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Two Positive Solutions for a Fourth-Order Three-Point BVP with Sign-Changing Green's Function

Habib Djourdem[1]*, Slimane Benaicha[2], and Noureddine Bouteraa[3]

Abstract
This paper concerns the fourth-order three-point boundary value problem (BVP)

$$u^{(4)}(t) = f(t, u(t)), \quad t \in [0,1],$$

$$u'(0) = u''(0) = u(1) = 0, \; \alpha u''(1) - u'''(\eta) = 0,$$

where $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, $\alpha \in [0,1)$ and $\eta \in \left[\frac{2\alpha+10}{15-2\alpha}, 1\right)$. Although the corresponding Green's function is sign-changing, we still obtain the existence of at least two positive and decreasing solutions under some suitable conditions on f by applying the two-fixed-point theorem due to Avery and Henderson. An example is also given to illustrate the main results.

Keywords: Completely continuous, fourth-order boundary value problem, Green's function, two positive solutions.
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[1] *Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran1, Ahmed Benbella. Algeria, ORCID: 0000-0002-7992-581X*
[2] *Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran1, Ahmed Benbella. Algeria, ORCID: 0000-0002-8953-8709*
[3] *Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran1, Ahmed Benbella. Algeria, ORCID: 0000-0002-8772-1315*
***Corresponding author**: djourdem.habib7@gmail.com
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1. Introduction

Fourth-order differential equations arise from a variety of different areas of applied mathematics, physics, engineering, material mechanics, fluid mechanics and so on [1, 2]. Many authors studied the existence of positive solutions for fourth-order m-point boundary value problems using different methods see [3]-[6] and the references therein.

In recent years, the existence and multiplicity of positive solutions of the boundary value problems with sign-changing Green's function has received much attention from many authors; see [7, 8, 9, 10, 11, 12, 13, 14].

In [15] Li, Sun and Kong considered the following BVP with an indefinitely signed Green's function

$$\begin{cases} u'''(t) = a(t) f(t, u(t)) = 0, & t \in (0,1), \\ u'(0) = 0, \quad u(1) = \alpha u(\eta), & u''(\eta) = 0, \end{cases}$$

where $\alpha \in [0,2)$, $\eta \in \left[\frac{\sqrt{121+24\alpha}}{3(4+\alpha)}, 1\right)$. By means of the Guo-Krasnoselskii's fixed point theorem, existence results of positive solutions were obtained.

In [16] Xie et al. discuss the existence of triple positive solutions for the BVP

$$\begin{cases} u'''(t) = f(t,u(t)) = 0, & t \in (0,1), \\ u'(0) = 0,\ u(1) = \alpha u(\eta),\ u''(\eta) = 0, \end{cases}$$

where $0 < \alpha < 1$, $\max\left\{\frac{1+2\alpha}{1+4\alpha}, \frac{1}{2-\alpha}\right\} < \eta < 1$. The main tool used is the fixed point theorem due to Avery and Peterson.

It is to be observed that there are other types of works on sign-changing Green's functions which prove the existence of sign-changing solutions, positive in some cases; see the papers [17]-[21].

Inspired and motivated by the works mentioned above, in this paper we will study the following nonlinear fourth-order three-point BVP with sign-changing Green's function

$$\begin{cases} u^{(4)}(t) = f(t,u(t)), & t \in [0,1] \\ u'(0) = u''(0) = u(1) = 0,\ \alpha u''(1) - u'''(\eta) = 0, \end{cases} \tag{1.1}$$

where $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, $\alpha \in [0,1)$ and $\eta \in \left[\frac{2\alpha+10}{15-2\alpha}, 1\right)$. By imposing suitable conditions on f, we obtain the existence of at least two positive and decreasing solutions for the BVP (1.1).

To end this section, we state some fundamental definitions and the two-fixed-point theorem due to Avery and Henderson [22].

Let K be a cone in a real Banach space E.

