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SOME GENERALIZED HERMITE-HADAMARD TYPE INEQUALITIES BY USING THE HARMONIC CONVEXITY OF DIFFERENTIABLE MAPPINGS

M. A. LATIF AND S. HUSSAIN

Abstract. In this paper, a general identity involving a differentiable mapping is established. By using mathematical analysis, Hölder inequality and some auxiliary results, new generalized Hermite-Hadamard type inequalities for differentiable harmonically-convex functions are established. It is expected that the results established in this paper contain previously established results as special cases.

1. Introduction

A function $\mu : \Sigma \subset \mathbb{R} \to \mathbb{R}$ is stated convex function (in the classical sense) if the inequality

$$\mu(\tau u_1 + (1-\tau) u_2) \le \tau\mu(u_1) + (1-\tau)\mu(u_2)$$

holds for all u_1, $u_2 \in \Sigma$ and $\tau \in [0,1]$.

The historically the theory convex functions very old. The initiation of the theory of convex functions began at the end of the nineteenth century. The backgrounds of the theory of convex functions can be found in the major contributions of Hölder [4], Hadamard [5] and Stolz [15]. Jensen [8] was first mathematician who realized the prominence of the convex functions and started the symmetric study of the convex functions in the beginning of the twentieth century. In the subsequent years this research evolved in the emergence of the theory of convex functions as an independent domain of mathematical analysis.

Inequalities have been proven to be a vital tool in the development of new results in almost all the branches of mathematics as well as in the other areas of sciences. The theory of convex functions provides a dynamic role in the growth of the theory of inequalities and hence it has been a subject of broad research over the past few

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decades. A number of interesting results have been proved by using the concept of classical convexity but the most widely studied result for convex functions is stated as follows:

If $\mu:[\alpha_1,\alpha_2]\subset\mathbb{R}\rightarrow\mathbb{R}$ is a convex function, the following inequality

$$\mu\left(\frac{\alpha_1+\alpha_2}{2}\right)\leq\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mu\left(u\right)du\leq\frac{\mu\left(\alpha_1\right)+\mu\left(\alpha_2\right)}{2} \tag{1.1}$$

holds. The double inequality (1.1) is well-known in literature as Hermite-Hadamard inequality which is considered an independent discovery of the legendary mathematicians, J. Hadamard and Ch. Hermite, see for instance [5] and [6].

The concept of classical convexity has been extended and generalized in several directions. One of the generalizations of classical convexity is the harmonic convexity stated in the definition below.

Definition 1. [7] *Let $\Sigma\subset\mathbb{R}\backslash\{0\}$ be a real interval. A function $\mu:\Sigma\rightarrow\mathbb{R}$ is said to be harmonically convex, if*

$$\mu\left(\frac{u_1u_2}{\varphi_\tau\left(u_1,u_2\right)}\right)\leq\tau\mu\left(u_2\right)+\left(1-\tau\right)\mu\left(u_1\right), \tag{1.2}$$

where $\varphi_\tau\left(u_1,u_2\right)=\tau u_1+\left(1-\tau\right)u_2$ for all u_1, $u_2\in\Sigma$ and $\tau\in[0,1]$. If the inequality in (1.2) is reversed, then μ is said to be harmonically concave.

The properties which explain how the usual convexity and the harmonic convexity are connected are studied in [7].

In recent years, a number of researchers have contributed to generalize and to extend the notion of harmonic convexity. Currently, a number of findings on Hermite-Hadamard type inequalities and their applications have been the result of applications of different generalizations and extensions of harmonic convex functions, see for example [1, 2, 3, 8, 10, 11, 12, 13, 14, 17] and [18].

In Section 2, we present some new generalized Hermite-Hadamard and Simpson's type inequalities differentiable harmonically-convex mappings.

2. Main Results

In order to present the main results of this paper, we first establish the following lemmas. Throughout this manuscript, we will use the notation $\kappa\alpha_1+\left(1-\kappa\right)\alpha_2=\varphi_\kappa\left(\alpha_1,\alpha_2\right)$ for our convenience

Lemma 1. *Let $\mu:\Sigma\subseteq\mathbb{R}\backslash\{0\}\rightarrow\mathbb{R}$ be a differentiable mapping on Σ° such that $\zeta_u=\frac{\alpha_2(u-\alpha_1)}{u(\alpha_2-\alpha_1)}$, where α_1, $\alpha_2\in\Sigma^\circ$ with $\alpha_1<\alpha_2$. If $\mu^{'}\in L[\alpha_1,\alpha_2]$, $u\in[\alpha_1,\alpha_2]$ and p_1, $p_2\in\mathbb{R}$, then the following equality holds*

$$\begin{aligned}&\Sigma_u\left(p_1,p_2\right)\\ &\equiv-\left[p_1\mu\left(\alpha_1\right)+\left(1-p_2\right)\mu\left(\alpha_2\right)+\left(p_2-p_1\right)\mu\left(u\right)-\frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\frac{\mu\left(u\right)}{u^2}du\right]\end{aligned}$$

$$= \alpha_1\alpha_2(\alpha_2-\alpha_1)\left\{\int_0^{\zeta_u}\frac{p_1-\kappa}{\varphi_\kappa^2(\alpha_1,\alpha_2)}\mu'\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)d\kappa\right.$$
$$\left.+\int_{\zeta_u}^1\frac{p_2-\kappa}{\varphi_\kappa^2(\alpha_1,\alpha_2)}\mu'\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)d\kappa\right\}. \quad (2.1)$$

