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# A Unified Family of Generalized $q$-Hermite Apostol Type Polynomials and its Applications

Subuhi Khan[1]* and Tabinda Nahid[1]

**Abstract**
The intended objective of this paper is to introduce a new class of generalized $q$-Hermite based Apostol type polynomials by combining the $q$-Hermite polynomials and a unified family of $q$-Apostol-type polynomials. The generating function, series definition and several explicit representations for these polynomials are established. The $q$-Hermite-Apostol Bernoulli, $q$-Hermite-Apostol Euler and $q$-Hermite-Apostol Genocchi polynomials are studied as special members of this family and corresponding relations for these polynomials are obtained.

**Keywords:** $q$-Hermite polynomials, Generalized $q$-Apostol type polynomials, Generalized $q$-Hermite Apostol type polynomials, Explicit representation
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[1] *Department of Mathematics, Aligarh Muslim University, Aligarh, India*
***Corresponding author**: subuhi2006@gmail.com
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## 1. Introduction and preliminaries

The $q$-calculus has been extensively studied for a long time by many mathematicians, physicists and engineers. The $q$-calculus is a generalization of many subjects, like the hypergeometric series, complex analysis and particle physics. The $q$-analogues of many orthogonal polynomials and functions assume a very pleasant form reminding directly of their classical counterparts. The $q$-calculus is mostly being used by physicists at a high level. In short, $q$-calculus is quite a popular subject today.

Throughout the present paper, $\mathbb{C}$ indicates the set of complex numbers, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_0$ indicates the set of non-negative integers. Further, the variable $q \in \mathbb{C}$ such that $|q| < 1$. The following $q$-standard notations and definitions are taken from [1].

The $q$-analogue of the shifted factorial $(a)_n$ is defined by

$$(a;q)_0 = 1, (a;q)_n = \prod_{m=0}^{n-1}(1-q^m a),\ n \in \mathbb{N}.$$

The $q$-analogues of a complex number $a$ and of the factorial function are defined by

$$[a]_q = \frac{1-q^a}{1-q},\ \ q \in \mathbb{C}-\{1\};\quad a \in \mathbb{C},$$

$$[n]_q! = \prod_{m=1}^{n}[m]_q = \frac{(q;q)_n}{(1-q)^n},\ q \neq 1;\ n \in \mathbb{N},\ \ [0]_q! = 1,\ q \in \mathbb{C}.$$

The $q$-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \quad k = 0,1,\ldots,n.$$

The $q$-exponential function is defined as:

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{((1-q)x;q)_\infty}, \quad |x| < |1-q|^{-1}. \tag{1.1}$$

The $q$-Hermite polynomials are special or limiting cases of the orthogonal polynomials as they contain no parameter other than $q$ and appears to be at the bottom of a hierarchy of the classical $q$-orthogonal polynomials [2]. The $q$-Hermite polynomials constitute a 1-parameter family of orthogonal polynomials, which for $q = 1$ reduce to the well known Hermite polynomials. We recall that the $q$-Hermite polynomials $H_{n,q}(x)$ is defined by the following generating function [3]:

$$F_q(x,t) := F_q(t)e_q(xt) = \sum_{n=0}^{\infty} H_{n,q}(x)\frac{t^n}{[n]_q!}, \tag{1.2}$$

$$F_q(t) := \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2} \frac{t^{2n}}{[2n]_q!!}, \quad [2n]_q!! = [2n]_q[2n-2]_q....[2]_q.$$

$$D_{q,x}H_{n,q}(x) = [n]_q H_{n-1,q}(x).$$

Recently, many mathematicians studied the unification of the Bernoulli and Euler polynomials. Luo and Srivastava [4, 5] introduced the generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha$. Further, the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x)$ of order $\alpha$ and the generalized Apostol-Genochhi polynomials $G_n^{(\alpha)}(x)$ of order $\alpha$ are investigated by Luo [6, 7]. Thereafter, in 2014 Ernst [8] defined the $q$-analogues of the generalized Apostol type polynomials.