Definition 1.1. *A functional $\psi : K \to \mathbb{R}$ is said to be increasing on K provided $\psi(x) \le \psi(y)$ for all $x, y \in K$ with $x \le y$, where $x \le y$ if and only if $y - x \in K$.*

Definition 1.2. *Let $\gamma : K \to [0,+\infty)$ be continuous. For each $d > 0$, one defines the set*

$$K(\gamma,d) = \{u \in K : \gamma(u) < d\}.$$

Theorem 1.3. *[22] Let ψ and γ be increasing, nonnegative, and continuous functionals on K, and let ω be a nonnegative continuous functional on K with $\omega(0) = 0$ such that, for some $c > 0$ and $M > 0$,*

$$\gamma(u) \le \omega(u) \le \psi(u), \qquad \|u\| \le M\gamma(u)$$

for all $u \in \overline{K(\gamma,c)}$. Suppose there exist a completely continuous operator $T : \overline{K(\gamma,c)} \to K$ and $0 < a < b < c$ such that

$$\omega(\lambda u) \le \lambda\omega(u) \quad for\ 0 \le \lambda \le 1,\ u \in \partial K(\omega,b),$$

and

(1) $\gamma(Tu) > c$ *for all* $u \in \partial K(\gamma,c)$;

(2) $\omega(Tu) < bc$ *for all* $u \in \partial K(\omega,b)$;

(3) $K(\psi,a) \ne \emptyset$ *and* $\psi(Tu) > a$ *for all* $u \in \partial K(\psi,a)$.

Then T has at least two fixed points u_1 and u_2 in $\overline{K(\gamma,c)}$ such that

$$a < \psi(u_1)\ with\ \omega(u_1) < b,$$
$$b < \omega(u_2)\ with\ \gamma(u_2) < c.$$

2. Preliminaries and lemmas

Let Banach space $E = C[0,1]$ be equipped with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$.

For the BVP

$$\begin{cases} u^{(4)}(t) = 0, & t \in (0,1), \\ u'(0) = u''(0) = u(1) = 0,\ \alpha u''(1) - u'''(\eta) = 0, \end{cases} \tag{2.1}$$

we have the following lemma.

Lemma 2.1. *The BVP (2.1) has only a trivial solution.*

Proof. It is simple to check. □

Now, for any $y \in E$, we consider the BVP

$$\begin{cases} u^{(4)}(t) = y(t) \quad t \in [0,1], \\ u'(0) = u''(0) = u(1) = 0,\ \alpha u''(1) - u'''(\eta) = 0, \end{cases} \tag{2.2}$$

After a direct computation, one may obtain the expression of Green's function of the BVP (2.2) as follows:
For $s \geq \eta$

$$G(t,s) = \begin{cases} -\frac{\alpha(1-t^3)(1-s)}{6(1-\alpha)} - \frac{(1-s)^3}{6}, & 0 \leq t \leq s \leq 1, \\ \frac{(t-s)^3}{6} - \frac{\alpha(1-t^3)(1-s)}{6(1-\alpha)} - \frac{(1-s)^3}{6}, & 0 \leq s \leq t \leq 1, \end{cases}$$

and for $s < \eta$

$$G(t,s) = \begin{cases} \frac{(1-t^3)(1-\alpha(1-s))}{6(1-\alpha)} - \frac{(1-s)^3}{6}, & 0 \leq t \leq s \leq 1, \\ \frac{(t-s)^3}{6} + \frac{(1-t^3)(1-\alpha(1-s))}{6(1-\alpha)} - \frac{(1-s)^3}{6}, & 0 \leq s \leq t \leq 1. \end{cases}$$

Remark 2.2. *$G(t,s)$ has the following properties:*

$$G(t,s) \geq 0 \quad for\ \ 0 \leq s < \eta, \qquad G(t,s) \leq 0 \quad for\ \eta \leq s \leq 1.$$

