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Analysis of the Dynamical System $\dot{x}(t) = A\,x(t) + h(t,x(t)),\ x(t_0) = x_0$ in a Special Time-Dependent Norm

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Abstract
As the main new result, we show that one can construct a time-dependent positive definite matrix $R(t,t_0)$ such that the solution $x(t)$ of the initial value problem $\dot{x}(t) = A\,x(t) + h(t,x(t)),\ x(t_0) = x_0$, under certain conditions satisfies the equation $\|x(t)\|_{R(t,t_0)} = \|x_A(t)\|_R$ where $x_A(t)$ is the solution of the above IVP when $h \equiv 0$ and R is a constant positive definite matrix constructed from the eigenvectors and principal vectors of A and A^* and where $\|\cdot\|_{R(t,t_0)}$ and $\|\cdot\|_R$ are weighted norms. Applications are made to dynamical systems, and numerical examples underpin the theoretical findings.

Keywords: Nonlinear initial value problem with linear principal part, Vibration suppression, Monotonicity behavior, Two-sided bounds, Weighted norm
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1. Introduction

In this paper, the solution of the nonlinear initial value problem (for short: IVP) with linear principal part $\dot{x}(t) = Ax(t) + h(t,x(t)),\ x(t_0) = x_0$ is investigated in a special time-dependent weighted norm $\|\cdot\|_{R(t,t_0)}$ with positive definite matrix $R(t,t_0)$. It will be shown that under certain conditions, $R(t,t_0)$ can be constructed such that $\|x(t)\|_{R(t,t_0)} = \|x_A(t)\|_R$ where $x_A(t)$ is the solution of the initial value problem $\dot{x}_A(t) = A\,x_A(t),\ x_A(t_0) = x_0$ and R is a constant positive definite matrix constructed from the eigenvectors and principal vectors of A and A^*. In other words, the solution $x(t)$ in the time-dependent weighted norm $\|\cdot\|_{R(t,t_0)}$ is equal to the solution $x_A(t)$ of the pertinent linear IVP in the weighted norm $\|\cdot\|_R$. As a consequence, since $x_A(t)$ shows vibration suppression and monotonicity behavior under certain conditions, the same holds for $\|x(t)\|_{R(t,t_0)}$. This is the main new result.

The paper is structured as follows. In Section 2, the weighted norm $\|\cdot\|_R$ and, in Section 3, the biorthogonality of eigenvectors and principal vectors of the matrices A and A^* are recapitulated. Section 4 contains two fundamental matrices, namely one for the nonlinear IVP and one for the associated linear IVP. In Section 5, the matrix $R(t,t_0)$ is constructed, and the equation $\|x(t)\|_{R(t,t_0)} = \|x_A(t)\|_R$ is derived. Section 6 contains an expression for $\|x(t)\|_{R(t,t_0)}$ in the norm $\|\cdot\|_2$, and Section 7 two-sided bounds on $\|x(t)\|$ in any vector norm $\|\cdot\|$. In Section 8, Applications to free nonlinear dynamical systems with linear principal part are given including numerical examples. Section 9 is the conclusion section.

2. The weighted norm $\|\cdot\|_R$ revisited

In this section, we revisit the results of [1] concerning the weighted norm $\|\cdot\|_R$, where R is a special positive definite matrix constructed from the eigenvectors and principal vectors of the adjoint A^* of a given system matrix A.

2.1 The case of a diagonalizable matrix A

We first turn to diagonalizable matrices A.

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$ be diagonalizable. Let $\alpha_j = \lambda_j(A)$ be the eigenvalues and u_j be the associated left eigenvectors of A for $j = 1,\cdots,n$; further, let $A^* \in \mathbb{C}^{n\times n}$ be the adjoint matrix of A so that u_j^* are the right eigenvectors of A^* corresponding to the eigenvalues $\overline{\alpha_j}$ of A^* for $j = 1,\cdots,n$, i.e.,*

$$u_j A = \alpha_j u_j,\ j = 1,\cdots,n$$

and

$$A^* u_j^* = \overline{\alpha_j}\, u_j^*,\ j = 1,\cdots,n.$$

Let

$$\rho_j = \overline{\alpha_j} + \alpha_j = 2\,Re\,\alpha_j = 2\,Re\,\overline{\alpha_j},\ j = 1,\cdots,n$$

and

$$R_j = u_j^* u_j,\ j = 1,\cdots,n. \tag{2.1}$$

Then,

$$A^* R_j + R_j A = \rho_j R_j,\ j = 1,\cdots,n.$$

In other words: The matrix eigenvalue problem

$$A^* V + VA = \mu V$$

has the n solution pairs

$$(\mu, V) = (\rho_j, R_j)$$

with real ρ_j and positive semi-definite matrix $R_j \in \mathbb{C}^{n\times n}$ for $j = 1,\cdots,n$. Further,

$$R := \sum_{j=1}^{n} R_j \tag{2.2}$$

is positive definite.

Proof. See [1, Theorems 4 - 6]. □

Remark 2.2. *Since R in (2.2) is positive definite, by*

$$\|u\|_R := (Ru,u)^{\frac{1}{2}},\ u \in \mathbb{C}^n,$$

a weighted norm $\|\cdot\|_R$ is defined.

2.2 The case of a general square matrix A

In this subsection, we consider general square matrices A.

Theorem 2.3. *Let $A \in \mathbb{C}^{n\times n}$ have a canonical Jordan form consisting of r Jordan blocks. Let $\alpha_j = \lambda_j(A)$ be the eigenvalues and $u_1^{(j)},\cdots,u_{m_j}^{(j)}$ be a chain of associated left principal vectors for $j = 1,\cdots,r$. Further, let $A^* \in \mathbb{C}^{n\times n}$ be the adjoint matrix of A so that ${u_1^{(j)}}^*,\cdots,{u_{m_j}^{(j)}}^*$ is a chain of right principal vectors of A^* corresponding to the eigenvalues $\overline{\alpha_j} = \lambda_j(A^*)$ for $j = 1,\cdots,r$, i.e.*

$$u_k^{(j)} A = \alpha_j u_k^{(j)} + u_{k-1}^{(j)}$$

with $u_0^{(j)} = 0,\ k = 1,\cdots,m_j,\ j = 1,\cdots,r$ *and*

$$A^* u_k^{(j)*} = \overline{\alpha_j} u_k^{(j)*} + u_{k-1}^{(j)}{}^*$$

with $u_0^{(j)*} = 0,\ k = 1,\cdots,m_j,\ j = 1,\cdots,r.$

Let

$$\rho_j = \overline{\alpha_j} + \alpha_j = 2\,Re\,\alpha_j = 2\,Re\,\overline{\alpha_j},\ j = 1,\cdots,r$$

and

$$R_j^{(k,k)} := u_k^{(j)*} u_k^{(j)},\ k = 1,\cdots,m_j,\ j = 1,\cdots,r. \tag{2.3}$$

Then,

$$A^* R_j^{(1,1)} + A R_j^{(1,1)} = \rho_j R_j^{(1,1)},\ j = 1,\cdots,r.$$

In other words: The matrix eigenvalue problem

$$A^* V + VA = \mu V$$

has r solution pairs

$$(\mu, V) = (\rho_j, R_j^{(1,1)})$$

with real ρ_j*. Moreover, the matrices* $R_j^{(k,k)}$ *are positive semi-definite for* $k = 1,\cdots,m_j,\ j = 1,\cdots,r$*. Further,*

$$R_j := \sum_{k=1}^{m_j} R_j^{(k,k)}, j = 1,\cdots,r \quad \text{and} \quad R := \sum_{j=1}^{r}\sum_{k=1}^{m_j} R_j^{(k,k)} \tag{2.4}$$

is positive definite.

Proof. See [1, Theorems 7 - 8]. □

Remark 2.4. *With (2.4), also a weighted norm* $\|\cdot\|_R$ *can be defined.*

3. Biorthogonality system of principal vectors of A and A^* revisited

First, we investigate the case of a diagonalizable matrix A and then the case of a general square matrix. Even though the result for a diagonalizable matrix will be included in that for the case of a general square matrix, it seems nevertheless be worthwhile to study this case separately. This is also a review section.

3.1 Diagonalizable matrix A

In this subsection, we summarize a known result on the biorthogonality of the eigenvectors of matrices A and A^*. It can be shown that − for diagonalizable matrices A − the eigenvectors of A and A^* are biorthogonal (so that there is nothing to construct in this case).

For the sequel, we formulate the following *conditions* :

(C1) $A \in \mathbb{C}^{n\times n}$.

(C2) A is diagonalizable, and $\lambda_i,\ i = 1,\cdots,n$ are the eigenvalues of A as well as $p_i,\ i = 1,\cdots,n$ the associated eigenvectors.

(C3) $u_i^*,\ i = 1,\cdots,n$ are the eigenvectors of A^* corresponding to the eigenvalues $\overline{\lambda_i},\ i = 1,\cdots,n$ of A^*.

(C4) $\lambda_i \neq \lambda_j,\ i \neq j,\ i,j = 1,\cdots,n.$

Then, we have the following theorem.