The generalized $q$-Apostol-Bernoulli polynomials $B_{n,q,\lambda}^{(\alpha)}(x)$ of order $\alpha \in \mathbb{N}_0$ are defined by the following generating function [8]:

$$\left(\frac{t}{\lambda e_q(t)-1}\right)^{\alpha} e_q(xt) = \sum_{n=0}^{\infty} B_{n,q,\lambda}^{(\alpha)}(x)\frac{t^n}{[n]_q!}. \tag{1.3}$$

The generalized $q$-Apostol-Euler polynomials $E_{n,q,\lambda}^{(\alpha)}(x)$ of order $\alpha \in \mathbb{N}_0$ are defined by the following generating function [8]:

$$\left(\frac{2}{\lambda e_q(t)+1}\right)^{\alpha} e_q(xt) = \sum_{n=0}^{\infty} E_{n,q,\lambda}^{(\alpha)}(x)\frac{t^n}{[n]_q!}. \tag{1.4}$$

The generalized $q$-Apostol-Genocchi polynomials $G_{n,q,\lambda}^{(\alpha)}(x)$ of order $\alpha \in \mathbb{N}_0$ are defined by the following generating function [8]:

$$\left(\frac{2t}{\lambda e_q(t)+1}\right)^{\alpha} e_q(xt) = \sum_{n=0}^{\infty} G_{n,q,\lambda}^{(\alpha)}(x)\frac{t^n}{[n]_q!}. \tag{1.5}$$

In view of equations (1.3)-(1.5), the generalized $q$-Apostol type polynomials $\mathscr{P}_{n,q,\beta}^{(\alpha)}(x;k,a,b)$ $(\alpha \in \mathbb{N}_0, \lambda, a, b \in \mathbb{C})$ of order $\alpha$ are defined by the following generating function:

$$\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} e_q(xt) = \sum_{n=0}^{\infty} \mathscr{P}_{n,q,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{[n]_q!}, \tag{1.6}$$

where $\mathscr{P}_{n,q,\beta}^{(\alpha)}(k,a,b) = \mathscr{P}_{n,q,\beta}^{(\alpha)}(0;k,a,b)$ are the $q$-Apostol type numbers of order $\alpha$.

If we take the limit $q \to 1$, the generalized $q$-Apostol type polynomials defined by equation (1.6) reduces to the unified Apostol type polynomials [9]. In fact, the following special cases hold:

$$\lim_{q\to 1} \mathscr{P}_{n,q,\lambda}^{(\alpha)}(x;1,1,1) = B_{n,\lambda}^{(\alpha)}(x),$$

$$\lim_{q\to 1}\mathscr{P}^{(\alpha)}_{n,q,\lambda}(x;0,-1,1)=E^{(\alpha)}_{n,\lambda}(x),$$

$$\lim_{q\to 1}\mathscr{P}^{(\alpha)}_{n,q,\frac{\lambda}{2}}(x;1,-1/2,1)=G^{(\alpha)}_{n,\lambda}(x),$$

where $B^{(\alpha)}_{n,\lambda}(x),E^{(\alpha)}_{n,\lambda}(x)$ and $G^{(\alpha)}_{n,\lambda}(x)$ are the generalized forms of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials.

In the current article, the $q$-Hermite-Apostol type polynomials are introduced and their explicit relations are proved. The corresponding results for the $q$-Hermite-Apostol Bernoulli, $q$-Hermite-Apostol Euler and $q$-Hermite-Apostol Genocchi polynomials are established.

## 2. Generalized $q$-Hermite Apostol type polynomials

In this section, a new hybrid class of the generalized $q$-Hermite-Apostol type polynomials (GqHATyP), denoted by ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ is introduced by convoluting the $q$-Hermite polynomials and generalized $q$-Apostol type polynomials. In order to establish the generating function for the these polynomials, the following result is proved:

**Theorem 2.1.** *The following generating function for the generalized q-Hermite based Apostol type polynomials* ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)(\alpha\in\mathbb{N}_0,\lambda,a,b\in\mathbb{C})$ *holds true:*

$$\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha}F_q(t)e_q(xt)=\sum_{n=0}^{\infty}{}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\frac{t^n}{[n]_q!},\tag{2.1}$$

*Proof.* Expanding the exponential function $e_q(xt)$ and then replacing the powers of $x,i.e.\ x^0;x^1;x^2;\cdots;x^n$ by the correlating q-Hermite polynomials $H_{0,q}(x);H_{1,q}(x);\cdots;H_{n,q}(x)$ in the l.h.s. of equation (1.6) and after summing up the terms of the resultant equation and denoting the resultant GqHATyP in the r.h.s. by ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$, assertion (2.1) is proved. □

Taking $x=0$ in equation (2.1), we get

$${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(k,a,b)={}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(0;k,a,b),$$

where ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(k,a,b)$ are the $q$-Hermite Apostol type numbers of order $\alpha$.