Proof. By integration by parts, we have

$$\int_0^{\zeta_u}\frac{p_1-\kappa}{\varphi_\kappa^2(\alpha_1,\alpha_2)}\mu'\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)d\kappa$$
$$=\frac{1}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}\int_0^{\zeta_u}(p_1-\kappa)\,d\left[\mu\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)\right]$$
$$=\frac{(p_1-\kappa)}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}\mu\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)\Bigg|_0^{\zeta_u}+\frac{1}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}\int_0^{\zeta_u}\mu\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)d\kappa$$
$$=\frac{(p_1-\zeta_u)}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}\mu\left(\frac{\alpha_1\alpha_2}{\varphi_{\zeta_u}(\alpha_1,\alpha_2)}\right)-\frac{p_1\mu(\alpha_1)}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}$$
$$+\frac{1}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}\int_0^{\zeta_u}\mu\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)d\kappa$$
$$=\frac{(p_1-\zeta_u)\mu(u)}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}-\frac{p_1\mu(\alpha_1)}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}+\frac{1}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}\int_0^{\zeta_u}\mu\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)d\kappa. \quad (2.2)$$

and

$$\int_{\zeta_u}^1\frac{p_2-\kappa}{\varphi_\kappa^2(\alpha_1,\alpha_2)}\mu'\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)d\kappa$$
$$=\frac{1}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}\int_{\zeta_u}^1(p_2-\kappa)\,d\left[\mu\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)\right]$$
$$=\frac{(p_2-\kappa)}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}\mu\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)\Bigg|_{\zeta_u}^1+\frac{1}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}\int_{\zeta_u}^1\mu\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)d\kappa$$
$$=\frac{(p_2-1)\mu(\alpha_2)}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}-\frac{(p_2-\zeta_u)\mu(u)}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}+\frac{1}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}\int_{\zeta_u}^1\mu\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)d\kappa. \quad (2.3)$$

Adding (2.2) and (2.3), making use of the substitution $u=\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}$ and multiplying the resulting equality by $\alpha_1\alpha_2(\alpha_2-\alpha_1)$, we get the required identity. □

Lemma 2. *Let* $\xi\geq 0$*,* $c>0$*,* $0<x_1<x_2$ *with* $1-\frac{x_1}{x_2}>\frac{1}{c}$ *and* $\alpha,\beta\in\mathbb{R}$*. Then*

$$A(\alpha,\beta;x_1,x_2,\xi,c)$$

$$= \int_0^c \frac{(\alpha+\beta u)\left|\xi-u\right|}{\varphi_u\left(x_1,x_2\right)}du$$

$$= \begin{cases} \frac{c[\alpha x_1\xi - x_1x_2(\alpha(2\xi-1)+\beta(c+\xi))+x_2(\alpha(\xi-1)+\beta(c+\xi-2))]}{x_2(x_1-x_2)^2(x_1c-x_2c+x_2)} \\ \quad -\frac{[x_2(\alpha+2\beta-\beta\xi)+x_1(-\alpha+\beta\xi)]\ln\left(\frac{x_2}{cx_1+x_2-cx_2}\right)}{(x_1-x_2)^3}, & \xi \geq c, \\ \\ -\frac{c[x_1\alpha\xi - x_1x_2(\alpha(2\xi-1)+\beta(c+\xi))+x_2(\alpha(\xi-1)+\beta(c+\xi-2))]}{x_2(x_1-x_2)^2(x_1c-x_2c+x_2)} \\ \quad +\frac{[x_2(\alpha+2\beta-\beta\xi)+x_1(-\alpha+\beta\xi)]\ln\left(\frac{x_2}{cx_1+x_2-cx_2}\right)}{(x_1-x_2)^3}, & 0 \leq \xi \leq c. \end{cases}$$

Proof. The proof follows by straightforward computations. □

Lemma 3. *Let ξ, $c \geq 0$, $\alpha, \beta \in \mathbb{R}$, $r \in \{0\} \cup \mathbb{N}$. Then the following equality holds*

$$B\left(\alpha,\beta;\xi,c,r\right)$$
$$= \int_0^c (\alpha+\beta u)\left|\xi-u\right|^r du$$
$$= \begin{cases} \frac{\xi^{r+1}((r+2)\alpha+\beta\xi)-(\xi-c)^{r+1}((r+2)\alpha+\beta(c+cr+\xi))}{(r+1)(r+2)}, & \xi \geq c, \\ \\ \frac{(-\xi)^{r+1}((r+2)\alpha+\beta\xi)+(c-\xi)^{r+1}((r+2)\alpha+\beta(c+cr+\xi))}{(r+1)(r+2)}, & 0 \leq \xi \leq c. \end{cases}$$

Proof. It can be easily proved using properties of absolute value and simple integration techniques. □

Lemma 4. *Let $c \geq 0$, $\alpha, \beta \in \mathbb{R}$, $r > 2$ and $0 < x_1 < x_2$. Then*

$$C\left(\alpha,\beta,x_1,x_2;c,r\right)$$
$$= \int_0^c \frac{(\alpha+\beta u)}{\varphi_u^r\left(x_1,x_2\right)}du$$
$$= \frac{1}{\left(x_1-x_2\right)^2\left(r-1\right)\left(r-2\right)}\left\{x_2\left[\alpha x_1\left(r-2\right)+x_2\left(\beta-\alpha\left(r-2\right)\right)\right]\right.$$
$$\left.+\left(x_2+x_1c-x_2c\right)^{1-r}\left[\left(\alpha x_1\left(r-2\right)+c\beta\left(r-1\right)\right)+x_2\left(\left(1+c-cr\right)-\alpha\left(r-2\right)\right)\right]\right\}.$$

Proof. The proof follows by using simple integration techniques. □

We are now in position to set up the results of this paper.

