gamma$. Since $\rho_z(C_i) = \mathcal{O}(\rho_z(D_i))$, there exists an $R > 0$ such that

$$|\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L| \leq R$$

for each $z \in Z$ $(i = 1,2,\ldots;\ n = 1,2,\ldots)$. So, this implies that

$$T_z^1(m,n) \leq \frac{R}{m} |\{1 \leq i \leq m : |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L| \geq \gamma\}|$$

$$\leq R \frac{\max_n |\{1 \leq i \leq m : |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L| \geq \gamma\}|}{m}$$

$$= R \frac{S_m}{m}.$$

Then, due to our hypothesis, $C_i \overset{WV_\sigma^L}{\sim} D_i$.

Definition 2.3 Let $0 < p < \infty$. Two sequences $\{C_i\}$ and $\{D_i\}$ are said to be Wijsman asymptotically strongly p-invariant equivalent of multiple L if for each $z \in Z$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L|^p = 0,$$

uniformly in m. In this case, we write $C_i \overset{[W_{V_\sigma}^L]_p}{\sim} D_i$ and if $L = 1$, simply Wijsman asymptotically strongly p-invariant equivalent.

Theorem 2.4 If $C_i \overset{[W_{V_\sigma}^L]_p}{\sim} D_i$, then $C_i \overset{W_{\mathcal{J}_\sigma}^L}{\sim} D_i$.

Proof. Let $C_i \overset{[W_{V_\sigma}^L]_p}{\sim} D_i$ and $\gamma > 0$ is given. For each $z \in Z$, we can write

$$\sum_{i=1}^{m} |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L|^p$$

$$\geq \sum_{\substack{i=1 \\ |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L| \geq \gamma}}^{m} |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L|^p$$

$$\geq \gamma^p |\{1 \leq i \leq m : |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L| \geq \gamma\}|$$

$$\geq \gamma^p \max_n |\{1 \leq i \leq m : |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L| \geq \gamma\}|$$

and so

$$\frac{1}{m} \sum_{i=1}^{m} |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L|^p$$

$$\geq \gamma^p \frac{\max_n |\{1 \leq i \leq m : |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L| \geq \gamma\}|}{m}$$

$$= \gamma^p \frac{S_m}{m},$$

for all n. This implies that $\lim_{m \to \infty} \frac{S_m}{m} = 0$ and consequently $C_i \overset{W_{\mathcal{J}_\sigma}^L}{\sim} D_i$.

Theorem 2.5 Let $\rho_z(C_i) = \mathcal{O}(\rho_z(D_i))$. If $C_i \overset{W_{\mathcal{J}_\sigma}^L}{\sim} D_i$, then $C_i \overset{[W_{V_\sigma}^L]_p}{\sim} D_i$.

Proof. Let $\rho_z(C_i) = \mathcal{O}(\rho_z(D_i))$ and $\gamma > 0$ is given. Also, we suppose that $C_i \overset{W_{\mathcal{J}_\sigma}^L}{\sim} D_i$. By assumption, we have $\mathcal{V}(B_{\gamma,z}^{\sim}) = 0$. Since

$\rho_z(C_i) = \mathcal{O}(\rho_z(D_i))$, there exists an $R > 0$ such that

$$|\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L| \leq R$$

for each $z \in Z$ $(i = 1,2,\ldots;\ n = 1,2,\ldots)$.

Then, for each $z \in Z$ we get

$$\frac{1}{m}\sum_{i=1}^{m} |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L|^p$$

$$= \frac{1}{m} \sum_{\substack{i=1 \\ |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L| \geq \gamma}}^{m} |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L|^p$$

$$+ \frac{1}{m} \sum_{\substack{i=1 \\ |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L| < \gamma}}^{m} |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L|^p$$

$$\leq R \frac{\max_n \left|\{i \leq m : \left|\rho_z\left(C_{\sigma^i(n)}, D_{\sigma^i(n)}\right) - L\right| \geq \gamma\}\right|}{m} + \gamma^p$$

$$\leq R \frac{S_m}{m} + \gamma^p,$$

for all n. Hence, for each $z \in Z$ we have

$$\lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} |\rho_z(C_{\sigma^i(n)}, D_{\sigma^i(n)}) - L|^p = 0.$$

This completes the proof.

Considering the Theorem 2.4 and Theorem 2.5 together we can give the following corollary:

Corollary 2.6 Let $\rho_z(C_i) = \mathcal{O}(\rho_z(D_i))$. Then, $C_i \overset{W_{\mathcal{J}_\sigma}^L}{\sim} D_i$ if and only if $C_i \overset{[W V_\sigma^L]_p}{\sim} D_i$.

Now, without proof, we shall state a theorem that gives a relation between $W_{\mathcal{J}_\sigma}^L$ and WS_σ^L.

Theorem 2.7 $C_i \overset{W_{\mathcal{J}_\sigma}^L}{\sim} D_i \Leftrightarrow C_i \overset{WS_\sigma^L}{\sim} D_i$.