Moreover, for $s \geq \eta$,

$$\max\{G(t,s) : t \in [0,1]\} = G(1,s) = 0,$$

$$\min\{G(t,s) : t \in [0,1]\} = G(0,s) = -\frac{\alpha(1-s)}{6(1-\alpha)} - \frac{(1-s)^3}{6}$$

and for $s < \eta$,

$$\max\{G(t,s) : t \in [0,1]\} = G(0,s) = \frac{1-\alpha(1-s)}{6(1-\alpha)} - \frac{(1-s)^3}{6},$$

$$\min\{G(t,s) : t \in [0,1]\} = G(1,s) = 0.$$

Let

$$K_0 = \{y \in E : y(t) \text{ is nonnegative and decreasing on } [0,1]\}.$$

Then K_0 is a cone in E.

Lemma 2.3. *Let $y \in K_0$ and $u(t) = \int_0^1 G(t,s)y(s)\,ds$, $t \in [0,1]$. Then u is the unique solution of the BVP (2.2) and $u \in K_0$. Moreover, $u(t)$ is concave on $[0,\eta]$.*

Proof. For $t \in [0,\eta]$, we have

$$\begin{aligned} u(t) &= \int_0^t \left[\frac{(t-s)^3}{6} + \frac{(1-t^3)(1-\alpha(1-s))}{6(1-\alpha)} - \frac{(1-s)^3}{6}\right] y(s)\,ds \\ &+ \int_t^\eta \left[\frac{(1-t^3)(1-\alpha(1-s))}{6(1-\alpha)} - \frac{(1-s)^3}{6}\right] y(s)\,ds \\ &+ \int_\eta^1 \left[-\frac{\alpha(1-t^3)(1-s)}{6(1-\alpha)} - \frac{(1-s)^3}{6}\right] y(s)\,ds. \end{aligned}$$

Since $y \in K_0$ and $\eta \geq \frac{2\alpha+10}{15-2\alpha}$ implies that $\eta > \frac{3\alpha}{4+2\alpha}$, we get

$$\begin{aligned}
u'(t) &= -\frac{\alpha t^2}{2(1-\alpha)}\int_0^\eta sy(s)\,ds - \frac{t^2}{2}\int_t^\eta y(s)\,ds \\
&\quad + \int_0^t \left[\frac{s^2-2ts}{2}\right] y(s)\,ds + \frac{\alpha t^2}{2(1-\alpha)}\int_\eta^1 (1-s)\,y(s)\,ds \\
&\leq y(\eta)[-\frac{\alpha t^2}{2(1-\alpha)}\int_0^\eta s ds - \frac{t^2}{2}\int_t^\eta y(s)\,ds \\
&\quad + \int_0^t \left[\frac{s^2-2ts}{2}\right] ds + \frac{\alpha t^2}{2(1-\alpha)}\int_\eta^1 \left(1-s(1-s)^3\right) ds] \\
&= \frac{t^2}{2}y(\eta)[\frac{\alpha(1-2\eta)}{2(1-\alpha)} + \frac{t}{3} - \eta] \\
&\leq \frac{t^2}{2}y(\eta)[\frac{\alpha(1-2\eta)}{2(1-\alpha)} - \frac{2\eta}{3}] \\
&\leq 0, \quad t \in [0,\eta].
\end{aligned}$$

At the same time, $y \in K_0$ and $\eta \geq \frac{2\alpha+10}{15-2\alpha} > \frac{1}{2}$ shows that

$$\begin{aligned}
u''(t) &= -\frac{\alpha t}{1-\alpha}\int_0^\eta sy(s)\,ds - t\int_t^\eta y(s)\,ds \\
&\quad - \int_0^t sy(s)\,ds + \frac{\alpha t}{1-\alpha}\int_\eta^1 (1-s)\,y(s)\,ds \\
&\leq y(\eta)[-\frac{\alpha t}{1-\alpha}\int_0^\eta s ds - t\int_t^\eta ds \\
&\quad - \int_0^t s ds + \frac{\alpha t}{1-\alpha}\int_\eta^1 (1-s)\,ds] \\
&= t y(\eta)\left[\frac{\alpha(1-2\eta)}{2(1-\alpha)} + \frac{t}{2} - \eta\right] \\
&\leq t y(\eta)\left[\frac{\alpha(1-2\eta)}{2(1-\alpha)}\right] \\
&\leq 0, \quad t \in [0,\eta].
\end{aligned}$$