Theorem 1. *Let $\mu : \Sigma \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on Σ° with α_1, $\alpha_2 \in \Sigma^\circ$ and $\alpha_1 < \alpha_2$. If $\mu' \in L[\alpha_1,\alpha_2]$ and $\left|\mu'\right|^\sigma$ is harmonically-convex on $[\alpha_1,\alpha_2]$ for $\sigma \geq 1$, then the following inequality holds true for all $u \in [\alpha_1,\alpha_2]$ and p_1, $p_2 \in [0,1]$*

$$\left|\Sigma_u\left(p_1,p_2\right)\right| \leq \alpha_1\alpha_2\left(\alpha_2-\alpha_1\right)\left\{\left[A\left(1,0;\alpha_1,\alpha_2,p_1,\zeta_u\right)\right]^{1-\frac{1}{\sigma}}\right.$$

$$\times\left(A(1,-1;\alpha_1,\alpha_2,p_1,\zeta_u)\left|\mu'(\alpha_1)\right|^\sigma + A(0,1;\alpha_1,\alpha_2,p_1,\zeta_u)\left|\mu'(\alpha_2)\right|^\sigma\right)^{\frac{1}{\sigma}}$$
$$+\left[A(1,0;\alpha_2,\alpha_1,1-p_2,1-\zeta_u)\right]^{1-\frac{1}{\sigma}}\left(A(0,1;\alpha_2,\alpha_1,1-p_2,1-\zeta_u)\left|\mu'(\alpha_1)\right|^\sigma\right.$$
$$\left.\left.+A(1,-1;\alpha_2,\alpha_1,1-p_2,1-\zeta_u)\left|\mu'(\alpha_2)\right|^\sigma\right)^{\frac{1}{\sigma}}\right\}, \quad (2.4)$$

where $A(\alpha,\mu;\alpha_1,\alpha_2,\xi,c)$ *is defined in Lemma 2 and* ζ_u *is defined in Lemma 1.*

Proof. Taking the absolute value on both sides of the result in Lemma 1, applying the power-mean inequality and using the harmonic-convexity of $\left|\mu'\right|^\sigma$, $\sigma \geq 1$, we have

$$\left|\Sigma_u(p_1,p_2)\right| \leq \alpha_1\alpha_2(\alpha_2-\alpha_1)\left\{\int_0^{\zeta_u}\frac{|p_1-\kappa|}{\varphi_\kappa^2(\alpha_1,\alpha_2)}\left|\mu'\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)\right|d\kappa\right.$$
$$\left.+\int_{\zeta_u}^1\frac{|p_2-\kappa|}{\varphi_\kappa^2(\alpha_1,\alpha_2)}\left|\mu'\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)\right|d\kappa\right\} \leq \alpha_1\alpha_2(\alpha_2-\alpha_1)$$
$$\times\left\{\left(\int_0^{\zeta_u}\frac{|p_1-\kappa|}{\varphi_\kappa^2(\alpha_1,\alpha_2)}d\kappa\right)^{1-\frac{1}{\sigma}}\left(\int_0^{\zeta_u}\frac{|p_1-\kappa|\left[(1-\kappa)\left|\mu'(\alpha_1)\right|^\sigma+\kappa\left|\mu'(\alpha_2)\right|^\sigma\right]}{\varphi_\kappa^2(\alpha_1,\alpha_2)}d\kappa\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+\left(\int_{\zeta_u}^1\frac{|p_2-\kappa|}{\varphi_\kappa^2(\alpha_1,\alpha_2)}d\kappa\right)^{1-\frac{1}{\sigma}}\left(\int_{\zeta_u}^1\frac{|p_2-\kappa|\left[(1-\kappa)\left|\mu'(\alpha_1)\right|^\sigma+\kappa\left|\mu'(\alpha_2)\right|^\sigma\right]}{\varphi_\kappa^2(\alpha_1,\alpha_2)}d\kappa\right)^{\frac{1}{\sigma}}\right\}. \quad (2.5)$$

By applying Lemma 2, we observe that

$$\int_0^{\zeta_u}\frac{|p_1-\kappa|}{\varphi_\kappa^2(\alpha_1,\alpha_2)}d\kappa = A(1,0;\alpha_1,\alpha_2,p_1,\zeta_u),$$
$$\int_{\zeta_u}^1\frac{|p_2-\kappa|}{\varphi_\kappa^2(\alpha_1,\alpha_2)}d\kappa = \int_0^{1-\zeta_u}\frac{|1-p_2-\kappa|}{\varphi_\kappa^2(\alpha_2,\alpha_1)}d\kappa = A(1,0;\alpha_2,\alpha_1,1-p_2,1-\zeta_u),$$
$$\int_0^{\zeta_u}\frac{\kappa|p_1-\kappa|}{\varphi_\kappa^2(\alpha_1,\alpha_2)}d\kappa = A(0,1;\alpha_1,\alpha_2,p_1,\zeta_u),$$
$$\int_0^{\zeta_u}\frac{(1-\kappa)|p_1-\kappa|}{\varphi_\kappa^2(\alpha_1,\alpha_2)}d\kappa = A(1,-1;\alpha_1,\alpha_2,p_1,\zeta_u),$$
$$\int_{\zeta_u}^1\frac{\kappa|p_2-\kappa|}{\varphi_\kappa^2(\alpha_1,\alpha_2)}d\kappa = \int_0^{1-\zeta_u}\frac{(1-\kappa)|1-p_2-\kappa|}{\varphi_\kappa^2(\alpha_2,\alpha_1)}d\kappa$$
$$= A(1,-1;\alpha_2,\alpha_1,1-p_2,1-\zeta_u)$$

and

$$\int_{\zeta_u}^{1} \frac{(1-\kappa)\left|p_2-\kappa\right|}{\varphi_\kappa^2(\alpha_1,\alpha_2)} d\kappa = \int_0^{1-\zeta_u} \frac{\kappa\left|1-p_2-\kappa\right|}{\varphi_\kappa^2(\alpha_2,\alpha_1)} d\kappa$$
$$= A\left(0,1;\alpha_2,\alpha_1,1-p_2,1-\zeta_u\right).$$