Theorem 3.1. *(Biorthogonality relations of eigenvectors)*

Let the conditions (C1) - (C4) be fulfilled. Then, after appropriate normalization of the eigenvectors $p_i,\ i = 1,\cdots,n$ and $u_i^*, i = 1,\cdots,n$, one has the biorthogonality relations

$$(p_i, u_j^*) = \delta_{ij},\ i,j = 1,\cdots,n, \tag{3.1}$$

where $(\cdot,\cdot)$ is the usual scalar product on $\mathbb{C}^n \times \mathbb{C}^n$.

Proof. See [2, Theorem 1]. □

Remark 3.2. *The condition (C4) is not essential so that it can be omitted. For this, see [3, Theorem 3]. But, we keep it because it is fulfilled in our Numerical Example 1 in Section 8.*

3.2 General square matrix A

In this subsection (more precisely, in Theorem 3.3), we exploit the fact that a principal vector of stage k of matrix A resp. A^* remains a principal vector of stage k if one adds a linear combination of principal vectors of stages 1 to $k-1$ of A resp. A^*, as the case may be. Hereby, we can construct a biorthogonal set of principal vectors of A resp. A^* (provided that they are not already biorthogonal, in which case there is nothing to construct).

Like in Subsection 3.1, we formulate the following *conditions* :

$(C1')$ $A \in \mathbb{C}^{n\times n}$.

$(C2')$ $\lambda_i,\ i = 1,\cdots,r$ are the eigenvalues of A corresponding to the Jordan blocks $J_i(\lambda_i) \in \mathbb{C}^{m_i\times m_i},\ i = 1,\cdots,r$ with the chains of principal vectors $p_1^{(i)},\cdots,p_{m_i}^{(i)},\ i = 1,\cdots,r$.

$(C3')$ $u_1^{(i)^*},\cdots,u_{m_i}^{(i)^*},\ i = 1,\cdots,r$ are the principal vectors of A^* corresponding to the eigenvalues $\overline{\lambda_i},\ i = 1,\cdots,r$ of the Jordan blocks $J_i(\overline{\lambda_i}) \in \mathbb{C}^{m_i\times m_i},\ i = 1,\cdots,r$.

$(C4')$ $\lambda_i \neq \lambda_j,\ i \neq j,\ i,j = 1,\cdots,r$.

Then, we have

Theorem 3.3. *(Biorthogonality relations for principal vectors)*

Let the conditions $(C1')-(C4')$ be fulfilled. Then, the systems $\{p_1^{(l)},\cdots,p_{m_1}^{(l)};\cdots;p_1^{(r)},\cdots,p_{m_r}^{(r)}\}$ and $\{u_1^{(1)^*},\cdots,u_{m_1}^{(1)^*};\cdots;u_1^{(r)^*},\cdots,u_{m_r}^{(r)^*}\}$ can be constructed such that the following biorthogonality relations hold:

$$(p_k^{(i)}, u_l^{(i)^*}) = \begin{cases} 1, & l = m_i - k + 1 \\ 0, & l \neq m_i - k + 1 \end{cases}$$

$k = 1,\cdots,m_i,\ i = 1,\cdots,r$ *and*

$$(p_k^{(i)}, u_l^{(j)^*}) = 0,\ i \neq j,$$

$k = 1,\cdots,m_i,\ l = 1,\cdots,m_j,\ i,j = 1,\cdots,r.$

So, with

$$v_l^{(i)^*} := u_{m_i-l+1}^{(i)}{}^*, \tag{3.2}$$

$l = 1,\cdots,m_i,\ i = 1,\cdots,r$ *one has the biorthogonality relations*

$$(p_k^{(i)}, v_l^{(i)^*}) = \delta_{kl}, \tag{3.3}$$

$k,l = 1,\cdots,m_i,\ i = 1,\cdots,r$, *and*

$$(p_k^{(i)}, v_l^{(j)^*}) = 0,\ i \neq j, \tag{3.4}$$

$k = 1,\cdots,m_i,\ l = 1,\cdots,m_j,\ i,j = 1,\cdots,r.$

Proof. See [2, Theorem 2]. □

Remark 3.4. *The properties (3.3) and (3.4) can also be written as*

$$(p_k^{(i)}, v_l^{(j)^*}) = \delta_{ij}\delta_{kl}, \tag{3.5}$$

$k = 1,\cdots,m_i,\ i = 1,\cdots,r;\ l = 1,\cdots,m_j,\ j = 1,\cdots,r.$

4. Representations of the solution $x(t)$ of the IVP $\dot{x}(t) = Ax(t) + h(t,x(t)),\ t \ge t_0,\ x(t_0) = x_0$ and of $x_A(t)$ when $h \equiv 0$ by fundamental matrices

In the following, we discuss the existence, uniqueness, and boundedness of the solution of the initial value problem $\dot{x}(t) = Ax(t) + h(t,x(t)),\ t \ge t_0,\ x(t_0) = x_0$ as well as pertinent representations of $x(t)$ and $x_A(t)$ by use of fundamental matrices.

Let $t_0 \in \mathbb{R}_0^+$, let $\mathbb{F}$ be the field of real or complex numbers and $\mathbb{F}^n$ be the set of n-tuples with elements in $\mathbb{F}$. Further, let $\|\cdot\|$ be a norm on $\mathbb{F}^n$, let $\gamma > 0$ and $\mathbb{F}^n_\gamma = \{u \in \mathbb{F}^n \,|\, \|u\| \le \gamma\}$. Finally, let $A \in \mathbb{F}^{n\times n}$ and $h(t,u) \in \mathbb{F}^n$, $t \ge t_0$, $u \in \mathbb{F}^n_\gamma$, and continuous. We investigate the initial value problem

$$\dot{x}(t) = Ax(t) + h(t,x(t)),\ t \ge t_0,\ x(t_0) = x_0. \tag{4.1}$$

Let $\lambda_j(A),\ j = 1,\cdots,n$ be the eigenvalues of matrix A. The *spectral abscissa* $\nu[A]$ is defined as the maximum of the real parts of the eigenvalues, i.e.,

$$\nu[A] = \max_{j=1,\cdots,n} Re\,\lambda_j(A).$$

We suppose that $\nu[A] < 0$. Further, if the eigenvalues of matrix A play a role, we implicitly assume that $\mathbb{F} = \mathbb{C}$. For $\nu_{x_0}[A]$ and the index $\iota(\lambda(A))$ of an eigenvalue $\lambda(A)$, we refer the reader, e.g., to [4].

Let $\Phi_A(t,t_0)$ be the *fundamental matrix* (or *evolution*) pertinent to the problem $\dot{x}_A(t) = Ax_A(t)$ with the property $\Phi_A(t_0,t_0) = E$, where $E \in \mathbb{F}^{n\times n}$ is the identity matrix. Then, the initial value problem is equivalent to the integral equation

$$x(t) = \Phi_A(t,t_0)x_0 + \int_{t_0}^{t} \Phi_A(t,s)\,h(s,x(s))\,ds,\ t \ge t_0. \tag{4.2}$$

This is a common implicit representation of $x(t)$ using $\Phi_A(t,t_0) = \exp(A\,(t-t_0)$. But, we shall employ a different explicit one, below.

For the sequel, we state the following conditions:

(C_0) The function $h(\cdot,\cdot)$ is continuous on $D_\gamma := \{(t,u)\,|\,t \ge t_0,\ u \in \mathbb{F}^n_\gamma\} = \{(t,u)\,|\,t \ge t_0,\ u \in \mathbb{F}^n,\ \|u\| \le \gamma\}$.

(C_1) The function $h(\cdot,\cdot)$ satisfies the (uniform) Lipschitz condition

$$\|h(t,u) - h(t,u')\| \le L_h\,\|u - u'\|,\ t \ge t_0,\ u,u' \in \mathbb{F}^n_\gamma$$

with a positive constant L_h.

(C_2) For every $u \in \mathbb{F}^n_\gamma$,

$$\lim_{u\to 0} \frac{\|h(t,u)\|}{\|u\|} = 0 \qquad \text{uniformly with respect to t} \ge \text{t}_0.$$

Herewith, we have the following theorem:

Theorem 4.1. *(Existence, uniqueness, and boundedness of the solution)*

Let the conditions (C_0),(C_1), and (C_2) be fulfilled. Further, let the spectral abscissa $\nu[A] < 0$, and $x_0 \ne 0$, $\|x_0\|$ as well as L_h be sufficiently small.

Then, integral equation (4.2) and thus initial value problem (4.1) has a unique bounded solution for all $t \ge t_0$.

Proof. See [5, Theorem 1]. □

Remark 4.2. *Sufficient for (C_2) is the following condition:*

(C_2') There exists a constant $c_h > 0$ such that

$$\|h(t,u)\| \le c_h\,\|u\|^\kappa, ; t \ge t_0,\ u \in \mathbb{F}^n,\ \|u\| \le \gamma,$$

with $\kappa > 1$.

We need this stronger condition for the derivation of a lower bound on the solution $x(t)$ of (4.1).

Now, let $\Phi_{A,h}(t,t_0)$ be the fundamental matrix with $\Phi_{A,h}(t_0,t_0) = E$ pertinent to the IVP (4.1). Then, the representation of $x(t)$ using $\Phi_{A,h}(t,t_0)$ is given by

$$x(t) = \Phi_{A,h}(t,t_0)\,x_0. \tag{4.3}$$

The representation (4.3) for the solution $x(t)$ of the IVP (4.1) plays a major role in the subsequent sections.