For $t \in [\eta, 1)$, we have

$$\begin{aligned}
u(t) &= \int_0^\eta \left[\frac{(t-s)^3}{6} - \frac{\left(1-t^3\right)(1-\alpha(1-s))}{6(1-\alpha)} - \frac{(1-s)^3}{6}\right] y(s)\,ds \\
&\quad + \int_\eta^t \left[\frac{(t-s)^3}{6} - \frac{\alpha\left(1-t^3\right)(1-s)}{6(1-\alpha)} - \frac{(1-s)^3}{6}\right] y(s)\,ds \\
&\quad + \int_t^1 \left[-\frac{\alpha\left(1-t^3\right)(1-s)}{6(1-\alpha)} - \frac{(1-s)^3}{6}\right] y(s)\,ds.
\end{aligned}$$

In view of $y \in K_0$ and $\eta > \frac{1}{2}$, we get

$$\begin{aligned}
u'(t) &= -\frac{\alpha t^2}{2(1-\alpha)} \int_0^\eta s y(s)\,ds + \int_0^\eta \left[\frac{s^2-2ts}{2}\right] y(s)\,ds \\
&\quad + \int_\eta^t \left[\frac{(t-s)^2}{2}\right] y(s)\,ds + \frac{\alpha t^2}{2(1-\alpha)} \int_\eta^1 (1-s)\,y(s)\,ds \\
&\le y(\eta)[-\frac{\alpha t^2}{2(1-\alpha)} \int_0^\eta s\,ds + \int_0^\eta \left[\frac{s^2-2ts}{2}\right] ds \\
&\quad + \int_\eta^t \left[\frac{(t-s)^2}{2}\right] ds + \frac{\alpha t^2}{2(1-\alpha)} \int_\eta^1 (1-s)\,ds] \\
&= \frac{t^2}{2} y(\eta) \left[\frac{\alpha(1-2\eta)}{2(1-\alpha)} + \frac{t-3\eta}{3}\right] \\
&\le \frac{t^2}{2} y(\eta) [\frac{\alpha(1-2\eta)}{2(1-\alpha)} + \frac{t-2\eta}{2}] \\
&= \frac{t^2}{2} y(\eta) [\frac{(1-2\eta)}{2(1-\alpha)}] \\
&\le 0, \quad t \in (\eta, 1].
\end{aligned}$$

Obviously, $u^{(4)}(t) = y(t)$ for $t \in [0,1]$, $u'(0) = u''(0) = u(1) = 0$, $\alpha u''(1) - u'''(\eta) = 0$. This shows that u is a solution of the BVP (2.2). The uniquess follows immediately from Lemma 2.1. Since $u'(t) \le 0$ for $t \in [0,1]$ and $u(1) = 0$, we have $u(t) \ge 0$ for $t \in [0,1]$. So, $u \in K_0$. In view of $u''(t) \le 0$ for $t \in [0,\eta]$, we know that $u(t)$ is concave on $[0,\eta]$. □

Lemma 2.4. *Let $y \in K_0$. Then the unique solution u of the BVP (2.2) satisfies*

$$\min_{t\in[0,\tau]} u(t) \ge \tau^* \|u\|,$$

where $\tau \in \left(0, \frac{1}{2}\right]$ and $\tau^* = \frac{\eta-\tau}{\eta}$.