Next, the series expansions for the GqHATyP ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ is obtained by proving the following result:

**Theorem 2.2.** *The following series expansions for the generalized q-Hermite based Apostol type polynomials* ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ *hold true:*

$${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q\mathscr{P}^{(\alpha)}_{r,q,\beta}(k,a,b)H_{n-r,q}(x),\tag{2.2}$$

$${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q{}_H\mathscr{P}^{(\alpha)}_{r,q,\beta}(k,a,b)x^{n-r}.\tag{2.3}$$

*Proof.* Utilizing equations (1.2) and (1.6) in the l.h.s. of generating function (2.1) and then using Cauchy-product rule in the l.h.s. of the resultant equation, it follows that

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q\mathscr{P}^{(\alpha)}_{r,q,\beta}(k,a,b)H_{n-r,q}(x)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}{}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\frac{t^n}{[n]_q!}.\tag{2.4}$$

Equating the coefficients of identical powers of $t$ in both sides of equation (2.4), assertion (2.2) follows.

Utilizing equation (1.1) in the l.h.s. of generating function (2.1), it follows that

$$\sum_{n=0}^{\infty}{}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}x^n\frac{t^n}{[n]_q!}\sum_{r=0}^{\infty}{}_H\mathscr{P}^{(\alpha)}_{r,q,\beta}(k,a,b)\frac{t^r}{[n]_r!},$$

which on applying the Cauchy product rule in the r.h.s. and then comparing the coefficients of same powers of $t$ in both sides of resultant equation yields assertion (2.3).

□

| S. No. | $k$; $a$; $b$; $\beta$ | Generating function | Name of the polynomials |
|---|---|---|---|
| **I.** | $k=1$; $a=1$; $b=1$; $\beta=\lambda$ | $\left(\frac{t}{\lambda e_q(t)-1}\right)^{(\alpha)} F_q(t)\ e_q(xt)=\sum_{n=0}^{\infty} {}_HB_{n,q,\lambda}^{(\alpha)}(x)\frac{t^n}{[n]_q!}$ | The generalized $q$-Hermite-Apostol Bernoulli polynomials (GqHABP) |
| **II.** | $k=0$; $a=-1$; $b=1$; $\beta=\lambda$ | $\left(\frac{2}{\lambda e_q(t)+1}\right)^{(\alpha)} F_q(t)e_q(xt)=\sum_{n=0}^{\infty} {}_HE_{n,q,\lambda}^{(\alpha)}(x)\frac{t^n}{[n]_q!}$ | The generalized $q$-Hermite-Apostol Euler polynomials (GqHAEP) |
| **III.** | $k=1$; $a=-1/2$; $b=1$; $\beta=\lambda/2$ | $\left(\frac{2t}{\lambda e_q(t)+1}\right)^{(\alpha)} F_q(t)e_q(xt)=\sum_{n=0}^{\infty} {}_HG_{n,q,\lambda}^{(\alpha)}(x)\frac{t^n}{[n]_q!}$ | The generalized $q$-Hermite-Apostol Genocchi polynomials (GqHAGP) |

**Table 1.** Certain members belonging to the generalized $q$-Hermite-Apostol family

Different members of the generalized $q$-Hermite-Apostol family can be obtained by making suitable selections of the parameters $k,a,b$ and $\beta$ in generating relation (2.1). Some of these members are listed in Table 1.