5. Representation of the solution vector $x(t)$ of $\dot{x}(t) = Ax(t) + h(t,x(t)),\ x(t_0) = x_0$ in the weighted time-dependent norm $\|\cdot\|_{R(t,t_0}$

Let the conditions (C_0), (C_1), and (C_2') from Section 4 be fulfilled. We remind that the solution of

$$\dot{x}_A(t) = A\,x_A(t),\ t \geq t_0,\ x_A(t_0) = x_0 \tag{5.1}$$

can be written as

$$x_A(t) = \Phi_A(t,t_0)\,x_0 = e^{A(t-t_0)}\,x_0. \tag{5.2}$$

This is the representation of $x_A(t)$ by the fundamental matrix that plays a role, in the sequel. From (4.3), it follows

$$x(t) = \Phi_{A,h}(t,t_0)\,e^{-A(t-t_0)}\,e^{A(t-t_0)}\,x_0 = \Psi(t,t_0)\,e^{A(t-t_0)}\,x_0 \tag{5.3}$$

with

$$\Psi(t,t_0) := \Phi_{A,h}(t,t_0)\,e^{-A(t-t_0)}. \tag{5.4}$$

Thus,

$$x(t) = \Psi(t,t_0)x_A(t). \tag{5.5}$$

5.1 The case of a diagonalizable matrix A

Let the conditions $(C1)-(C4)$ be fulfilled. Then, according to [2, Theorem 5], one has the representation

$$x_A(t) = \sum_{k=1}^{n}(x_0,u_k^*)\,p_k\,e^{\lambda_k(t-t_0)},\ t \geq t_0, \tag{5.6}$$

with

$$A\,p_k = \lambda_k\,p_k,\ k = 1,\cdots,n$$

where $p_k, k = 1,\cdots,n$ and u_k^*, $k = 1,\cdots,n$ are biorthogonal, that is, where (3.1) is satisfied. Inserting (5.6) into (5.5) gives

$$x(t) = \sum_{k=1}^{n}(x_0,u_k^*)\,\Psi(t,t_0)\,p_k\,e^{\lambda_k(t-t_0)} = \sum_{k=1}^{n}(x_0,u_k^*)\ p_k(t,t_0)\,e^{\lambda_k(t-t_0)},\ t \geq t_0$$

with

$$p_k(t,t_0) := \Psi(t,t_0)\,p_k. \tag{5.7}$$

Define

$$P(t,t_0) := [p_1(t,t_0),\cdots,p_n(t,t_0))]. \tag{5.8}$$

Then,

$$P^{-1}(t,t_0)P(t,t_0) = E, \tag{5.9}$$

where E is the identity matrix. Set

$$P^{-1}(t,t_0) =: U(t,t_0) =: \begin{bmatrix} u_1(t,t_0) \\ u_2t,t_0) \\ \cdots \\ u_nt,t_0) \end{bmatrix} \tag{5.10}$$

where $u_j(t,t_0)$, $j = 1,\cdots,n$ are row vectors of length n.

From (5.8), (5.9), and (5.10), we have

$$u_j(t,t_0)\,p_k(t,t_0) = \delta_{jk}$$

or

$$(p_k(t,t_0), u_j^*(t,t_0)) = \delta_{jk}.$$

With (5.7), this leads to

$$(\Psi(t,t_0)\, p_k, u_j^*(t,t_0)) = \delta_{jk}$$

or

$$(p_k, \Psi^*(t,t_0)\, u_j^*(t,t_0)) = \delta_{jk}.$$

On the other hand, also

$$(p_k, u_j^*) = \delta_{jk}.$$

Subtracting both relations implies

$$(p_k, u_j^* - \Psi^*(t,t_0)\, u_j^*(t,t_0)) = 0,\ j,k = 1,\cdots,n$$

and thus

$$u_j^* - \Psi^*(t,t_0)\, u_j^*(t,t_0) = 0,\ j = 1,\cdots,n$$

or

$$u_j^*(t,t_0) = [\Psi^*(t,t_0)]^{-1}\, u_j^* = [\Psi^{-1}(t,t_0)]^*\, u_j^*,\ j = 1,\cdots,n.$$

This leads to

$$u_j(t,t_0) = u_j\, \Psi^{-1}(t,t_0),\ j = 1,\cdots,n.$$

Now, define

$$R_j(t,t_0) = u_j^*(t,t_0)\, u_j(t,t_0),\ j = 1,\cdots,n \tag{5.11}$$

and

$$R(t,t_0) = \sum_{j=1}^{n} R_j(t,t_0). \tag{5.12}$$

It is left to the reader to show that $R_j(t,t_0),\ j = 1,\cdots,n$ are positive semi-definite and that $R(t,t_0)$ is positive definite.

With (5.11) and (5.12), we obtain

$$\begin{aligned}
\|x(t)\|^2_{R(t,t_0)} &= (R(t,t_0)\,x(t), x(t)) = \sum_{j=1}^{n} (R_j(t,t_0)\,x(t), x(t)) \\
&= \sum_{j=1}^{n} ([\Psi^{-1}(t,t_0)]^*\, u_j^*\, [u_j\, \Psi^{-1}(t,t_0)]\, x(t), x(t)) \\
&= \sum_{j=1}^{n} (u_j^*\, u_j\, [\Psi^{-1}(t,t_0)\, x(t)], [\Psi^{-1}(t,t_0)\, x(t)]) \\
&= \sum_{j=1}^{n} (R_j\, x_A(t), x_A(t)) = (R\, x_A(t), x_A(t)) \\
&= \|x_A(t)\|^2_R
\end{aligned}$$

so that we have

Theorem 5.1. *Let the conditions (C_0), (C_1), and (C_2') be fulfilled. Further, let the spectral abscissa $\nu[A] < 0$ and $x_0 \neq 0$, $\|x_0\|$ as well as L_h be sufficiently small. Moreover, let the conditions (C1) - C4) be satisfied.*

Then, with (2.2) and (5.12),

$$\|x(t)\|^2_{R(t,t_0)} = \|x_A(t)\|^2_R$$

where, according to [1, (47)],

$$\|x_A(t)\|^2_R = \sum_{i=1}^{n} \|x_0\|^2_{R_i} e^{2\,\lambda_i(A)\,(t-t_0)},\; t \geq t_0.$$

In other words: The solution $x(t)$ to the nonlinear problem (4.1) in the time-dependent weighted norm $\|\cdot\|_{R(t,t_0)}$ is equal to the solution $x_A(t)$ of the pertinent linear problem (5.1) in the norm $\|\cdot\|_R$.

Remark 5.2. *We mention that the vectors $p_i, i = 1,\cdots,n$ and $u_i^*, i = 1,\cdots,n$ themselves need not be normed. For the representation (5.6), we only have to demand that relation (3.1) be satisfied.*

5.2 The case of a general square matrix A

Let the conditions $(C1') - (C4')$ from Section 3 be fulfilled. The relations (5.1)-(5.5) remain valid. But, here, instead of (5.6), according to [2, Theorem 6], we have

$$x_A(t) = \sum_{i=1}^{r}\sum_{k=1}^{m_i} (x_0, u^{(i)}_{m_i-k+1}{}^*) x^{(i)}_k(t),\; t \geq t_0 \tag{5.13}$$

with

$$x^{(i)}_k(t) = [p^{(i)}_1 \frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + p^{(i)}_{k-1}(t-t_0) + p^{(i)}_k]\, e^{\lambda_i(t-t_0)},\; t \geq t_0, \tag{5.14}$$

$k = 1,\cdots,m_i,\; i = 1,\cdots,r$ where $p^{(i)}_k,\; k = 1,\cdots,m_i,\; i = 1,\cdots,r$ and $v^{(j)}_l{}^* := u^{(j)}_{m_j-l+1}{}^*,\; l = 1,\cdots,m_j,\; j = 1,\cdots,r$ from (3.2) satisfy (3.5).

In the semi-norm $\|\cdot\|_{R^{(k,k)}_i}$, $x_A(t)$ has, according to [1, 4.2,(56), (57)], the form

$$\|x(t)\|^2_{R^{(k,k)}_i} = |p^{(i)}_{x_0,k-1}(t-t_0)|^2\, e^{2\,\lambda_i(A)(t-t_0)}\;, t \geq t_0$$

with $p^{(i)}_{x_0,k-1}(t-t_0)$ in (5.21) below. Next, we proceed as in Section 5.1. From (5.5) and (5.13), (5.14) we conclude that

$$\begin{aligned} x(t) &= \Psi(t,t_0)\,x_A(t) = \sum_{i=1}^{r}\sum_{k=1}^{m_i} (x_0, u^{(i)}_{m_i-k+1}{}^*) \times \\ &\quad [\Psi(t,t_0)\,p^{(i)}_1 \tfrac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + \Psi(t,t_0)\,p^{(i)}_{k-1}(t-t_0) + \Psi(t,t_0)\,p^{(i)}_k]\, e^{\lambda_i(t-t_0)} \end{aligned}$$

so that

$$\begin{aligned} x(t) &= \sum_{i=1}^{r}\sum_{k=1}^{m_i} (x_0, u^{(i)}_{m_i-k+1}{}^*) \times \\ &\quad [p^{(i)}_1(t,t_0) \tfrac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + p^{(i)}_{k-1}(t,t_0)\,(t-t_0) + p^{(i)}_k(t,t_0)]\, e^{\lambda_i(t-t_0)} \end{aligned}$$

with

$$p^{(i)}_j(t,t_0) = \Psi(t,t_0)\, p^{(i)}_j,\; j = 1,\cdots,m_i, i = 1,\cdots,r. \tag{5.15}$$