Proof. From Lemma 2.3, we know that $u(t)$ is concave on $[0,\eta]$; thus for $t \in [0,\eta]$,

$$u(t) \ge \frac{\eta-t}{\eta} u(0) + \frac{t}{\eta} u(\eta).$$

At the same time, it follows from $u \in K_0$ that $\|u\| = u(0)$ which

$$u(t) \ge \frac{\eta-t}{\eta} \|u\|.$$

Therefore,

$$\min_{t\in[0,\tau]} u(t) = u(\tau) \ge \frac{\eta-\tau}{\eta} \|u\| = \tau^* \|u\|.$$

□

3. Main results

In what follows, we assume that f satisfies the following two conditions:

$(C1)$ For each $u \in [0,+\infty)$, the mapping $t \longmapsto f(t,u)$ is decreasing;

$(C2)$ For each $t \in [0,1]$, the mapping $u \longmapsto f(t,u)$ is increasing.

Let

$$K = \left\{ u \in K_0 : \min_{t\in[0,\tau]} u(t) \ge \tau^* \|u\| \right\}.$$

Then it is easy to see that K is a cone in E.

Now, we define an operator T as follows:

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s))\, ds, \quad u \in K,\ t \in [0,1].$$

First, it is obvious that if u is a fixed point of T in K, then u is a nonnegative and decreasing solution of the BVP (1.1). Next, by Lemmas 2.3 and 2.4, we know that $T : K \to K$. Furthermore, although $G(t,s)$ is not continuous, it follows from known textbook results, for example, see [23], that $T : K \to K$ is completely continuous.

For convenience, we denote

$$A = \int_0^\tau G(\eta,s)\, ds, \qquad B = \int_0^\eta G(\tau,s)\, ds.$$

Theorem 3.1. *Assume that $(C1)$ and $(C2)$ hold. Moreover, suppose that there exist numbers a, b and c with $0 < a < b < \tau^* c$ such that*

$$f(\tau,c) > \frac{c}{A}, \tag{3.1}$$

$$f\left(0, \frac{b}{\tau^*}\right) < \frac{b}{B}, \tag{3.2}$$

$$f(\tau,\tau^* a) > \frac{a}{A}. \tag{3.3}$$

Then the BVP (1.1) has at least two positive and decreasing solutions.

Proof. First, we define the increasing, nonnegative, and continuous functionals γ, ω and ψ on K as follows:

$$\begin{aligned} \gamma(u) &= \min_{t\in[0,\tau]} u(t) = u(\tau), \\ \omega(u) &= \max_{t\in[\tau,1]} u(t) = u(\tau), \\ \psi(u) &= \max_{t\in[0,1]} u(t) = u(0). \end{aligned}$$

Obviously, for any $u \in K$, $\gamma(u) = \omega(u) \le \psi(u)$. At the same time, for each $u \in K$, in view of $\gamma(u) = \min_{t\in[0,\tau]} u(t) \ge \tau^* \|u\|$, we have

$$\|u\| \le \frac{1}{\tau^*}\gamma(u) \quad for\, u \in K.$$

Furthermore, we also note that

$$\omega(\lambda u) = \lambda\omega(u) \quad for \quad 0 \le \lambda \le 1,\ u \in K.$$

Next, for any $u \in K$, we claim that

$$\int_\tau^1 G(\eta,s) f(s,u(s))\, ds \ge 0. \tag{3.4}$$

In fact, it follows from $(C1)$, $(C2)$, and $\eta \ge \frac{2\alpha+10}{15-2\alpha}$ that