**Proposition 2.3.** *The following relations for the generalized q-Hermite based Apostol type polynomials* ${}_H\mathscr{P}_{n,q,\beta}^{(\alpha)}(x;k,a,b)$ *holds true:*

$$D_{q,t}e_q(xt)=x\ e_q(xt),$$
$$D_{q,x}\left({}_H\mathscr{P}_{n,q,\beta}^{(\alpha)}(x;k,a,b)\right)=[n]_q\ {}_H\mathscr{P}_{n-1,q,\beta}^{(\alpha)}(x;k,a,b).$$

**Theorem 2.4.** *For each $n\in\mathbb{N}$ and for the q-commuting variables x and u such that $xu=qux$, the generalized q-Hermite based Apostol type polynomials* ${}_H\mathscr{P}_{k,q,\beta}^{(\alpha)}(x;k,a,b)$ *satisfy the following relations:*

$${}_H\mathscr{P}_{n,q,\beta}^{(\alpha+\gamma)}(x;k,a,b)=\sum_{r=0}^{n}\begin{bmatrix} n\\ r\end{bmatrix}_q {}_H\mathscr{P}_{r,q,\beta}^{(\alpha)}(x;k,a,b)\mathscr{P}_{n-r,q,\beta}^{(\gamma)}(k,a,b). \tag{2.5}$$

$${}_H\mathscr{P}_{n,q,\beta}^{(\alpha+\gamma)}(x+u;k,a,b)=\sum_{r=0}^{n}\begin{bmatrix} n\\ r\end{bmatrix}_q {}_H\mathscr{P}_{r,q,\beta}^{(\alpha)}(x;k,a,b)\mathscr{P}_{n-r,q,\beta}^{(\gamma)}(u;k,a,b). \tag{2.6}$$

*Proof.* Replacing $\alpha$ by $\alpha+\gamma$ in definition (2.1), we have

$$\sum_{n=0}^{\infty}{}_H\mathscr{P}_{n,q,\beta}^{(\alpha+\gamma)}(x;k,a,b)\frac{t^n}{[n]_q!}=\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha+\gamma}F_q(t)e_q(xt)$$
$$=\left(\sum_{r=0}^{\infty}{}_H\mathscr{P}_{r,q,\beta}^{(\alpha)}(x;k,a,b)\frac{t^r}{[r]_q!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_{n,q,\beta}^{(\gamma)}(k,a,b)\frac{t^n}{[n]_q!}\right).$$

Using Cauchy-product rule in the r.h.s. of above equation, it follows that

$$\sum_{n=0}^{\infty}{}_H\mathscr{P}_{n,q,\beta}^{(\alpha+\gamma)}(x;k,a,b)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\begin{bmatrix} n\\ r\end{bmatrix}_q {}_H\mathscr{P}_{r,q,\beta}^{(\alpha)}(x;k,a,b)\mathscr{P}_{n-r,q,\beta}^{(\gamma)}(k,a,b)\frac{t^n}{[n]_q!} \tag{2.7}$$

Equating the coefficients of identical powers of $t$ in both sides of equation (2.7), assertion (2.5) follows. Further, replacing $\alpha$ by $\alpha+\gamma$ and $x$ by $x+u$ in Definition 2.1 and proceeding on the same lines of proof as above, assertion (2.6) follows. □

**Theorem 2.5.** *For each $n\in\mathbb{N}$ and for the q-commuting variables x and u such that $xu=qux$, the generalized q-Hermite based Apostol type polynomials* ${}_H\mathscr{P}_{k,q,\beta}^{(\alpha)}(x;k,a,b)$ *satisfy the following relation:*

$$\beta^b\,{}_H\mathscr{P}_{n,q,\beta}^{(\alpha)}(x+1;k,a,b)-a^b\,{}_H\mathscr{P}_{n,q,\beta}^{(\alpha)}(x;k,a,b)=\frac{2^{1-k}[n]_q!}{[n-k]_q!}{}_H\mathscr{P}_{n-k,q,\beta}^{(\alpha-1)}(x;k,a,b). \tag{2.8}$$