Next, define

$$P(t,t_0) := [p^{(1)}_1(t,t_0),\cdots,p^{(1)}_{m_1}(t,t_0);\cdots;p^{(r)}_1(t,t_0),\cdots,p^{(r)}_{m_r}(t,t_0)]. \tag{5.16}$$

Then,

$$P^{-1}(t,t_0)\,P(t,t_0) = E. \tag{5.17}$$

Set

$$P^{-1}(t,t_0) =: V(t,t_0) =: \left[\begin{array}{c} v_1^{(1)}(t,t_0) \\ \vdots \\ v_{m_1}^{(1)}(t,t_0) \\ \hline \vdots \\ \hline v_1^{(r)}(t,t_0) \\ \vdots \\ v_{m_r}^{(r)}(t,t_0) \end{array}\right]$$

where $v_k^{(j)}(t,t_0)$ are row vectors of length n. From (5.15), (5.16), and (5.17), we have

$$v_k^{(i)}(t,t_0)\,p_s^{(j)}(t,t_0) = \delta_{ij}\,\delta_{ks},$$

$k = 1,\cdots,m_i,\ 0 = 1,\cdots,r;\ s = 1,\cdots,m_j,\ j = 1,\cdots,r$ or

$$(p_s^{(j)}(t,t_0), v_k^{(i)^*}(t,t_0)) = \delta_{ij}\,\delta_{ks}.$$

With (5.15), this leads to

$$(\Psi(t,t_0)\,p_s^{(j)}, v_k^{(i)^*}(t,t_0)) = \delta_{ij}\,\delta_{ks}$$

or

$$(p_s^{(j)}, \Psi^*(t,t_0)\,v_k^{(i)^*}(t,t_0)) = \delta_{ij}\,\delta_{ks}.$$

On the other hand, also

$$(p_s^{(j)}, v_k^{(i)^*}) = \delta_{ij}\,\delta_{ks}.$$

Subtracting both relations implies

$$(p_s^{(j)}, v_k^{(i)^*} - \Psi^*(t,t_0)\,v_k^{(i)^*}(t,t_0)) = 0,\ j,k = 1,\cdots,n$$

and thus

$$v_k^{(i)^*} - \Psi^*(t,t_0)\,v_k^{(i)^*}(t,t_0) = 0,\ k = 1,\cdots,m_i,\ i = 1,\cdots,r$$

or

$$v_k^{(i)^*}(t,t_0) = [\Psi^*(t,t_0))]^{-1}\,v_k^{(i)^*} = [\Psi^{-1}(t,t_0)]^*\,v_k^{(i)^*},$$

$k = 1,\cdots,m_i,\ i = 1,\cdots,r$. This leads to

$$v_k^{(i)}(t,t_0) = v_k^{(i)}\,\Psi^{-1}(t,t_0)$$

$k = 1,\cdots,m_i,\ i = 1,\cdots,r$. Similarly to (3.2), define

$$v_l^{(i)^*}(t,t_0) := {u_{m_i-l+1}^{(i)}}^*(t,t_0),$$

as well as

$$R_j^{(k,k)}(t,t_0) = u_j^*(t,t_0)\,u_k^{(i)}(t,t_0) \tag{5.18}$$

$k = 1,\cdots,m_i,\ i = 1,\cdots,r,$

$$R_j(t,t_0) = \sum_{k=1}^{m_i} R_j^{(k,k)}(t,t_0),\ j = 1,\cdots,n \tag{5.19}$$

and

$$R(t,t_0) = \sum_{j=1}^{r} R_j(t,t_0). \tag{5.20}$$

Again, it is left to the reader to show that $R_j^{(k,k)}(t,t_0)$, $R_j(t,t_0)$ are positive semi-definite and that $R(t,t_0)$ is positive definite.

Herewith,

$$\begin{aligned}
\|x(t)\|^2_{R(t,t_0)} &= (R(t,t_0)x(t),x(t)) = \sum_{j=1}^{r}\sum_{k=1}^{m_i}(R_j^{(k,k)}(t,t_0)x(t),x(t)) \\
&= \sum_{j=1}^{r}\sum_{k=1}^{m_i}([\Psi^{-1}(t,t_0)]^* u_l^{(i)^*}[u_l^{(i)}\Psi^{-1}(t,t_0)]x(t),x(t)) \\
&= \sum_{j=1}^{r}\sum_{k=1}^{m_i}(u_l^{(i)^*}u_l^{(i)}[\Psi^{-1}(t,t_0)x(t)],[\Psi^{-1}(t,t_0)x(t)]) \\
&= \sum_{j=1}^{r}\sum_{k=1}^{m_i}(R_j^{(k,k)}x_A(t),x_A(t)) \\
&= \sum_{j=1}^{r}(R_j(t,t_0)x_A(t),x_A(t)) \\
&= (Rx_A(t),x_A(t)) \\
&= \|x_A(t)\|^2_R
\end{aligned}$$

so that we have

Theorem 5.3. *Let the conditions (C_0), (C_1), and (C_2') be fulfilled. Further, let the spectral abscissa $\nu[A] < 0$ and $x_0 \neq 0$, $\|x_0\|$ as well as L_h be sufficiently small. Moreover, let the conditions $(C1') - (C4')$ be satisfied.*

Then, with (2.4) and (5.20),

$$\|x(t)\|^2_{R(t,t_0)} = \|x_A(t)\|^2_R$$

where, according to [1, (57), (56)],

$$\|x_A(t)\|^2_R = \sum_{j=1}^{r}\sum_{k=1}^{m_i}|p^{(i)}_{x_0,k-1}(t-t_0)|^2 e^{2\lambda_i(A)(t-t_0)},\ t \geq t_0,$$

with

$$p^{(i)}_{x_0,k-1}(t-t_0) := (x_0, u_1^{(i)^*}\frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + u_{k-1}^{(i)^*}(t-t_0) + u_k^{(i)^*}). \tag{5.21}$$

In other words: The solution $x(t)$ to the nonlinear problem (4.1) in the time-dependent weighted norm $\|\cdot\|_{R(t,t_0)}$ is equal to the solution $x_A(t)$ of the pertinent linear problem (5.1) in the norm $\|\cdot\|_R$.

Remark 5.4. *We mention that the vectors $p_k^{(i)}$, $k = 1,\cdots,m_i$, $i = 1,\cdots,r$ and $v_k^{(i)^*}$, $i = 1,\cdots,r$ themselves need not be normed. For the representation (5.13), (5.14), we only have to demand that relation (3.5) be satisfied.*

6. An expression for $\|x(t)\|_{R(t,t_0)}$ in the norm $\|\cdot\|_2$

According to Section 5, under the respective conditions, one has

$$\|x(t)\|^2_{R(t,t_0)} = \|x_A(t)\|^2_R.$$

As a consequence of this, one obtains a series of corollaries. The first one follows from [6, Section 3].

6.1 The case of a diagonalizable matrix A

We first turn to diagonalizable matrices A.

Corollary 6.1. *Let the conditions (C_0), (C_1), and (C_2') as well as conditions (C1) - C4) be satisfied. Let R_j be given by (2.1) and R by (2.2) as well as $R_j(t,t_0)$ by (5.11) and $R(t,t_0)$ by (5.12). Further, let*

$$\psi_j(t) := (x_0, u_j^*)\, e^{Re\,\lambda_j(A)(t-t_0)},\ t \geq t_0, \tag{6.1}$$

$j = 1, \cdots, n$, *as well as*

$$\psi(t) := [\psi_1(t), \psi_2(t), \cdots, \psi_n(t)]^T. \tag{6.2}$$

Then,

$$|(x_0, u_j^*)| = \|x_0\|_{R_j},$$

$j = 1, \cdots, n$, *and*

$$\|x(t)\|_{R(t,t_0)} = \|x_A(t)\|_R = \|\psi(t)\|_2,\ t \geq t_0.$$

Proof. See proof of [6, Lemma 3]. □

6.2 The case of a general square matrix A

In this subsection, we consider the general square matrices A.