Definition 2.8 Two sequences $\{C_i\}$ and $\{D_i\}$ are said to be Wijsman asymptotically $\mathcal{J}^*$-invariant equivalent or Wijsman asymptotically $\mathcal{J}_\sigma^*$-equivalent of multiple L if and only if there exists a set $M = \{m_1 < m_2 <\ldots< m_i <\ldots\} \in \mathcal{F}(\mathcal{J}_\sigma)$ such that for each $z \in Z$,

$$\lim_{i\to\infty} \rho_z(C_{m_i}, D_{m_i}) = L.$$

In this case, we write $C_i \overset{W_{\mathcal{J}_\sigma^*}^L}{\sim} D_i$ and if $L = 1$, simply Wijsman asymptotically $\mathcal{J}_\sigma^*$-equivalent.

Theorem 2.9 If $C_i \overset{W_{\mathcal{J}_\sigma^*}^L}{\sim} D_i$, then $C_i \overset{W_{\mathcal{J}_\sigma}^L}{\sim} D_i$.

Proof. Let $C_i \overset{W_{\mathcal{J}_\sigma^*}^L}{\sim} D_i$. Then, there exists a set $H \in \mathcal{J}_\sigma$ such that for $M = \mathbb{N}\backslash H = \{m_1 < m_2 <\ldots< m_i <\ldots\}$ and each $z \in Z$,

$$\lim_{i\to\infty} \rho_z\left(C_{m_i}, D_{m_i}\right) = L. \quad (2.1)$$

Given $\gamma > 0$. By (2.1), there exists an $i_0 \in \mathbb{N}$ such that

$$|\rho_z(C_{m_i}, D_{m_i}) - L| < \gamma,$$

for each $i > i_0$. Hence, for every $\gamma > 0$ and each $z \in Z$ it is obvious that

$$\{i \in \mathbb{N} : |\rho_z(C_i, D_i) - L| \geq \gamma\} \subset H \cup \{m_1 < m_2 <\ldots< m_{i_0}\}. \quad (2.2)$$

Since $\mathcal{J}_\sigma$ is admissible, the set on the right-hand side of (2.2) belongs to $\mathcal{J}_\sigma$. Therefore, $C_i \overset{W_{\mathcal{J}_\sigma}^L}{\sim} D_i$.

The converse of Theorem 2.9 holds if $\mathcal{J}_\sigma$ has property (AP).

Theorem 2.10 Let $\mathcal{J}_\sigma$ has property (AP). If $C_i \overset{W_{\mathcal{J}_\sigma}^L}{\sim} D_i$, then $C_i \overset{W_{\mathcal{J}_\sigma^*}^L}{\sim} D_i$.

Proof. Suppose that $\mathcal{J}_\sigma$ satisfies condition (AP). Let $C_i \overset{W_{\mathcal{J}_\sigma}^L}{\sim} D_i$. Then, for every $\gamma > 0$ and each $z \in Z$ we have

$$\{i : |\rho_z(C_i, D_i) - L| \geq \gamma\} \in \mathcal{J}_\sigma.$$

Put

$$X_1 = \{i : |\rho_z(C_i, D_i) - L| \geq 1\}$$

and

$$X_n = \{i : \frac{1}{n} \leq |\rho_z(C_i, D_i) - L| < \frac{1}{n-1}\},$$

for $n \geq 2$ $(n \in \mathbb{N})$. Obviously, $X_i \cap X_j = \emptyset$ for each $z \in Z$ and $i \neq j$. By condition (AP), there exists a sequence of $\{Y_n\}_{n\in\mathbb{N}}$ such that $X_j \Delta Y_j$ are finite sets for $j \in \mathbb{N}$ and $Y = (\bigcup_{i=1}^{\infty} Y_j) \in \mathcal{J}_\sigma$. It is enough to prove that for $G = \mathbb{N} \backslash Y$ and each $z \in Z$, we have

$$\lim_{\substack{i\to\infty \\ i\in G}} \rho_z(C_i, D_i) = L. \tag{2.3}$$

Let $\delta > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \delta$. Hence, for each $z \in Z$

$$\{i : |\rho_z(C_i, D_i) - L| \geq \delta\} \subset \bigcup_{j=1}^{n+1} X_j.$$

Since $X_j \Delta Y_j$ $(j = 1,2,\ldots,n+1)$ are finite sets, there exists an $i_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{n+1} Y_j\right) \cap \{i : i > i_0\} = \left(\bigcup_{j=1}^{n+1} X_j\right) \cap \{i : i > i_0\}. \tag{2.4}$$

If $i > i_0$ and $i \notin Y$, then $i \notin \bigcup_{j=1}^{n+1} Y_j$ and by (2.4), $i \notin \bigcup_{j=1}^{n+1} X_j$. But then, for each $z \in Z$ we get

$$|\rho_z(C_i, D_i) - L| < \frac{1}{n+1} < \delta$$

and so (2.3) holds. Consequently, we have $C_i \overset{W_{\mathcal{J}_\sigma^*}^L}{\sim} D_i$.

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