| S. No. | Special polynomials | Results |
|---|---|---|
| I. | GqHABP ${}_HB^{(\alpha)}_{n,q,\lambda}(x)$ | ${}_HB^{(\alpha)}_{n,q,\lambda}(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q B^{(\alpha)}_{r,q,\lambda}\ H_{n-r,q}(x)$ <br> ${}_HB_{n,q,\lambda}(\alpha)(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q {}_HB^{(\alpha)}_{r,q,\lambda}\ x^{n-r}$ <br> ${}_HB^{(\alpha+\gamma)}_{n,q,\lambda}(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q {}_HB^{(\alpha)}_{r,q,\lambda}(x)\ B^{(\gamma)}_{n-r,q,\lambda}$ <br> ${}_HB^{(\alpha+\gamma)}_{n,q,\lambda}(x+u)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q {}_HB^{(\alpha)}_{r,q,\lambda}(x)\ B^{(\gamma)}_{n-r,q,\lambda}(u)$ <br> $\beta^b{}_HB^{(\alpha)}_{n,q,\lambda}(x+1)-a^b{}_HB^{(\alpha)}_{n,q,\lambda}(x)=\frac{2^{1-k}[n]_q!}{[n-k]_q!}\ {}_HB^{(\alpha-1)}_{n-k,q,\lambda}(x)$ |
| II. | GqHAEP ${}_HE^{(\alpha)}_{n,q,\lambda}(x)$ | ${}_HE^{(\alpha)}_{n,q,\lambda}(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q E^{(\alpha)}_{r,q,\lambda}\ H_{n-r,q}(x)$ <br> ${}_HE_{n,q,\lambda}(\alpha)(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q {}_HE^{(\alpha)}_{r,q,\lambda}\ x^{n-r}$ <br> ${}_HE^{(\alpha+\gamma)}_{n,q,\lambda}(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q {}_HE^{(\alpha)}_{r,q,\lambda}(x)\ E^{(\gamma)}_{n-r,q,\lambda}$ <br> ${}_HE^{(\alpha+\gamma)}_{n,q,\lambda}(x+u)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q {}_HE^{(\alpha)}_{r,q,\lambda}(x)\ E^{(\gamma)}_{n-r,q,\lambda}(u)$ <br> $\beta^b{}_HE^{(\alpha)}_{n,q,\lambda}(x+1)-a^b{}_HE^{(\alpha)}_{n,q,\lambda}(x)=\frac{2^{1-k}[n]_q!}{[n-k]_q!}\ {}_HE^{(\alpha-1)}_{n-k,q,\lambda}(x)$ |
| II. | GqHAGP ${}_HG^{(\alpha)}_{n,q,\lambda}(x)$ | ${}_HG^{(\alpha)}_{n,q,\lambda}(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q G^{(\alpha)}_{r,q,\lambda}\ H_{n-r,q}(x)$ <br> ${}_HG_{n,q,\lambda}(\alpha)(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q {}_HG^{(\alpha)}_{r,q,\lambda}\ x^{n-r}$ <br> ${}_HG^{(\alpha+\gamma)}_{n,q,\lambda}(x)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q {}_HG^{(\alpha)}_{r,q,\lambda}(x)\ G^{(\gamma)}_{n-r,q,\lambda}$ <br> ${}_HG^{(\alpha+\gamma)}_{n,q,\lambda}(x+u)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_q {}_HG^{(\alpha)}_{r,q,\lambda}(x)\ G^{(\gamma)}_{n-r,q,\lambda}(u)$ <br> $\beta^b{}_HG^{(\alpha)}_{n,q,\lambda}(x+1)-a^b{}_HG^{(\alpha)}_{n,q,\lambda}(x)=\frac{2^{1-k}[n]_q!}{[n-k]_q!}\ {}_HG^{(\alpha-1)}_{n-k,q,\lambda}(x)$ |

**Table 2.** Certain results for the GqHABP ${}_HB^{(\alpha)}_{n,q,\lambda}(x)$, GqHAEP ${}_HE^{(\alpha)}_{n,q,\lambda}(x)$ and GqHAGP ${}_HG^{(\alpha)}_{n,q,\lambda}(x)$

*Proof.* From generating relation (2.1), we have

$$\sum_{n=0}^{\infty}\beta^b{}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x+1;k,a,b)\frac{t^n}{[n]_q!}-\sum_{n=0}^{\infty}a^b{}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\frac{t^n}{[n]_q!}$$
$$=\beta^b\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha}F_q(t)e_q((x+1)t)-a^b\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha}F_q(t)e_q(xt)$$
$$=\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha}F_q(t)e_q(xt)\left(\beta^b e_q(t)-a^b\right)$$
$$\sum_{n=0}^{\infty}\left(\beta^b{}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x+1;k,a,b)-a^b{}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\right)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}2^{1-k}{}_H\mathscr{P}^{(\alpha-1)}_{n,q,\beta}(x;k,a,b)\frac{t^{n+k}}{[n]_q!}.$$

Equating the coefficients of same powers of $t$ in both sides of the above equation, assertion (2.8) follows. □

In view of Table 1, ceratin results for the GqHABP ${}_HB^{(\alpha)}_{n,q,\lambda}(x)$, GqHAEP ${}_HE^{(\alpha)}_{n,q,\lambda}(x)$ and GqHAGP ${}_HG^{(\alpha)}_{n,q,\lambda}(x)$ are established and are given in Table 2.