Corollary 6.2. *Let the conditions (C_0), (C_1), and (C_2') as well as conditions $(C1') - (C4')$ be fulfilled. Let $R_j^{(k,k)}$, R_j, and R be given by (2.3) and (2.4), respectively, as well as $R_j^{(k,k)}(t,t_0)$, $R_j(t,t_0)$, and $R(t,t_0)$ by (5.18), (5.19), and (5.20), as the case may be. Moreover, let $p_{x_0,k-1}^{(j)}(t-t_0)$ be given by (5.21), and let*

$$\psi_k^{(j)}(t) := p_{x_0,k-1}^{(j)}(t-t_0)\, e^{Re\,\lambda_j(A)\,(t-t_0)}, \tag{6.3}$$

$k = 1, \cdots, m_j$, $j = 1, \cdots, r$, *as well as*

$$\psi^{(j)}(t) := [\psi_1^{(j)}(t), \cdots, \psi_{m_j}^{(j)}(t)]^T, \tag{6.4}$$

$i = 1, \cdots, r$ *and*

$$\psi(t) := \begin{bmatrix} \psi^{(1)}(t) \\ \psi^{(2)}(t) \\ \vdots \\ \psi^{(r)}(t) \end{bmatrix}. \tag{6.5}$$

Then,

$$|(x_0, u_k^{(j)*})| = \|x_0\|_{R_j^{(k,k)}},$$

$k = 1, \cdots, m_j$, $j = 1, \cdots, r$, *and*

$$\|x(t)\|_{R(t,t_0)} = \|x_A(t)\|_R = \|\psi(t)\|_2,\ t \geq t_0.$$

Proof. See [6, Lemma 4]. □

7. Two-sided bounds on $x(t)$ in any norm $\|\cdot\|$ based on $\psi(t)$

In [6, Sections 4.1 and 4.2], under certain conditions, we have established the two-sided bounds

$$X_0\,\|\psi(t)\| \le \|x_A(t)\| \le X_1\,\|\psi(t)\|,\ t \ge t_0, \tag{7.1}$$

with $\psi(t)$ from (6.1), (6.2) for the case of diagonalizable matrices A and with $\psi(t)$ from (6.3)-(6.5) for general square matrices A.

The same two-sided bounds will be derived for $x(t)$ instead of $x_A(t)$. As a preparation for this, we prove the following lemma.

Lemma 7.1. *(Two-sided bound on $\|x(t)\|$ by $\|x_A(t)\|$)*

Let the conditions (C_0), (C_1), and (C_2') be satisfied; further, let $\nu_{x_0}[A] = \nu[A]$ and let the spectral abscissa $\nu[A]$ of matrix A be negative and for every eigenvalue $\lambda(A)$ with $Re\lambda(A) = \nu[A]$ let the index $\iota(\lambda(A))$ be $\iota(\lambda(A)) = 1$, let $x_0 \neq 0$, and $\|x_0\|$ as well as c_h and L_h be sufficiently small.

Then, there exist positive constants X_0 and X_1 such that

$$X_0\,\|x_A(t)\| \le \|x(t)\| \le X_1\,\|x_A(t)\|,\ t \ge t_0. \tag{7.2}$$

Proof. From [5, Corollary 4], we obtain

$$X_{A,h,0}\,e^{\nu[A]\,(t-t_0)} \le \|x(t)\| \le X_{A,h,1}\,e^{\nu[A]\,(t-t_0)},\ t \ge t_0 \tag{7.3}$$

with $x(t) = \Phi_{A,h}(t,t_0)\,x_0$ and positive constants $X_{A,h,0}$ and $X_{A,h,1}$. For the special case $h \equiv 0$, this leads to

$$X_{A,0}\,e^{\nu[A]\,(t-t_0)} \le \|x_A(t)\| \le X_{A,1}\,e^{\nu[A]\,(t-t_0)},\ t \ge t_0$$

with $x_A(t) = \Phi_A(t,t_0)\,x_0$ and positive constants $X_{A,0}$ and $X_{A,1}$ or

$$\frac{1}{X_{A,1}}\,\|x_A(t)\| \le e^{\nu[A]\,(t-t_0)} \le \frac{1}{X_{A,0}}\,\|x_A(t)\|,\ t \ge t_0. \tag{7.4}$$

From (7.3) and (7.4), we conclude that ((7.2) is valid with

$$X_0 = \frac{X_{A,h,0}}{X_{A,1}}$$

and

$$X_1 = \frac{X_{A,h,1}}{X_{A,0}}.$$

□

7.1 The case of a diagonalizable matrix A

In order to obtain the two-sided bounds in (7.1) with $x(t)$ instead of $x_A(t)$, we first turn to diagonalizable matrices A. Here, we have

Corollary 7.2. *Let the conditions (C_0), (C_1), and (C_2') as well as conditions (C1) - (C4) be satisfied and let $\|\cdot\|$ be any vector norm. Let $\psi(t)$ be defined by (6.1) and (6.2). Let $x(t)$ be the solution of the initial value problem (4.1). Then, there exist positive constants X_0 and X_1 such that*

$$X_0\|\psi(t)\| \le \|x(t)\| \le X_1\|\psi(t)\|,\ t \ge t_0\,. \tag{7.5}$$

Proof. The proof of (7.5) follows from Lemma 7.1 and relation (7.1) which, in turn, is stated in [6, Section 4.1, Theorem 5]. □

Remark 7.3. *The two-sided bound (7.5) turns out to be much better than (7.3).*

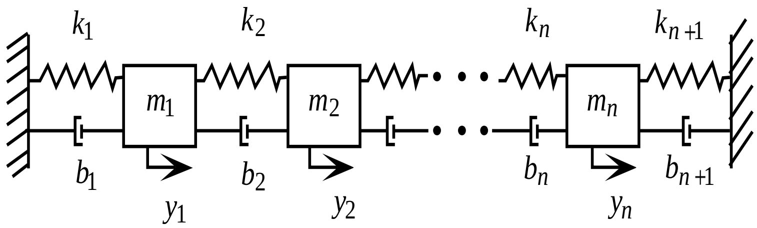

Figure 8.1. Multi-mass vibration model.

7.2 The case of a general square matrix A

In this subsection, we consider the general square matrices A.

Corollary 7.4. *Let the conditions* (C_0), (C_1), *and* (C_2') *as well as conditions* $(C1') - (C4')$ *be fulfilled, and let* $\|\cdot\|$ *be any vector norm. Let* $p^{(i)}_{x_0,k-1}(t-t_0)$ *be given by* (5.21) *and* $\psi(t)$ *by* (6.3)-(6.5). *Let* $x(t)$ *be the solution of the initial value problem* (4.1). *Then, there exist positive constants* X_0 *and* X_1 *such that*

$$X_0\|\psi(t)\| \leq \|x(t)\| \leq X_1\|\psi(t)\|,\ t \geq t_0\,. \tag{7.6}$$

Proof. The proof of (7.6) follows from Lemma 7.1 and relation (7.1) which, in turn, is stated in [6, Section 4.2, Theorem 6]. □

Remark 7.5. *The two-sided bound* (7.6) *turns out to be much better than* (7.3).

8. Applications to free nonlinear dynamical systems with linear principal part

In this section, we consider applications to free nonlinear dynamical systems with linear principal part represented by a mechanical multi-mass vibratory system. Both the case of a diagonalizable and the case of a non-diagonalizable system matrix A is considered. For both cases, numerical examples illustrate the obtained results.

8.1 The multi-mass vibration model with nonlinear stiffness functions

We consider the multi-mass vibration model in Figure 8.1.

Here, k_i means the nonlinear stiffness function

$$k_i(v) = k_i^{(0)}\,(v + \eta\, v^3),\ v \in \mathbb{R}$$

with positive constants $k_i^{(0)}$, $i = 1, \cdots, n+1$ and with some parameter $\eta \geq 0$. For $\eta = 0$, we obtain a linear model, and otherwise a nonlinear model.

The equation of motion in vector form is given by

$$M\,\ddot{y} + B\,\dot{y} + q(y) = 0, \quad y(0) = y_0,\, \dot{y}(0) = \dot{y}_0$$

with

$$q(y) = K^{(0)}\,y + \eta\, q^{(3)}(y)$$

and

$$q^{(3)}(y) = \begin{bmatrix} k_1^{(0)}(y_1 - y_0)^3 - k_2^{(0)}(y_2 - y_1)^3 \\ k_2^{(0)}(y_2 - y_1)^3 - k_3^{(0)}(y_3 - y_2)^3 \\ \vdots \\ k_{n-1}^{(0)}(y_{n-1} - y_{n-2})^3 - k_n^{(0)}(y_n - y_{n-1})^3 \\ k_n^{(0)}(y_n - y_{n-1})^3 - k_{n+1}^{(0)}(y_{n+1} - y_n)^3 \end{bmatrix},$$

where $y_0 = y_{n+1} = 0$; the matrices M, B and $K^{(0)}$ are given by

$$M = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & m_3 & & \\ & & & \ddots & \\ & & & & m_n \end{bmatrix},$$

$$B = \begin{bmatrix} b_1+b_2 & -b_2 & & & & \\ -b_2 & b_2+b_3 & -b_3 & & & \\ & -b_3 & b_3+b_4 & -b_4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -b_{n-1} & b_{n-1}+b_n & -b_n \\ & & & & -b_n & b_n+b_{n+1} \end{bmatrix},$$

$$K^{(0)} = \begin{bmatrix} k_1^{(0)}+k_2^{(0)} & -k_2^{(0)} & & & & \\ -k_2^{(0)} & k_2^{(0)}+k_3^{(0)} & -k_3^{(0)} & & & \\ & -k_3^{(0)} & k_3^{(0)}+k_4^{(0)} & -k_4^{(0)} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -k_{n-1}^{(0)} & k_{n-1}^{(0)}+k_n^{(0)} & -k_n^{(0)} \\ & & & & -k_n^{(0)} & k_n+k_{n+1}^{(0)} \end{bmatrix}$$

with the *mass, damping,* and *stiffness matrices* M, B, and $K^{(0)}$, as the case may be, and the *displacement vector* y. In *state-space description*, this problem takes the form

$$\dot{x}(t) = Ax(t) + h(t,x(t)), t \geq 0, x(0) = x_0$$

with $x = [y^T, z^T]^T$, $z = \dot{y}$, and where the *system matrix* A is given by

$$A = \left[\begin{array}{c|c} 0 & E \\ \hline -M^{-1}K^{(0)} & -M^{-1}B \end{array}\right]$$

and h by

$$h(t,u) = \eta \begin{bmatrix} 0 \\ \\ -M^{-1}\, q^{(3)}(v) \end{bmatrix} =: h_0(u)$$

with $t \geq 0$, $u = [v^T, w^T]^T$, $v, w \in \mathbb{F}^n$. We mention that $x, u \in \mathbb{F}^m$ and $A \in \mathbb{F}^{m \times m}$ with $m = 2n$. From [5], it follows that

$$\|h(t,u)\| = \|h_0(u)\| \leq c_h \|u\|^3, t \geq 0, \quad u \in \mathbb{F}^m,$$

where $c_h = \eta\, \tilde{c}_h$ with a constant $\tilde{c}_h$ independent of η as well as

$$\|h(t,u) - h(t,u')\| \leq L_h \|u - u'\|, t \geq 0,\ u, u' \in \mathbb{F}^m_\gamma,$$

where $L_h = \eta\, \tilde{L}_h$, with a constant $\tilde{L}_h$ independent of η.