$$
\begin{aligned}
&\int_\tau^1 G(\eta,s)\,f(s,u(s))\,ds\\
&=\int_\tau^\eta G(\eta,s)\,f(s,u(s))\,ds+\int_\eta^1 G(\eta,s)\,f(s,u(s))\,ds\\
&\geq f(\eta,u(\eta))\left[\int_\tau^\eta G(\eta,s)\,ds+\int_\eta^1 G(\eta,s)\,ds\right]\\
&=f(\eta,u(\eta))\\
&\quad\times[\int_\tau^\eta\left(\tfrac{(\eta-s)^3}{6}+\tfrac{(1-n^3)(1-\alpha(1-s))}{6(1-\alpha)}-\tfrac{(1-s)^3}{6}\right)ds+\int_\eta^1\left(-\tfrac{\alpha(1-s)(1-\eta^3)}{6(1-\alpha)}-\tfrac{(1-s)^3}{6}\right)ds]\\
&=\tfrac{(1-\eta)}{24(1-\alpha)}f(\eta,u(\eta))\\
&\quad\times[(3+\alpha)\eta^3+(3-\alpha)\eta^2+(3-\alpha)\eta-\alpha-1-2\left(3\eta-2\alpha\eta+\alpha\eta^2-2\alpha+3\right)\tau^2+4(1-\alpha)\tau^3]\\
&=\tfrac{(1-\eta)}{24(1-\alpha)}f(\eta,u(\eta))\\
&\quad\times[(3+\alpha)\eta^3+(3-\alpha)\eta^2+(3-\alpha)\eta-\alpha-1-2\left(3\eta-2\alpha\eta+\alpha\eta^2-2\alpha+3\right)\tau^2-4\alpha\tau^3]\\
&\geq\tfrac{(1-\eta)}{24(1-\alpha)}f(\eta,u(\eta))\times[(3+\alpha)\eta^3+\left(3-\tfrac{3}{2}\alpha\right)\eta^2+\tfrac{3}{2}\eta-\tfrac{\alpha}{2}-\tfrac{5}{2}]\\
&\geq\tfrac{(1-\eta)}{24(1-\alpha)}f(\eta,u(\eta))\times[\left(\tfrac{15}{4}-\tfrac{1}{2}\alpha\right)\eta-\tfrac{\alpha}{2}-\tfrac{5}{2}]\\
&\geq 0.
\end{aligned}
$$

Now, we assert that $\gamma(Tu)>c$ for all $u\in\partial K(\gamma,c)$.
To prove this, let $u\in\partial K(\gamma,c)$; that is, $u\in K$ and $\gamma(u)=u(\tau)=c$. Then

$$u(t)\geq u(\tau)=c,\quad t\in[0,\tau]. \tag{3.5}$$

Since $(Tu)(t)$ is decreasing on $[0,1]$, it follows from (3.1), (3.4), (3.5), $(C1)$ and $(C2)$ that

$$
\begin{aligned}
\gamma(Tu)&=(Tu)(\tau)\\
&\geq(Tu)(\eta)\\
&=\int_0^1 G(\eta,s)\,f(s,u(s))\,ds\\
&\geq\int_0^\tau G(\eta,s)\,f(s,u(s))\,ds\\
&\geq\int_0^\tau G(\eta,s)\,f(\tau,c)\,ds\\
&>\frac{c}{A}\int_0^\tau G(\eta,s)\,ds=c.
\end{aligned}
$$

Then, we assert that $\omega(Tu)<b$ for all $u\in\partial K(\omega,b)$.
To see this, suppose that $u\in\partial K(\omega,b)$; that is, $u\in K$ and $\omega(u)=b$. Since $\|u\|\leq\frac{1}{\tau^*}\gamma(u)=\frac{1}{\tau^*}\omega(u)$, we have

$$0\leq u(t)\leq\|u\|\leq\frac{b}{\tau^*},\qquad t\in[0,\eta]. \tag{3.6}$$

In view of Remark 2.2, (3.2), (3.6), $(C1)$ and $(C2)$, we get

$$
\begin{aligned}
\omega(Tu)&=(Tu)(\tau)\\
&=\int_0^1 G(\tau,s)\,f(s,u(s))\,ds\\
&\leq\int_0^\eta G(\tau,s)\,f(s,u(s))\,ds\\
&\leq\int_0^\eta G(\tau,s)\,f\left(0,\frac{b}{\tau^*}\right)ds\\
&<\frac{b}{B}\int_0^\eta G(\tau,s)\,ds\\
&=b.
\end{aligned}
$$