By using the above integrals in (2.5), we get the required inequality. □

Corollary 1. *If $0 \le p_1 \le \frac{1}{2} \le p_2 \le 1$, $\sigma = 1$ and $u = \frac{\alpha_1+\alpha_2}{2}$ in Theorem 1, the following inequality holds valid*

$$\left|p_1\mu(\alpha_1) + (1-p_2)\mu(\alpha_2) + (p_2-p_1)\mu\left(\frac{\alpha_1+\alpha_2}{2}\right) - \frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\frac{\mu(u)}{u^2}du\right|$$
$$\le \alpha_1\alpha_2(\alpha_2-\alpha_1)$$
$$\times\left\{\left[A\left(1,-1;\alpha_1,\alpha_2,p_1,\frac{\alpha_2}{\alpha_1+\alpha_2}\right) + A\left(0,1;\alpha_2,\alpha_1,1-p_2,\frac{\alpha_1}{\alpha_1+\alpha_2}\right)\right]\left|\mu'(\alpha_1)\right|\right.$$
$$\left.+\left[A\left(0,1;\alpha_1,\alpha_2,p_1,\frac{\alpha_2}{\alpha_1+\alpha_2}\right) + A\left(1,-1;\alpha_2,\alpha_1,1-p_2,\frac{\alpha_1}{\alpha_1+\alpha_2}\right)\right]\left|\mu'(\alpha_2)\right|\right\}, \tag{2.6}$$

where $A(\alpha,\mu;\alpha_1,\alpha_2,\xi,c)$ is defined in Lemma 2.

Proof. Since for $u = \frac{\alpha_1+\alpha_2}{2}$, $\zeta_u = \frac{\alpha_2}{\alpha_1+\alpha_2}$ and $1-\zeta_u = \frac{\alpha_1}{\alpha_1+\alpha_2}$. Hence the proof follows from the result of Theorem 1. □

Corollary 2. *If $p_1 = 0$ and $p_2 = 1$ in Corollary 1, the following Hermite-Hadamard type for harmonically convex functions holds*

$$\left|\mu\left(\frac{\alpha_1+\alpha_2}{2}\right) - \frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\frac{\mu(u)}{u^2}du\right| \le \alpha_1\alpha_2(\alpha_2-\alpha_1)$$
$$\times\left\{\left[A\left(1,-1;\alpha_1,\alpha_2,0,\frac{\alpha_2}{\alpha_1+\alpha_2}\right) + A\left(0,1;\alpha_2,\alpha_1,0,\frac{\alpha_1}{\alpha_1+\alpha_2}\right)\right]\left|\mu'(\alpha_1)\right|\right.$$
$$\left.+\left[A\left(0,1;\alpha_1,\alpha_2,0,\frac{\alpha_2}{\alpha_1+\alpha_2}\right) + A\left(1,-1;\alpha_2,\alpha_1,0,\frac{\alpha_1}{\alpha_1+\alpha_2}\right)\right]\left|\mu'(\alpha_2)\right|\right\}, \tag{2.7}$$

where $A(\alpha,\mu;\alpha_1,\alpha_2,\xi,c)$ is defined in Lemma 2.

Proof. It is a direct consequence of Corollary 1. □

Corollary 3. *If we set $p_1 = p_2 = \frac{1}{2}$ in Corollary 1, the following Hermite-Hadamard type for harmonically convex functions holds*

$$\left|\frac{\mu(\alpha_1)+\mu(\alpha_2)}{2} - \frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\frac{\mu(u)}{u^2}du\right| \le \alpha_1\alpha_2(\alpha_2-\alpha_1)$$

$$\times\left\{\left[A\left(1,-1;\alpha_1,\alpha_2,\frac{1}{2},\frac{\alpha_2}{\alpha_1+\alpha_2}\right)+A\left(0,1;\alpha_2,\alpha_1,\frac{1}{2},\frac{\alpha_1}{\alpha_1+\alpha_2}\right)\right]\left|\mu^{'}\left(\alpha_1\right)\right| + \left[A\left(0,1;\alpha_1,\alpha_2,\frac{1}{2},\frac{\alpha_2}{\alpha_1+\alpha_2}\right)+A\left(1,-1;\alpha_2,\alpha_1,\frac{1}{2},\frac{\alpha_1}{\alpha_1+\alpha_2}\right)\right]\left|\mu^{'}\left(\alpha_2\right)\right|\right\}, \tag{2.8}$$

where $A\left(\alpha,\mu;\alpha_1,\alpha_2,\xi,c\right)$ *is defined in Lemma 2.*

Proof. It follows from Corollary 1. □

Corollary 4. *Suppose* $p_1=\frac{1}{6}$ *and* $p_2=\frac{5}{6}$ *in the result of Corollary 1, the following Simpson type inequality holds for harmonically-convex functions*

$$\left|\frac{1}{3}\left[\frac{\mu\left(\alpha_1\right)+\mu\left(\alpha_2\right)}{2}+2\mu\left(\frac{\alpha_1+\alpha_2}{2}\right)\right]-\frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\frac{\mu\left(u\right)}{u^2}du\right|$$
$$\leq\alpha_1\alpha_2\left(\alpha_2-\alpha_1\right)\left\{\left[A\left(1,-1;\alpha_1,\alpha_2,\frac{1}{6},\frac{\alpha_2}{\alpha_1+\alpha_2}\right)+A\left(0,1;\alpha_2,\alpha_1,\frac{1}{6},\frac{\alpha_1}{\alpha_1+\alpha_2}\right)\right]\times\left|\mu^{'}\left(\alpha_1\right)\right| + \left[A\left(0,1;\alpha_1,\alpha_2,\frac{1}{6},\frac{\alpha_2}{\alpha_1+\alpha_2}\right)+A\left(1,-1;\alpha_2,\alpha_1,\frac{1}{6},\frac{\alpha_1}{\alpha_1+\alpha_2}\right)\right]\left|\mu^{'}\left(\alpha_2\right)\right|\right\}, \tag{2.9}$$

where $A\left(\alpha,\mu;\alpha_1,\alpha_2,\xi,c\right)$ *is defined in Lemma 2.*