8.2 Numerical examples

Numerical Example 1: Matrix A diagonalizable

(i) *Data:*

The values $m_j,\ j = 1,\cdots,n$ and $b_j,\ k_j^{(0)},\ j = 1,\cdots,n+1$ are also specified as in earlier papers, namely as

$$\begin{aligned} m_j &= 1,\ j = 1,\cdots,n \\ k_j^{(0)} &= 1,\ j = 1,\cdots,n+1 \end{aligned}$$

and

$$b_j = \begin{cases} 1/2, & j\ even \\ 1/4, & j\ odd. \end{cases}$$

Then,

$$M = E,$$

$$B = \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} & & & \\ -\frac{1}{2} & \frac{3}{4} & -\frac{1}{4} & & \\ & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & \\ & & \ddots & \ddots & \ddots \\ & & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ & & & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

(if n is even), and

$$K^{(0)} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

We add the details in order to make the paper more readable on its own. Further, we choose $n = 5$ in this paper so that the state-space vector has dimension $m = 2n = 10$. For the initial time, we take

$$t_0 = 0$$

and

$$\eta \in \{0, 0.5, 1.0\}.$$

Finally, the initial conditions for $y(t)$ and $\dot{y}(t)$ are chosen as

$$y_0 = [-1, 1, -1, 1, -1]^T$$

as well as

$$\dot{y}_0 = [-1, -1, -1, -1, -1]^T.$$

(ii) *Computation of important quantities:*
Using the Matlab routine *eig.m*, one obtains

$$\begin{aligned} \lambda_1(A^*) &= -0.699760638780536 + 1.795981478159753i, \\ \lambda_2(A^*) &= -0.699760638780536 - 1.795981478159753i, \\ \lambda_3(A^*) &= -0.562668374040742 + 1.616358701643860i, \\ \lambda_4(A^*) &= -0.562668374040742 - 1.616358701643860i, \\ \lambda_5(A^*) &= -0.375000000000000 + 1.363589014329465i, \\ \lambda_6(A^*) &= -0.375000000000000 - 1.363589014329465i, \\ \lambda_7(A^*) &= -0.050239361219464 + 0.516371450711010i, \\ \lambda_8(A^*) &= -0.050239361219464 - 0.516371450711010i, \\ \lambda_9(A^*) &= -0.187331625959257 + 0.994521686465592i, \\ \lambda_{10}(A^*) &= -0.187331625959257 - 0.994521686465592i. \end{aligned}$$

Therefore, $\lambda_j(A^*),\ j = 1,\cdots,m = 2n = 10$ and also $\lambda_j(A) = \overline{\lambda_j(A^*)},\ j = 1,\cdots,m = 2n = 10$ are distinct. Thus, matrix A is diagonalizable. Further, we obtain

$$U^* := [u_1^*,\cdots,u_{10}^*]$$

where

$$[u_1^*,\cdots,u_5^*] = \begin{bmatrix} 0.2680+0.0309i & 0.2680-0.0309i & -0.4533 & -0.4533 & 0.3314-0.2662i \\ -0.4491-0.0157i & -0.4491+0.0157i & 0.4039-0.0764i & 0.4039+0.0764i & 0.0539+0.0558i \\ 0.5119 & 0.5119 & 0.0563+0.1119i & 0.0563-0.1119i & -0.4393+0.1546i \\ -0.4370+0.0153i & -0.4370-0.0153i & -0.4321+0.0205i & -0.4321-0.0205i & -0.0354-0.0773i \\ 0.2439-0.0309i & 0.2439+0.0309i & 0.3970-0.1119i & 0.3970+0.1119i & 0.5101 \\ 0.0798-0.1237i & 0.0798+0.1237i & -0.1000+0.2519i & -0.1000-0.2519i & -0.1194-0.2759i \\ -0.0836+0.2122i & -0.0836-0.2122i & 0.0550-0.2290i & 0.0550+0.2290i & -0.0000+0.0000i \\ 0.0955-0.2474i & 0.0955+0.2474i & 0.1063-0.0140i & 0.1063+0.0140i & 0.0230+0.3285i \\ -0.0836+0.2122i & -0.0836-0.2122i & -0.0550+0.2290i & -0.0550-0.2290i & -0.0000-0.0000i \\ 0.0158-0.1237i & 0.0158+0.1237i & -0.0063-0.2379i & -0.0063+0.2379i & 0.0957-0.3478i \end{bmatrix},$$

and

$$[u_6^*,\cdots,u_{10}^*] = \begin{bmatrix} 0.3314+0.2662i & 0.0497-0.1441i & 0.0497+0.1441i & 0.3779 & 0.3779 \\ 0.0539-0.0558i & -0.0531-0.2199i & -0.0531+0.2199i & 0.3400-0.0254i & 0.3400+0.0254i \\ -0.4393-0.1546i & -0.0259-0.2635i & -0.0259+0.2635i & -0.0700-0.1592i & -0.0700+0.1592i \\ -0.0354+0.0773i & 0.0095-0.2322i & 0.0095+0.2322i & -0.3050+0.1050i & -0.3050-0.1050i \\ 0.5101 & -0.0755-0.1194i & -0.0755+0.1194i & -0.3079+0.1592i & -0.3079-0.1592i \\ -0.1194+0.2759i & -0.2568-0.0165i & -0.2568+0.0165i & 0.0314-0.3582i & 0.0314+0.3582i \\ -0.0000-0.0000i & -0.4417-0.0001i & -0.4417+0.0001i & -0.0079-0.3406i & -0.0079+0.3406i \\ 0.0230-0.3285i & -0.5104 & -0.5104 & -0.0857+0.0199i & -0.0857-0.0199i \\ -0.0000+0.0000i & -0.4417-0.0001i & -0.4417+0.0001i & 0.0079+0.3406i & 0.0079-0.3406i \\ 0.0957+0.3478i & -0.2536+0.0165i & -0.2536-0.0165i & 0.0543+0.3383i & 0.0543-0.3383i \end{bmatrix};$$

here the output results are given with only four decimal places for space reasons. The weighted matrix R is computed as

$$R = \begin{bmatrix} 1.2501 & -0.2868 & -0.1297 & 0.0135 & -0.0988 & 0.1966 & -0.1314 & -0.3356 & -0.0196 & 0.2662 \\ -0.2868 & 1.0887 & -0.3824 & -0.0842 & 0.0179 & -0.1567 & 0.2066 & 0.0440 & 0.0239 & 0.0319 \\ -0.1297 & -0.3824 & 1.1899 & -0.3531 & -0.1196 & 0.2780 & 0.0018 & 0.2200 & -0.1271 & -0.3402 \\ 0.0135 & -0.0842 & -0.3531 & 1.0873 & -0.3227 & -0.0173 & -0.0525 & -0.1892 & 0.1947 & 0.0505 \\ -0.0988 & 0.0179 & -0.1196 & -0.3227 & 1.2619 & -0.3017 & 0.0042 & 0.3092 & 0.0215 & 0.2699 \\ 0.1966 & -0.1567 & 0.2780 & -0.0173 & -0.3017 & 0.7620 & 0.2782 & 0.1039 & 0.0439 & -0.0256 \\ -0.1314 & 0.2066 & 0.0018 & -0.0525 & 0.0042 & 0.2782 & 0.8373 & 0.3358 & 0.1512 & 0.0458 \\ -0.3356 & 0.0440 & 0.2200 & -0.1892 & 0.3092 & 0.1039 & 0.3358 & 0.9171 & 0.3240 & 0.1085 \\ -0.0196 & 0.0239 & -0.1271 & 0.1947 & 0.0215 & 0.0439 & 0.1512 & 0.3240 & 0.8373 & 0.2919 \\ 0.2662 & 0.0319 & -0.3402 & 0.0505 & 0.2699 & -0.0256 & 0.0458 & 0.1085 & 0.2919 & 0.7685 \end{bmatrix},$$

We mention that the items in (i) and (ii) are already given in [6]. We have added them for the sake of completeness.