Finally, we assert that $K(\psi,a) \neq \emptyset$ and $\psi(Tu) > a$ for all $u \in \partial K(\psi,a)$.
In fact, the constant function $\frac{a}{2} \in K(\psi,a)$. Moreover, for $u \in \partial K(\psi,a)$, that is $u \in K$ and $\psi(u) = u(0) = a$. Then

$$u(t) \geq \tau^* \|u\| = \tau^* u(0) = \tau^* a, \qquad t \in [0,\tau]. \tag{3.7}$$

Since $(Tu)(t)$ is decreasing on $[0,1]$, it follows from (3.3), (3.4), (3.7), $(C1)$ and $(C2)$ that

$$\begin{aligned} \psi(Tu) &= (Tu)(0) \\ &\geq (Tu)(\eta) \\ &= \int_0^1 G(\eta,s) f(s,u(s))\,ds \\ &\geq \int_0^\tau G(\eta,s) f(\tau,\tau * a)\,ds \\ &> \frac{a}{A}\int_0^\tau G(\eta,s)\,ds = a. \end{aligned}$$

To sum up, all the hypotheses of Theorem 1.3 are satisfied. Hence, T has at least two fixed points u_1 and u_2; that is, the BVP (1.1) has at least two positive and decreasing solutions u_1 and u_2 satisfying

$$\begin{aligned} a &< \max_{t\in[0,1]} u_1(t) \quad \textit{with} \quad \max_{t\in[\tau,1]} u_1(t) < b \\ b &< \max_{t\in[\tau,1]} u_2(t) \quad \textit{with} \quad \min_{t\in[0,\tau]} u_2(t) < c. \end{aligned}$$

□

4. An example

Consider the BVP

$$\begin{cases} u^{(4)}(t) = f(t,u(t)), \quad t \in [0,1], \\ u'(0) = u''(0) = u(1) = 0, \ \frac{1}{5}u''(1) - u'''\left(\frac{4}{5}\right) = 0, \end{cases} \tag{4.1}$$

where

$$f(t,u) = \begin{cases} \sqrt{u} + 9752 + \frac{1-t^3}{6}, \quad (t,u) \in [0,1]\times[0,169], \\ 340u - 47695 + \frac{1-t^3}{6}, \quad (t,u) \in [0,1]\times[169,170], \\ \frac{2021u^2}{5780} + \frac{1-t^3}{6}, \quad (t,u) \in [0,1]\times[170,+\infty]. \end{cases}$$

Since $\alpha = \frac{1}{5}$ and $\eta = \frac{4}{5}$, if we choose $\tau = \frac{1}{3}$, then a simple calculation shows that

$$A = \frac{1603}{162000}, \quad B = \frac{994}{10125}, \quad \tau^* = \frac{7}{12}.$$

Thus, if we let $a = 80$, $b = 98.7$ and $c = 300$, then it is easy to verify that all the conditions of Theorem 3.1. are satisfied. So, it follows from Theorem 3.1 that the BVP (4.1) has at least two positive and decreasing solutions.

References

[1] A. R. Aftabizadeh, *Existence and uniqueness theorems for fourth-order boundary value problems*, J. Math. Anal. Appl., **116** (1986), 415-426.

[2] R.P. Agarwal, Y.M Chow, *Iterative methods for a fourth order boundary value problem*, J. Comput. Appl. Math., **10** (1984), 203-217.

[3] Z. Bai, *The upper and lower solution method for some fourth-order boundary value problem*, Nonlinear Anal., **67** (2007), 1704- 1709.

[4] Z. Bekri, S. Benaicha, *Existence of positive of solution for a nonlinear three-point boundary value problem*, Sib. Elektron. Math. Rep., **14** (2017).