Proof. Proof directly follows from the result of Corollary 1. □

Theorem 2. *Let* $\mu:\Sigma\subseteq(0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* Σ° *with* $\alpha_1,\ \alpha_2\in\Sigma^\circ$ *and* $\alpha_1<\alpha_2$. *If* $\mu^{'}\in L\left[\alpha_1,\alpha_2\right]$ *and* $\left|\mu^{'}\right|^\sigma$ *is harmonically-convex on* $\left[\alpha_1,\alpha_2\right]$ *for* $\sigma>1$, *the following inequality holds true for all* $u\in\left[\alpha_1,\alpha_2\right]$ *and* $p_2,p_1\in[0,1]$

$$\left|\Sigma_u\left(p_1,p_2\right)\right|\leq\alpha_1\alpha_2\left(\alpha_2-\alpha_1\right)\left\{\left[B\left(1,0;p_1,\zeta_u,\left\lfloor\frac{\sigma}{\sigma-1}\right\rfloor\right)\right]^{1-\frac{1}{\sigma}}\times\left(C\left(1,-1;\alpha_1,\alpha_2,\zeta_u,2\sigma\right)\left|\mu^{'}\left(\alpha_1\right)\right|^\sigma+C\left(0,1;\alpha_1,\alpha_2,\zeta_u,2\sigma\right)\left|\mu^{'}\left(\alpha_2\right)\right|^\sigma\right)^{\frac{1}{\sigma}} + \left[B\left(1,0;1-p_2,1-\zeta_u,\left\lfloor\frac{\sigma}{\sigma-1}\right\rfloor\right)\right]^{1-\frac{1}{\sigma}}\left(C\left(0,1;\alpha_2,\alpha_1,1-\zeta_u,2\sigma\right)\left|\mu^{'}\left(\alpha_1\right)\right|^\sigma + C\left(1,-1;\alpha_2,\alpha_1,1-\zeta_u,2\sigma\right)\left|\mu^{'}\left(\alpha_2\right)\right|^\sigma\right)^{\frac{1}{\sigma}}\right\}, \tag{2.10}$$

where $B\left(\alpha,\mu;\xi,c,r\right)$ *is defined in Lemma 3,* $C\left(\alpha,\mu;\alpha_1,\alpha_2,c,r\right)$ *is defined in Lemma 4,* ζ_u *is defined in Lemma 1 and* $\lfloor u\rfloor$ *is the floor function.*

Proof. From the result in Lemma 1, applying the Hölder inequality and using the harmonic-convexity of $\left|\mu^{'}\right|^{\sigma}$, $\sigma > 1$, we have

$$\left|\Sigma_u\left(p_1,p_2\right)\right| \le \alpha_1\alpha_2\left(\alpha_2-\alpha_1\right)\left\{\int_0^{\zeta_u}\frac{\left|p_1-\kappa\right|}{\varphi_\kappa^2\left(\alpha_1,\alpha_2\right)}\left|\mu^{'}\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa\left(\alpha_1,\alpha_2\right)}\right)\right|d\kappa\right.$$
$$\left.+\int_{\zeta_u}^1\frac{\left|p_2-\kappa\right|}{\varphi_\kappa^2\left(\alpha_1,\alpha_2\right)}\left|\mu^{'}\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa\left(\alpha_1,\alpha_2\right)}\right)\right|d\kappa\right\} \le \alpha_1\alpha_2\left(\alpha_2-\alpha_1\right)$$
$$\times\left\{\left(\int_0^{\zeta_u}\left|p_1-\kappa\right|^{\frac{\sigma}{\sigma-1}}d\kappa\right)^{1-\frac{1}{\sigma}}\left(\int_0^{\zeta_u}\frac{\left[\left(1-\kappa\right)\left|\mu^{'}\left(\alpha_1\right)\right|^{\sigma}+\kappa\left|\mu^{'}\left(\alpha_2\right)\right|^{\sigma}\right]}{\varphi_\kappa^{2\sigma}\left(\alpha_1,\alpha_2\right)}d\kappa\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+\left(\int_{\zeta_u}^1\left|p_2-\kappa\right|^{\frac{\sigma}{\sigma-1}}d\kappa\right)^{1-\frac{1}{\sigma}}\left(\int_{\zeta_u}^1\frac{\left[\left(1-\kappa\right)\left|\mu^{'}\left(\alpha_1\right)\right|^{\sigma}+\kappa\left|\mu^{'}\left(\alpha_2\right)\right|^{\sigma}\right]}{\varphi_\kappa^{2\sigma}\left(\alpha_1,\alpha_2\right)}d\kappa\right)^{\frac{1}{\sigma}}\right\}. \tag{2.11}$$

By applying Lemma 3 and Lemma 4 we have that

$$\int_0^{\zeta_u}\left|p_1-\kappa\right|^{\frac{\sigma}{\sigma-1}}d\kappa \le \int_0^{\zeta_u}\left|p_1-\kappa\right|^{\left\lfloor\frac{\sigma}{\sigma-1}\right\rfloor}d\kappa = B\left(1,0;p_1,\zeta_u,\left\lfloor\frac{\sigma}{\sigma-1}\right\rfloor\right),$$

$$\int_{\zeta_u}^1\left|p_2-\kappa\right|^{\frac{\sigma}{\sigma-1}}d\kappa = \int_0^{1-\zeta_u}\left|1-p_2-\kappa\right|^{\frac{\sigma}{\sigma-1}}d\kappa$$
$$\le B\left(1,0;1-p_2,1-\zeta_u,\left\lfloor\frac{\sigma}{\sigma-1}\right\rfloor\right),$$

$$\int_0^{\zeta_u}\frac{\kappa}{\varphi_\kappa^{2\sigma}\left(\alpha_1,\alpha_2\right)}d\kappa = C\left(0,1;\alpha_1,\alpha_2,\zeta_u,2\sigma\right),$$
$$\int_0^{\zeta_u}\frac{\left(1-\kappa\right)}{\varphi_\kappa^{2\sigma}\left(\alpha_1,\alpha_2\right)}d\kappa = C\left(1,-1;\alpha_1,\alpha_2,\zeta_u,2\sigma\right),$$
$$\int_{\zeta_u}^1\frac{\kappa}{\varphi_\kappa^{2\sigma}\left(\alpha_1,\alpha_2\right)}d\kappa = \int_0^{1-\zeta_u}\frac{\left(1-\kappa\right)}{\varphi_\kappa^{2\sigma}\left(\alpha_2,\alpha_1\right)}d\kappa = C\left(1,-1;\alpha_2,\alpha_1,1-\zeta_u,2\sigma\right)$$

and

$$\int_{\zeta_u}^1\frac{\left(1-\kappa\right)}{\varphi_\kappa^{2\sigma}\left(\alpha_1,\alpha_2\right)}d\kappa = \int_0^{1-\zeta_u}\frac{\kappa}{\varphi_\kappa^{2\sigma}\left(\alpha_2,\alpha_1\right)}d\kappa = C\left(0,1;\alpha_2,\alpha_1,1-\zeta_u,2\sigma\right).$$