(iii) *Graph of* $y = \|x(t)\|_{R(t,t_0)}$ *for* $\eta \in \{0, 0.5, 1.0\}$*:*

In Figure 8.2, the curve $y = \|x(t)\|_{R(t,t_0)}$ for $\eta \in \{0, 0.5, 1.0\}$ is plotted.

The result in all three cases $\eta \in \{0, 0.5, 1.0\}$ is numerically identical with that in [6, Fig.4], i.e. for $y = \|x_A(t)\|_R$. But, the method of computing Figure 8.2 is different from that used for [6, Fig.4]; for this, see Section 8.3 below. The result of Figure 8.2 underpins the theoretical findings in Corollary 6.1.

The curve $y = \|x(t)\|_{R(t,t_0)}$ for $\eta \in \{0, 0.5, 1.0\}$ behaves essentially like $y = e^{-t}$ and clearly shows vibration suppression. Thus the curve in Figure 8.2 may serve as a measure of the damping property of the system.

The vibration behavior is due to the fact that the eigenvalues are pairwise conjugate complex. Since $Re\,\lambda_j(A^*) = Re\,\lambda_j(A) < 0,\ j = 1,\cdots,10$, the system is asymptotically stable so that $\|x(t)\|_{R(t,t_0)} \to 0\ (t \to \infty)$.

(iv) *Two-sided bounds on* $y = \|x(t)\|_2$ *for* $\eta = 0$*:*

Now, we apply Corollary 7.2, first for $\eta = 0$ in order to check corresponding results from [6].

In Figure 8.3, the optimal upper and lower bounds $y = X_{1,2}\,\|\psi(t)\|_2$ and $y = X_{0,2}\,\|\psi(t)\|_2$ on $y = \|x(t)\|_2$ for $\eta = 0$ are shown where the optimal constants $X_{1,2}$ and $X_{0,2}$ are determined by the differential calculus of norms. Let $t_{s,u,2}$ and $t_{s,l,2}$ be the pertinent places of contact. For the initial guesses $t_{s,u,2_0} = 15.0$ and $t_{s,l,2_0} = 12.0$, the following results are obtained:

$$\begin{aligned} t_{s,u,2} &\doteq 15.204749, \\ X_{1,2} &\doteq 1.560408, \end{aligned}$$

and

$$\begin{aligned} t_{s,l,2} &\doteq 12.162025, \\ X_{0,2} &\doteq 0.803475. \end{aligned}$$

These results are numerically identical with those in [6]. However, the computational methods are different, see Section 8.3.

(v) *Two-sided bounds on* $y = \|x(t)\|_2$ *for* $\eta = 0.5$*:*

Now, we apply Corollary 7.2, for $\eta = 0.5$. In Figure 8.4, the pertinent optimal upper and lower bounds $y = X_{1,2}\,\|\psi(t)\|_2$ and $y = X_{0,2}\,\|\psi(t)\|_2$ on $y = \|x(t)\|_2$ are shown where the optimal constants $X_{1,2}$ and $X_{0,2}$ are determined by the differential calculus of norms. Let $t_{s,u,2}$ and $t_{s,l,2}$ be the pertinent places of contact. For the initial guesses $t_{s,u,2_0} = 19.0$ and $t_{s,l,2_0} = 16.0$, the following results are obtained:

$$\begin{aligned} t_{s,u,2} &\doteq 19.697311, \\ X_{1,2} &\doteq 1.444709, \end{aligned}$$

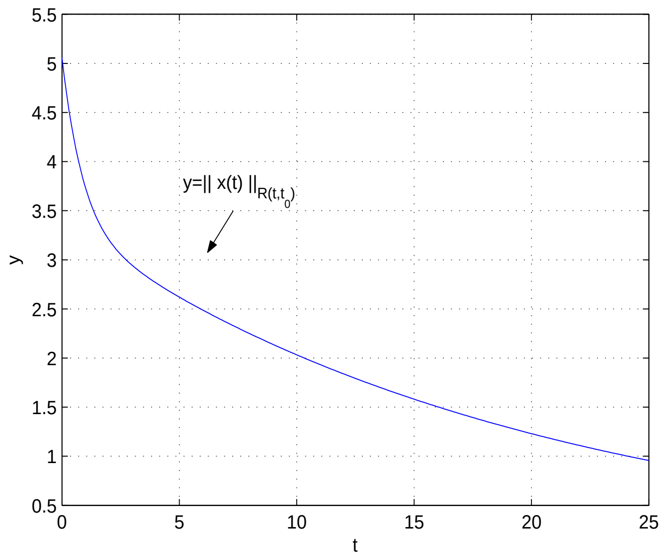

Figure 8.2. $y = \|x(t)\|_{R(t,t_0)}$ for $\eta \in \{0, 0.5, 1.0\}$ for diagonalizable system matrix A.

and

$$\begin{aligned} t_{s,l,2} &\doteq 16.737657, \\ X_{0,2} &\doteq 0.759622. \end{aligned}$$

These results are new. Computational details are given in Section 8.3.

Numerical Example 2: Matrix A non-diagonalizable

(i) *Construction of a non-diagonalizable matrix A:*

In the case $n = 2$ in Figure 8.1, we have

$$M = \left[\begin{array}{c|c} m_1 & 0 \\ \hline 0 & m_2 \end{array}\right],$$

$$B = \left[\begin{array}{c|c} b_1 + b_2 & -b_2 \\ \hline -b_2 & b_2 + b_3 \end{array}\right],$$

and

$$K^{(0)} = \left[\begin{array}{c|c} k_1^{(0)} + k_2^{(0)} & -k_2^{(0)} \\ \hline -k_2^{(0)} & k_2^{(0)} + k_3^{(0)} \end{array}\right],$$

so that the pertinent characteristic equation reads

$$|\lambda^2 M + \lambda B + K^{(0)}| = \left|\begin{array}{c|c} \lambda^2 m_1 + \lambda(b_1 + b_2) + (k_1^{(0)} + k_2^{(0)}) & \lambda(-b_2) - k_2^{(0)} \\ \hline \lambda(-b_2) - k_2^{(0)} & \lambda^2 m_2 + \lambda(b_2 + b_3) + (k_2^{(0)} + k_3^{(0)}) \end{array}\right| = 0.$$

For the construction of a case with non-diagonalizable matrix A, we choose

$$b_2 = 0,\ m_2 = m_1 = 1,\ b_3 = b_1,\ k_3^{(0)} = k_1^{(0)}.$$

Then,

$$\lambda^2 m_1 + \lambda b_1 + (k_1^{(0)} + k_2^{(0)}) = s k_2^{(0)} \quad \text{with} \quad s \in \{+1, -1\}.$$

Hence, with $m_1 = 1$,

$$\lambda = -\frac{b_1}{2} \pm \sqrt{(\frac{b_1}{2})^2 - k_1^{(0)} - k_2^{(0)} + s k_2^{(0)}}.$$

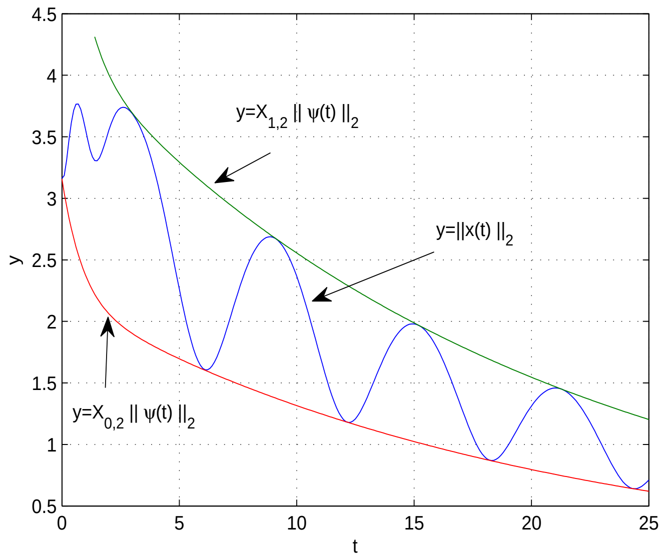

Figure 8.3. $y = \|x(t)\|_2$ for diagonalizable system matrix A and $\eta = 0$ as well as optimal upper and lower bounds.

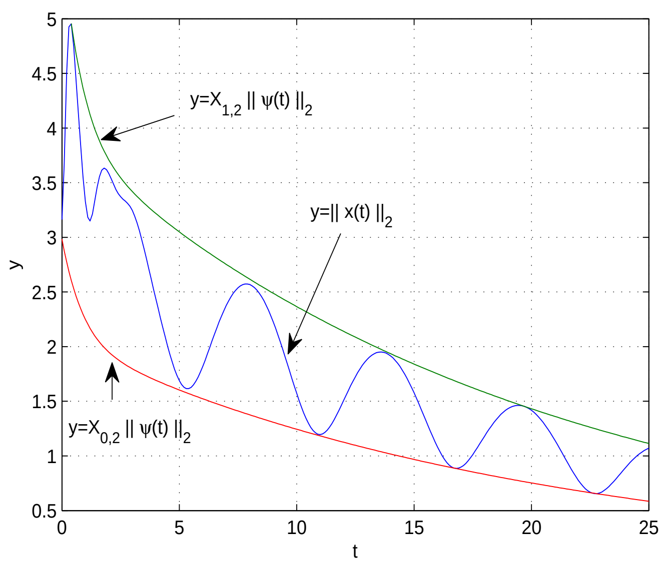

Figure 8.4. $y = \|x(t)\|_2$ for diagonalizable system matrix A and $\eta = 0.5$ as well as optimal upper and lower bounds.