In the next section, certain explicit representations for the GqHATyP ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ are established.

## 3. Explicit representations

In order to derive the explicit representations for the GqHATyP ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$, we recall the following definition:

**Definition 3.1.** *The generalized q-Stirling numbers $S_q(n,\nu,a,b,\beta)$ of the second kind of order $\nu$ is defined as [10]:*

$$\sum_{n=0}^{\infty} S_q(n,\nu,a,b,\beta)\frac{t^n}{[n]_q!} = \frac{(\beta^b e_q(t)-a^b)^\nu}{[\nu]_q!}.$$

**Theorem 3.2.** *The following explicit formula for the generalized q-Hermite based Apostol type polynomials ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ in terms of the generalized q-Stirling numbers of the second kind $S_q(n,\nu,a,b,\beta)$ holds true:*

$$ {}_H\mathscr{P}^{(\alpha)}_{n-\nu k,q,\beta}(x;k,a,b) = 2^{\nu(k-1)}\frac{[\nu]_q![n-\nu k]_q!}{[n]_q!}\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q {}_H\mathscr{P}^{(\nu-\alpha)}_{l,q,\beta}(x;k,a,b)S_q(n-l,\nu,a,b,\beta). \tag{3.1}$$

*Proof.* From generating relation (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\frac{t^n}{[n]_q!} &= \left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} F_q(t)e_q(xt)\frac{(\beta^b e_q(t)-a^b)^\nu}{[\nu]_q!}\left(\frac{[\nu]_q!}{(\beta^b e_q(t)-a^b)^\nu}\right)\\ &= \frac{[\nu]_q!}{(2^{1-k}t^k)^\nu}\sum_{l=0}^{\infty} {}_H\mathscr{P}^{(\alpha-\nu)}_{l,q,\beta}(x;k,a,b)\frac{t^l}{[l]_q!}\left(\sum_{n=0}^{\infty}S_q(n,\nu,a,b,\beta)\frac{t^n}{[n]_q!}\right).\end{aligned}$$

Applying the Cauchy-product rule on the r.h.s. of the above equation, it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} {}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\frac{t^{n+\nu k}}{[n]_q!} =&[\nu]_q!2^{(k-1)\nu}\sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q {}_H\mathscr{P}^{(\alpha-\nu)}_{l,q,\beta}(x;k,a,b)\right.\\ &\left.\times S_q(n-l,\nu,a,b,\beta)\right\}\frac{t^n}{[n]_q!}.\end{aligned}\tag{3.2}$$

Equating the coefficients of identical powers of $t$ in both sides of equation (3.2) yields assertion (3.1). □

**Theorem 3.3.** *The following explicit relation for the generalized q-Hermite based Apostol type polynomials ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ in terms of the generalized q-Apostol Bernoulli polynomials $B_{n,q,\lambda}(x)$ holds true:*

$$\begin{aligned}{}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b) =&\frac{1}{[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix} n+1\\ r\end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix} r\\ m\end{bmatrix}_q B_{n+1-r,q,\lambda}(x)\right.\\ &\left.-\sum_{m=0}^{n+1}\begin{bmatrix} n+1\\ m\end{bmatrix}_q B_{n+1-m,q,\lambda}(x)\right\}{}_H\mathscr{P}^{(\alpha)}_{m,q,\beta}(k,a,b).\end{aligned}\tag{3.3}$$

*Proof.* Consider generating function (2.1) in the following form:

$$\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} F_q(t)e_q(xt) = \left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} F_q(t)\left(\frac{t}{\lambda e_q(t)-1}\right)\frac{\lambda e_q(t)-1}{t}e_q(xt),$$

which on simplifying and rearranging the terms becomes

$$\begin{aligned}\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} F_q(t)e_q(xt) =& \left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} F_q(t)\left(\frac{t}{\lambda e_q(t)-1}\,e_q(xt)\right)\frac{\lambda}{t}e_q(t)\\ &-\frac{1}{t}\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} F_q(t)\left(\frac{t}{\lambda e_q(t)-1}\,e_q(xt)\right).\end{aligned}\tag{3.4}$$