By using the values of the above integrals in (2.11), we get the required inequality. □

Theorem 3. *Let $\mu : \Sigma \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on Σ° with $\alpha_1,\ \alpha_2 \in \Sigma^\circ$ and $\alpha_1 < \alpha_2$. If $\mu^{'} \in L[\alpha_1,\alpha_2]$ and $\left|\mu^{'}\right|^\sigma$ is harmonically-convex on $[\alpha_1,\alpha_2]$ for $\sigma > 1$, the following inequality holds true for all $u \in [\alpha_1,\alpha_2]$ and $p_2, p_1 \in [0,1]$*

$$\begin{aligned}|\Sigma_u(p_1,p_2)| \le \alpha_1\alpha_2(\alpha_2-\alpha_1)\Bigg\{&\left[C\left(1,0;\alpha_1,\alpha_2,\zeta_u,\frac{\sigma}{\sigma-1}\right)\right]^{1-\frac{1}{\sigma}} \\ &\times\left(B(1,-1;\zeta_u,\lfloor\sigma\rfloor)\left|\mu^{'}(\alpha_1)\right|^\sigma + B(0,1;\zeta_u,\lfloor\sigma\rfloor)\left|\mu^{'}(\alpha_2)\right|^\sigma\right)^{\frac{1}{\sigma}} \\ &+\left[C\left(1,0;\alpha_2,\alpha_1,1-\zeta_u,\frac{\sigma}{\sigma-1}\right)\right]^{1-\frac{1}{\sigma}}\left(B(0,1;1-\zeta_u,\lfloor\sigma\rfloor)\left|\mu^{'}(\alpha_1)\right|^\sigma\right. \\ &\left.+B(1,-1;1-\zeta_u,\lfloor\sigma\rfloor)\left|\mu^{'}(\alpha_2)\right|^\sigma\right)^{\frac{1}{\sigma}}\Bigg\}, \quad (2.12)\end{aligned}$$

where $B(\alpha,\mu;\xi,c,r)$ is defined in Lemma 3, $C(\alpha,\mu;\alpha_1,\alpha_2,c,r)$ is defined in Lemma 4, ζ_u is defined in Lemma 1 and $\lfloor u\rfloor$ is the floor function.

Proof. Taking the absolute value on both sides of the result in Lemma 1, applying the Hölder inequality and using the harmonic-convexity of $\left|\mu^{'}\right|^\sigma$, $\sigma > 1$, we have

$$\begin{aligned}|\Sigma_u(p_1,p_2)| \le \alpha_1\alpha_2(\alpha_2-\alpha_1)\Bigg\{&\left(\int_0^{\zeta_u}\frac{d\kappa}{\varphi_\kappa^{\frac{\sigma}{\sigma-1}}(\alpha_1,\alpha_2)}\right)^{1-\frac{1}{\sigma}} \\ \times\left(\int_0^{\zeta_u}|p_1-\kappa|^\sigma\left[(1-\kappa)\left|\mu^{'}(\alpha_1)\right|^\sigma+\kappa\left|\mu^{'}(\alpha_2)\right|^\sigma\right]d\kappa\right)^{\frac{1}{\sigma}} &+\left(\int_{\zeta_u}^1\frac{d\kappa}{\varphi_\kappa^{\frac{\sigma}{\sigma-1}}(\alpha_1,\alpha_2)}\right)^{1-\frac{1}{\sigma}} \\ \times\left(\int_{\zeta_u}^1|p_2-\kappa|^\sigma\left[(1-\kappa)\left|\mu^{'}(\alpha_1)\right|^\sigma+\kappa\left|\mu^{'}(\alpha_2)\right|^\sigma\right]d\kappa\right)^{\frac{1}{\sigma}}&\Bigg\}. \quad (2.13)\end{aligned}$$

By applying Lemma 3 and Lemma 4 and using similar arguments as in proving Theorem 2, we get the desired result. □

Theorem 4. *Let $\mu : \Sigma \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on Σ° with $\alpha_1,\ \alpha_2 \in \Sigma^\circ$ and $\alpha_1 < \alpha_2$. If $\mu^{'} \in L[\alpha_1,\alpha_2]$ and $\left|\mu^{'}\right|^\sigma$ is harmonically-convex on $[\alpha_1,\alpha_2]$ for $\sigma > 1$ and $0 < p < \sigma$, then the following inequality holds true for all $u \in [\alpha_1,\alpha_2]$ and $p_2, p_1 \in [0,1]$*

$$|\Sigma_u(p_1,p_2)| \le \alpha_1\alpha_2(\alpha_2-\alpha_1)\Bigg\{\left[D\left(1,0;\alpha_1,\alpha_2,p_1,\zeta_u,\left\lfloor\frac{\sigma-p}{\sigma-1}\right\rfloor,\frac{2(\sigma-p)}{\sigma-1}\right)\right]^{1-\frac{1}{\sigma}}$$

$$\times\left(D\left(1,-1;\alpha_1,\alpha_2,p_1,\zeta_u,\lfloor p\rfloor,2p\right)\left|\mu^{'}\left(\alpha_1\right)\right|^{\sigma}+D\left(0,1;\alpha_1,\alpha_2,p_1,\zeta_u,\lfloor p\rfloor,2p\right)\left|\mu^{'}\left(\alpha_2\right)\right|^{\sigma}\right)^{\frac{1}{\sigma}}$$

$$+\left[D\left(1,0;\alpha_2,\alpha_1,1-p_2,1-\zeta_u,\left\lfloor\frac{\sigma-p}{\sigma-1}\right\rfloor,\frac{2\left(\sigma-p\right)}{\sigma-1}\right)\right]^{1-\frac{1}{\sigma}}$$

$$\times\left(D\left(0,1;\alpha_2,\alpha_1,1-p_2,1-\zeta_u,\lfloor p\rfloor,2p\right)\left|\mu^{'}\left(\alpha_1\right)\right|^{\sigma}\right.$$

$$\left.\left.+D\left(1,-1,\alpha_2,\alpha_1,1-p_2,1-\zeta_u,\lfloor p\rfloor,2p\right)\left|\mu^{'}\left(\alpha_2\right)\right|^{\sigma}\right)^{\frac{1}{\sigma}}\right\} \quad (2.14)$$