Now, in order to get one real solution, we set

$$k_1^{(0)} := (\frac{b_1}{2})^2.$$

This implies

$$\lambda = \begin{cases} -\frac{b_1}{2}, & s = +1, \\ -\frac{b_1}{2} \pm i\sqrt{2k_2^{(0)}}, & s = -1. \end{cases}$$

(ii) *Data:*

As numerical values for the quantities not yet specified, we choose $b_1 = 1/4$, $k_2^{(0)} = 2^3 = 8$. On the whole, this delivers the following data:

$$m_1 = m_2 = 1;\ b_1 = 1/4, b_2 = 0, b_3 = 1/4;\ k_1^{(0)} = 1/64 = 1/2^4, k_2^{(0)} = 8, k_3^{(0)} = 1/64 = 1/2^4,$$

which leads to

$$M = \left[\begin{array}{c|c} m_1 & 0 \\ \hline 0 & m_2 \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right],$$

$$B = \left[\begin{array}{c|c} b_1 + b_2 & -b_2 \\ \hline -b_2 & b_2 + b_3 \end{array}\right] = \left[\begin{array}{c|c} 0.25 & 0 \\ \hline 0 & 0.25 \end{array}\right],$$

and

$$K^{(0)} = \left[\begin{array}{c|c} k_1^{(0)} + k_2^{(0)} & -k_2^{(0)} \\ \hline -k_2^{(0)} & k_2^{(0)} + k_3^{(0)} \end{array}\right] = \left[\begin{array}{c|c} 1/64 + 8 & -8 \\ \hline -8 & 8 + 1/64 \end{array}\right] = \left[\begin{array}{c|c} 8.015625 & -8 \\ \hline -8 & 8.015625 \end{array}\right].$$

Further, we choose

$$t_0 = 0$$

as well as

$$y_0 = [-1, 1]^T$$

and

$$\dot{y}_0 = [-1, -1]^T.$$

(iii) *Computation of important quantities:*

Using the Matlab routine *jordan*, one obtains

$$\begin{aligned} \lambda_1(A^*) &= -0.1250 + 4.0000i, \\ \lambda_2(A^*) &= -0.1250 - 4.0000i, \\ \lambda_3(A^*) &= -0.1250, \\ \lambda_4(A^*) &= \lambda_3(A^*). \end{aligned}$$

Here, $m_1 = m_2 = 1$, and $m_3 = 2$. Thus, matrix A^* and therefore also A is not diagonalizable, and the computation of

$$U^* := [u_1^*, u_2^*, u_1^{(3)^*}, u_2^{(3)^*}]$$

gives

$$U^* = \begin{bmatrix} 0.25 + 0.0078125i & 0.25 - 0.0078125i & 0.0625 & 0.5 \\ -0.25 - 0.0078125i & -0.25 + 0.0078125i & 0.0625 & 0.5 \\ 0.0625i & 0.0625i & 0.5 & 0 \\ 0.0625i & 0.0625i & 0.5 & 0 \end{bmatrix}.$$

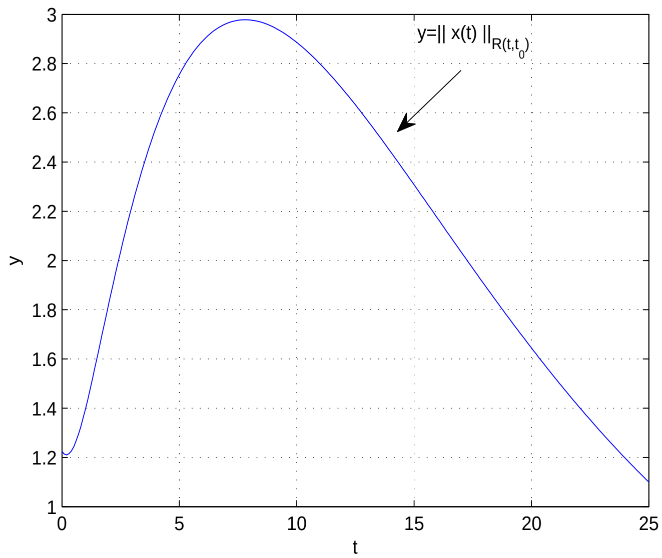

Figure 8.5. $y = \|x(t)\|_{R(t,t_0)}$ for non-diagonalizable system matrix A and $\eta \in \{0, 0.5, 1.0\}$.

The weighted matrix R is computed as

$$R = \begin{bmatrix} 0.379028320312500 & 0.128784179687500 & 0.032226562500000 & 0.030273437500000 \\ 0.128784179687500 & 0.379028320312500 & 0.030273437500000 & 0.032226562500000 \\ 0.032226562500000 & 0.030273437500000 & 0.257812500000000 & 0.242187500000000 \\ 0.030273437500000 & 0.032226562500000 & 0.242187500000000 & 0.257812500000000 \end{bmatrix}.$$

We mention that the items (i) - (iii) are already given in [6]. We have added them for the sake of completeness.

(iv) *Graph of* $y = \|x(t)\|_{R(t,t_0)}$ *for* $\eta \in \{0, 0.5, 1.0\}$*:*

In the norm $\|\cdot\|_{R(t,t_0)}$, we obtain Figure 8.5.

The result in all three cases is numerically identical with that in [6, Fig.9], i.e., for $y = \|x_A(t)\|_R$. But, again, the method of computing Figure 8.5 is different from that used for [6, Fig.8]; for this, see the computational aspects discussed in Section 8.3.

Remark 8.1. *The computation of the optimal two-sided bounds* $y = X_{1,2}\|\psi(t)\|_2$ *and* $y = X_{0,2}\|\psi(t)\|_2$ *on* $y = \|x(t)\|_2$ *for* $\eta = 0.5$ *corresponding to Figure 8.4 is left to the reader.*

8.3 Computational aspects

In this subsection, we say something about the used computer equipment, used Matlab programs, and the computation time.

(i) As to the *computer equipment*, the following hardware was available: an Intel Core2 Duo Processor at 3166 GHz, a 500 GB mass storage facility, and two 2048 MB high-speed memories. As software package for the computations, we used MATLAB, Version 7.11.

(ii) Whereas in [6], the computations were based on the representation $x_A(t) = e^{A(t-t_0)}x_0$ of the solution of the initial value problem $\dot{x}_A(t) = Ax_A(t),\ t \geq t_0,\ x_A(t_0) = x_0$, here for the solution of the nonlinear IVP $\dot{x}(t) = Ax(t) + h(t, x(t)),\ t \geq t_0,\ x(t_0) = x_0$, the Matlab program ODE45 is applied. In the case of $\eta = 0$, we obtain the same numerical result as in the linear case. The computation time is larger, however.

(iii) The *computation time t* of an operation was determined by the command sequence *t1=clock; operation; t=etime(clock,t1)*. It is put out in seconds, rounded to four decimal places. For the computation of the eigenvalues of matrix A, we used the command [XA,DA]=eig(A); the pertinent computation time for *Example 1* is less than 0.0001 s. For the computation of the 251 values $t, y(t)$ with $y(t) = \|x(t)\|_{R(t,t_0)},\ t = 0(0.1)25$ for, say, Figure 8.5, it took $t(table for Figure 8.5) = 7.8930s$. The computation times for the other figure are of a similar order.

9. Conclusion

In this paper, it is shown that one can construct a time-dependent positive definite matrix $R(t, t_0)$ such that the solution $x(t)$ of the nonlinear initial value problem with linear principal part $\dot{x}(t) = Ax(t) + h(t, x(t)),\ t \geq t_0,\ x(t_0) = x_0$ in the weighted norm $\|\cdot\|_{R(t,t_0)}$ is equal to the solution $x_A(t)$ of $\dot{x}_A(t) = Ax_A(t),\ t \geq t_0,\ x_A(t_0) = x_0$ in the weighted norm $\|\cdot\|_R$ where R is a constant positive definite matrix. As a consequence, if $\|x_A(t)\|_R$ shows vibration suppression or monotonicity behavior, so does $\|x(t)\|_{R(t,t_0)}$. Further, since $\|x_A(t)\|_R$ can be used to assess the damping behavior of the underlying dynamical problem, by the equation $\|x(t)\|_{R(t,t_0)} = \|x_A(t)\|_R$ also the damping behavior of the nonlinear IVP can be assessed. The results are applied to dynamical systems, and examples underpin the theoretical findings.

One might object that incase of matrix A is not diagonalizable, the Jordan canonical form has to be calculated. But, the determination of the Jordan canonical form can be done by the *jordan routine* of MATLAB. Further, engineers usually reduce an originally large matrix A by a process called *condensation*. For these reduced matrices, it is usually no numerical problem to determine the canonical Jordan form, and it is then not costly to compute $\|x(t)\|_{R(t,t_0)}$. In addition, in engineering practice, often models with small matrices A are applied. For these models, the new method is likewise of major interest. Moreover, the matrices used in practice are in most cases diagonalizable. In these cases, no numerical problem at all exists.

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