[5] N. Bouteraa, S. Benaicha, H. Djourdem and M-E. Benattia, *Positive solutions of nonlinear fourth-order two-point boundary value problem with a parameter*, Romanian J. Math. Comput. Sci., **8**(1) (2018), 17-30.

[6] N. Bouteraa, S. Benaicha, *Triple positive solutions of higher-order nonlinear boundary value problems*, J. Comput. Sci. Comput. Math., **7**(2) (2017), 25-31.

[7] A. Cabada, R. Enguica, L. Lopez-Somoza, *Positive solutions for second-order boundary value problems with sign changing Green's functions*, Electron. J. Differential Equations, **45** (2017), 1-17.

[8] C. Gao, F. Zhang, R. Ma, *Existence of positive solutions of second-order periodic boundary value problems with sign-changing Green's function*, Acta Math. Appl. Sin. Engl. Ser., **33**(2) (2017), 263–268.

[9] R. Ma, *Nonlinear periodic boundary value problems with sign-changing Green's function*, Nonlinear Anal., **74**(5)(2011), 1714–1720.

[10] B.-W. Niu, J.-P. Sun, Q.-Y. Ren, *Two positive solutions of third-order BVP with integral boundary condition and sign-changing Green's function*, J. Funct. Spaces., (2015), Article ID 491423, 8 pages.

[11] A. P. Palamides, A. Veloni, S. Alatsathianos, *Positive solutions of a third-order BVP independent of the sign of the Green's function*, Differ. Equ. Dyn. Syst. **21**(3)(2013), 261–279.

[12] J.-P. Sun, J. Zhao, *Iterative technique for a third-order three-point BVP with sign-changing Green's function*, Electron. J. Differential Equations **215** (2013), 1–9.

[13] J. Wang, C. Gao, *Positive solutions of discrete third-order boundary value problems with sign-changing Green's function*, Adv. Differ. Equ. **56** (2015).

[14] Y.H. Zhao, X.L. Li, *Iteration for a third-order three-point BVP with sign-changing Green's function*, J. Appl. Math., vol. (2014), Article ID 541234, 6 pages.

[15] X. Li, J. Sun, F. Kong, *Existence of positive solution for a third-order three-point BVP with sign-changing Green's function*, Electron. J. Qual. Theory Differ. Equ., **30** (2013), 1-11.

[16] D. Xie, H. Zhou, C. Bai, Y. Liu and X. Wu, *Triple positive solutions for a third-order three-point boundary value problem with sign-changing Green's function*, Ital. J. Pure Appl. Math., **38** (2017), 519-530.

[17] L.-J. Gao, J.-P. Sun, *Positive solutions of a third-order three-point BVP with sign-changing Green's function*, Math. Probl. Eng., (2014), Article ID 406815, 6 pages.

[18] G. Infante, J. R. L.Webb, *Three-point boundary value problems with solutions that change sign*, J. Integral Equations Appl., **15** (2003), 37-57.

[19] A. P. Palamides, G. Smyrlis, *Positive solutions to a singular third-order three-point boundary value problem with an indefinitely signed Green's function*, Nonlinear Anal., **68**(7) (2008), 2104–2118.

[20] J.-P. Sun, J. Zhao, *Positive solution for a third-order three-point boundary value problem with sign-changing Green's function*, Commun. Appl. Anal., **16**(2) (2012), 219–228.

[21] Y. Zhang, J-P. Sun and J. Zhao, *Positive solutions for a fourth-order three-point BVP with sign-changing Green's function*, Electron. J. Qual. Theory Differ., **5** (2018), 1–11.

[22] R. I. Avery, J. Henderson, *"Two positive fixed points of nonlinear operators on ordered Banach spaces*, Comm. Appl. Nonlinear Anal., **8**(1) (2001), 27–36.

[23] R. H. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, John Wiley and Sons, New York, NY, USA, 1976.