Proof. Using Lemma 1, applying the Hölder inequality and using the harmonic-convexity of $\left|\mu^{'}\right|^{\sigma}$, $\sigma>1$, we have

$$\left|\Sigma_u\left(p_1,p_2\right)\right|\leq\alpha_1\alpha_2\left(\alpha_2-\alpha_1\right)\left\{\left(\int_0^{\zeta_u}\frac{\left|p_1-\kappa\right|^{\frac{\sigma-p}{\sigma-1}}}{\varphi_\kappa^{\frac{2(\sigma-p)}{\sigma-1}}\left(\alpha_1,\alpha_2\right)}d\kappa\right)^{1-\frac{1}{\sigma}}\right.$$

$$\times\left(\int_0^{\zeta_u}\frac{\left|p_1-\kappa\right|^p\left[\left(1-\kappa\right)\left|\mu^{'}\left(\alpha_1\right)\right|^{\sigma}+\kappa\left|\mu^{'}\left(\alpha_2\right)\right|^{\sigma}\right]d\kappa}{\varphi_\kappa^{2p}\left(\alpha_1,\alpha_2\right)}\right)^{\frac{1}{\sigma}}+\left(\int_{\zeta_u}^1\frac{\left|p_2-\kappa\right|^{\frac{\sigma-p}{\sigma-1}}d\kappa}{\varphi_\kappa^{\frac{2(\sigma-p)}{\sigma-1}}\left(\alpha_1,\alpha_2\right)}\right)^{1-\frac{1}{\sigma}}$$

$$\left.\times\left(\int_{\zeta_u}^1\frac{\left|p_2-\kappa\right|^p\left[\left(1-\kappa\right)\left|\mu^{'}\left(\alpha_1\right)\right|^{\sigma}+\kappa\left|\mu^{'}\left(\alpha_2\right)\right|^{\sigma}\right]}{\varphi_\kappa^{2p}\left(\alpha_1,\alpha_2\right)}d\kappa\right)^{\frac{1}{\sigma}}\right\}. \quad (2.15)$$

Let

$$D\left(\alpha,\mu;\alpha_1,\alpha_2,\xi,c,r,\kappa\right)=\int_0^c\frac{\left(\alpha+\mu u\right)\left|\xi-u\right|^r}{\varphi_u^{\kappa}\left(\alpha_1,\alpha_2\right)}du$$

$$=\begin{cases}\int_0^c\frac{(\alpha+\mu u)(\xi-u)^r}{\varphi_u^{\kappa}(\alpha_1,\alpha_2)}du, & \xi\geq c,\\ \int_0^c\frac{(\alpha+\mu u)(u-\xi)^r}{\varphi_u^{\kappa}(\alpha_1,\alpha_2)}du, & 0\leq\xi\leq c,\end{cases}$$

where κ, $\alpha,\mu\in\mathbb{R}$, $0<\alpha_1<\alpha_2$, ξ, $c\geq0$ and $r\in\{0\}\cup\mathbb{N}$. The above integral can be solved numerically by using the software Matlab or Mathematica. By using this integral, we have the following observations

$$\int_0^{\zeta_u}\frac{\left|p_1-\kappa\right|^{\frac{\sigma-p}{\sigma-1}}}{\varphi_\kappa^{\frac{2(\sigma-p)}{\sigma-1}}\left(\alpha_1,\alpha_2\right)}d\kappa\leq D\left(1,0;\alpha_1,\alpha_2,p_1,\zeta_u,\left\lfloor\frac{\sigma-p}{\sigma-1}\right\rfloor,\frac{2\left(\sigma-p\right)}{\sigma-1}\right),$$

$$\int_{\zeta_u}^1\frac{\left|p_2-\kappa\right|^{\frac{\sigma-p}{\sigma-1}}}{\varphi_\kappa^{\frac{2(\sigma-p)}{\sigma-1}}\left(\alpha_1,\alpha_2\right)}d\kappa=\int_0^{1-\zeta_u}\frac{\left|1-p_2-\kappa\right|^{\frac{\sigma-p}{\sigma-1}}}{\varphi_\kappa^{\frac{2(\sigma-p)}{\sigma-1}}\left(\alpha_2,\alpha_1\right)}d\kappa$$

$$\leq D\left(1,0;\alpha_2,\alpha_1,1-p_2,1-\zeta_u,\left\lfloor\frac{\sigma-p}{\sigma-1}\right\rfloor,\frac{2(\sigma-p)}{\sigma-1}\right),$$

$$\int_0^{\zeta_u}\frac{|p_1-\kappa|^p(1-\kappa)}{\varphi_\kappa^{2p}(\alpha_1,\alpha_2)}d\kappa\leq D(1,-1;\alpha_1,\alpha_2,p_1,\zeta_u,\lfloor p\rfloor,2p),$$

$$\int_0^{\zeta_u}\frac{\kappa|p_1-\kappa|^p}{\varphi_\kappa^{2p}(\alpha_1,\alpha_2)}d\kappa\leq D(0,1;\alpha_1,\alpha_2,p_1,\zeta_u,\lfloor p\rfloor,2p),$$

$$\int_{\zeta_u}^1\frac{(1-\kappa)|p_2-\kappa|^p}{\varphi_\kappa^{2p}(\alpha_1,\alpha_2)}d\kappa\leq D(0,1;\alpha_2,\alpha_1,1-p_2,1-\zeta_u,\lfloor p\rfloor,2p)$$

and

$$\int_{\zeta_u}^1\frac{\kappa|p_2-\kappa|^p}{\varphi_\kappa^{2p}(\alpha_1,\alpha_2)}d\kappa\leq D(1,-1;\alpha_2,\alpha_1,1-p_2,1-\zeta_u,\lfloor p\rfloor,2p),$$

where $\lfloor u\rfloor$ is the floor function. By using the above inequalities in (2.15), we get the inequality (2.14). □