Using equations (1.3) and (2.1) in equation (3.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)\frac{t^n}{[n]_q!} =& \frac{1}{t}\left(\lambda\sum_{m=0}^{\infty} {}_H\mathscr{P}^{(\alpha)}_{m,q,\beta}(k,a,b)\frac{t^m}{[m]_q!}\sum_{n=0}^{\infty}B_{n,q}(x;\lambda)\frac{t^n}{[n]_q!}\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}\right.\\ &\left.-\sum_{m=0}^{\infty} {}_H\mathscr{P}^{(\alpha)}_{m,q,\beta}(k,a,b)\frac{t^m}{[m]_q!}\sum_{n=0}^{\infty}B_{n,q}(x;\lambda)\frac{t^n}{[n]_q!}\right).\end{aligned}\tag{3.5}$$

| S. No. | Special polynomials | Explicit representations |
|---|---|---|
| I. | GqHABP ${}_HB^{(\alpha)}_{n,q,\lambda}(x)$ | ${}_HB^{(\alpha)}_{n-\nu,q,\lambda}(x)=\frac{[\nu]_q![n-\nu]_q!}{[n]_q!}\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_q\,{}_HB^{\nu-\alpha}_{l,q,\lambda}(x)\;S(n-l,\nu,1,1,\lambda)$ <br> ${}_HB^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix}r\\m\end{bmatrix}_qB_{n+1-r,q,\lambda}(x)-\sum_{m=0}^{n+1}\begin{bmatrix}n+1\\m\end{bmatrix}_qB_{n+1-m,q,\lambda}(x)\right\}{}_HB^{(\alpha)}_{m,q,\lambda}$ <br> ${}_HB^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{2}\sum_{m=0}^{n}\left\{\lambda\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_qE_{n-r,q,\lambda}(x)+E_{n-m,q,\lambda}(x)\right\}\,{}_HB^{(\alpha)}_{m,q,\lambda}$ <br> ${}_HB^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{2[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix}r\\m\end{bmatrix}_qG_{n+1-r,q,\lambda}(x)-\sum_{m=0}^{n+1}\begin{bmatrix}n+1\\m\end{bmatrix}_qG_{n+1-m,q,\lambda}(x)\right\}{}_HB^{(\alpha)}_{m,q,\lambda}$ |
| II. | GqHAEP ${}_HE^{(\alpha)}_{n,q,\lambda}(x)$ | ${}_HE^{(\alpha)}_{n,q,\lambda}(x)=\frac{[\nu]_q!}{2^\nu}\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_q\,{}_HE^{\nu-\alpha}_{l,q,\lambda}(x)\;S(n-l,\nu,-1,1,\lambda)$ <br> ${}_HE^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix}r\\m\end{bmatrix}_qB_{n+1-r,q,\lambda}(x)-\sum_{m=0}^{n+1}\begin{bmatrix}n+1\\m\end{bmatrix}_qB_{n+1-m,q,\lambda}(x)\right\}{}_HE^{(\alpha)}_{m,q,\lambda}$ <br> ${}_HE^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{2}\sum_{m=0}^{n}\left\{\lambda\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_qE_{n-r,q,\lambda}(x)+E_{n-m,q,\lambda}(x)\right\}\,{}_HE^{(\alpha)}_{m,q,\lambda}$ <br> ${}_HE^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{2[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix}r\\m\end{bmatrix}_qG_{n+1-r,q,\lambda}(x)-\sum_{m=0}^{n+1}\begin{bmatrix}n+1\\m\end{bmatrix}_qG_{n+1-m,q,\lambda}(x)\right\}{}_HE^{(\alpha)}_{m,q,\lambda}$ |
| III. | GqHAGP ${}_HG^{(\alpha)}_{n,q,\lambda}(x)$ | ${}_HG^{(\alpha)}_{n-\nu,q,\lambda}(x)=\frac{[\nu]_q![n-\nu]_q!}{[n]_q!}\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_q\,{}_HG^{\nu-\alpha}_{l,q,\lambda}(x)\;S(n-l,\nu,-1/2,1,\lambda/2)$ <br> ${}_HG^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix}r\\m\end{bmatrix}_qB_{n+1-r,q,\lambda}(x)-\sum_{m=0}^{n+1}\begin{bmatrix}n+1\\m\end{bmatrix}_qB_{n+1-m,q,\lambda}(x)\right\}{}_HG^{(\alpha)}_{m,q,\lambda}$ <br> ${}_HG^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{2}\sum_{m=0}^{n}\left\{\lambda\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_qE_{n-r,q,\lambda}(x)+E_{n-m,q,\lambda}(x)\right\}\,{}_HG^{(\alpha)}_{m,q,\lambda}$ <br> ${}_HG^{(\alpha)}_{n,q,\lambda}(x)=\frac{1}{2[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix}r\\m\end{bmatrix}_qG_{n+1-r,q,\lambda}(x)-\sum_{m=0}^{n+1}\begin{bmatrix}n+1\\m\end{bmatrix}_qG_{n+1-m,q,\lambda}(x)\right\}{}_HG^{(\alpha)}_{m,q,\lambda}$ |