Theorem 5. *Let $\mu:\Sigma\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable mapping on Σ° with α_1, $\alpha_2\in\Sigma^\circ$ and $\alpha_1<\alpha_2$. If $\mu^{'}\in L[\alpha_1,\alpha_2]$ and $\left|\mu^{'}\right|^\sigma$ is harmonically-convex on $[\alpha_1,\alpha_2]$ for $\sigma>1$, then the following inequality holds true for all $u\in[\alpha_1,\alpha_2]$ and $p_2,p_1\in[0,1]$*

$$\begin{aligned}|\Sigma_u(p_1,p_2)|\leq\alpha_1\alpha_2(\alpha_2-\alpha_1)\Bigg\{&\left[D\left(1,0;\alpha_1,\alpha_2,p_1,\zeta_u,\left\lfloor\frac{\sigma}{\sigma-1}\right\rfloor,\frac{\sigma}{\sigma-1}\right)\right]^{1-\frac{1}{\sigma}}\\ \times\Big(C(1,-1;\alpha_1,\alpha_2,p_1,\zeta_u,\sigma)\left|\mu^{'}(\alpha_1)\right|^\sigma&+C(0,1;\alpha_1,\alpha_2,p_1,\zeta_u,\sigma)\left|\mu^{'}(\alpha_2)\right|^\sigma\Big)^{\frac{1}{\sigma}}\\ &+\left[D\left(1,0;\alpha_2,\alpha_1,1-p_2,1-\zeta_u,\left\lfloor\frac{\sigma}{\sigma-1}\right\rfloor,\frac{\sigma}{\sigma-1}\right)\right]^{1-\frac{1}{\sigma}}\\ &\times\Big(C(0,1;\alpha_2,\alpha_1,1-p_2,1-\zeta_u,\sigma)\left|\mu^{'}(\alpha_1)\right|^\sigma\\ &+C(1,-1;\alpha_2,\alpha_1,1-p_2,1-\zeta_u,\sigma)\left|\mu^{'}(\alpha_2)\right|^\sigma\Big)^{\frac{1}{\sigma}}\Bigg\}\end{aligned}\tag{2.16}$$

Proof. From Lemma 1, applying the Hölder inequality and using the harmonic-convexity of $\left|\mu^{'}\right|^\sigma$, $\sigma>1$, we have

$$\begin{aligned}|\Sigma_u(p_1,p_2)|\leq\alpha_1\alpha_2(\alpha_2-\alpha_1)\Bigg\{&\int_0^{\zeta_u}\frac{|p_1-\kappa|}{\varphi_\kappa^2(\alpha_1,\alpha_2)}\left|\mu^{'}\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)\right|d\kappa\\ +\int_{\zeta_u}^1\frac{|p_2-\kappa|}{\varphi_\kappa^2(\alpha_1,\alpha_2)}\left|\mu^{'}\left(\frac{\alpha_1\alpha_2}{\varphi_\kappa(\alpha_1,\alpha_2)}\right)\right|d\kappa\Bigg\}&\leq\alpha_1\alpha_2(\alpha_2-\alpha_1)\end{aligned}$$

$$\times\left\{\left(\int_0^{\zeta_u}\frac{|p_1-\kappa|^{\frac{\sigma}{\sigma-1}}}{\varphi_\kappa^{\frac{\sigma}{\sigma-1}}(\alpha_1,\alpha_2)}d\kappa\right)^{1-\frac{1}{\sigma}}\left(\int_0^{\zeta_u}\frac{\left[(1-\kappa)\left|\mu'(\alpha_1)\right|^\sigma+\kappa\left|\mu'(\alpha_2)\right|^\sigma\right]}{\varphi_\kappa^\sigma(\alpha_1,\alpha_2)}d\kappa\right)^{\frac{1}{\sigma}}\right.$$

$$\left.+\left(\int_{\zeta_u}^{1}\frac{|p_2-\kappa|^{\frac{\sigma}{\sigma-1}}}{\varphi_\kappa^{\frac{\sigma}{\sigma-1}}(\alpha_1,\alpha_2)}d\kappa\right)^{1-\frac{1}{\sigma}}\left(\int_{\zeta_u}^{1}\frac{\left[(1-\kappa)\left|\mu'(\alpha_1)\right|^\sigma+\kappa\left|\mu'(\alpha_2)\right|^\sigma\right]}{\varphi_\kappa^\sigma(\alpha_1,\alpha_2)}d\kappa\right)^{\frac{1}{\sigma}}\right\}. \tag{2.17}$$

By applying Lemma 4 and using similar arguments as in proving Theorem 4, we get the result given in (2.16). □

Remark 1. *If $0 \le p_1 \le \frac{1}{2} \le p_2 \le 1$, $\sigma = 1$ and $u = \frac{\alpha_1+\alpha_2}{2}$, one can derive some interesting Hermite-Hadamard type and Simpson type inequalities form Theorem 2-Theorem 5.*

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Current address: Department of Mathematics, Deanship of Preparatory Year, University of Hail, Hail 2440, Saudi Arabia

E-mail address: m_amer_latif@hotmail.com

ORCID Address: http://orcid.org/0000-0003-2349-3445

Current address: Department of Mathematics, University of Engineering and Technology Lahore, Pakistan

E-mail address: sabirhus@gmail.com