**Table 3.** Explicit representations for the GqHABP ${}_HB^{(\alpha)}_{n,q,\lambda}(x)$, GqHAEP ${}_HE^{(\alpha)}_{n,q,\lambda}(x)$ and GqHAGP ${}_HG^{(\alpha)}_{n,q,\lambda}(x)$

Comparing the coefficients of identical powers of $t$ in both sides of equation (3.5) yields assertion (3.3). □

Similarly, we can prove the following results:

**Corollary 3.4.** *The following explicit relation for the generalized q-Hermite based Apostol type polynomials ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ in terms of the the generalized q-Apostol Euler polynomials $E_{n,q,\lambda}(x)$ holds true:*

$$ {}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)=\frac{1}{2}\sum_{m=0}^{n}\left\{\lambda\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_qE_{n-r,q,\lambda}(x)+E_{n-m,q,\lambda}(x)\right\}{}_H\mathscr{P}^{(\alpha)}_{m,q,\beta}(k,a,b). $$

**Corollary 3.5.** *The following explicit relation for the generalized q-Hermite based Apostol type polynomials ${}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ in terms of the generalized q-Apostol Genocchi polynomials $G_{n,q,\lambda}(x)$ holds true:*

$$ \begin{aligned} {}_H\mathscr{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)=&\frac{1}{2[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_q\sum_{m=0}^{r}\begin{bmatrix}r\\m\end{bmatrix}_qG_{n+1-r,q,\lambda}(x)\right.\\ &\left.-\sum_{m=0}^{n+1}\begin{bmatrix}n+1\\m\end{bmatrix}_qG_{n+1-m,q,\lambda}(x)\right\}{}_H\mathscr{P}^{(\alpha)}_{m,q,\beta}(k,a,b). \end{aligned} $$

In view of Table 1, ceratin explicit representations for the GqHABP ${}_HB^{(\alpha)}_{n,q,\lambda}(x)$, GqHAEP ${}_HE^{(\alpha)}_{n,q,\lambda}(x)$ and GqHAGP ${}_HG^{(\alpha)}_{n,q,\lambda}(x)$ are established and are given in Table 3.

**Note.** It is to be observed that for $\lambda = 1$, the results derived above for the generalized $q$-Hermite-Apostol Bernoulli polynomials ${}_HB^{(\alpha)}_{n,q,\lambda}(x)$, the generalized $q$-Hermite-Apostol Euler polynomials ${}_HE^{(\alpha)}_{n,q,\lambda}(x)$ and generalized $q$-Hermite-Apostol Genocchi polynomials ${}_HG^{(\alpha)}_{n,q,\lambda}(x)$ gives the analogous results for the generalized $q$-Hermite Bernoulli polynomials ${}_HB^{(\alpha)}_{n,q}(x)$, the generalized $q$-Hermite Euler polynomials ${}_HE^{(\alpha)}_{n,q}(x)$ and generalized $q$-Hermite Genocchi polynomials ${}_HG^{(\alpha)}_{n,q}(x)$.

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