



Lucas polynomial solution of nonlinear differential equations with variable delays

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Abstract

In this study, a novel matrix method based on Lucas series and collocation points has been used to solve nonlinear differential equations with variable delays. The application of the method converts the nonlinear equation to a matrix equation which corresponds to a system of nonlinear algebraic equations with unknown Lucas coefficients. The method is tested on three problems to show that it allows both analytical and approximate solutions.

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1. Introduction

Ordinary differential equations fail to model many physical phenomena when the model is not only determined by its present state but also by a certain past state. Consequently, differential equations with time delays are used in modeling of real life situations such as human body control and multibody control systems, electric circuits, dynamical behaviour of a system in fluid mechanics, chemical engineering [21], spread of bacteriophage infection [38], stage structured populations [37], epidemic model in biology and dynamic diseases model in physiology [35].

Some numerical methods have been developed to solve nonlinear differential equations (NDE) with proportional and constant delays; among them, one can mention Aboodh transformation method [5], Adomian decomposition method [12, 31], Power series method [11], Decomposition method [39], Differential transform method [28], Hermite wavelet based method [36], Variational iteration method [25, 29, 41], Power and Padé series based method [24], Spectral method [6], Variable multistep methods [27], Quasilinearization technique [34], Runge-Kutta-Fehlberg methods [30], Polynomial least squares method [13], Homotopy perturbation method [35], and First Boubaker polynomial approach [16].

On the other hand, there are few studies about nonlinear differential equations with variable delays. A study on the existence of positive ω -periodic solutions has been carried

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out by Dorociaková and Olach [19]. Chen et.al. [14] presented new criteria for asymptotic stability. Asymptotic behaviour of solutions is studied by Dix [18]. Fixed points and stability are studied [7, 17, 23, 44]. Only a few numerical techniques have been applied to solve such kind of equations: A new multi-step technique [10], Legendre-Gauss collocation method [40] and Runge-Kutta method using Hermite interpolation [22].

Numerical solutions of ODEs, fractional differential equations (FDE) and integro-differential equations are of great interest. Recently, methods based on Lucas, Fibonacci and Fermat polynomials have been proposed to solve FDEs [1–4, 8, 42, 43]. In these studies, they derived the operational matrix of fractional derivatives and observed that the numerical solutions have smaller errors than those obtained by using orthogonal polynomials.

In the present study, we consider the NDE with variable delays of the form

$$\sum_{k=0}^2 \sum_{j=0}^1 P_{kj}(t) y^{(k)}(t - \tau_{kj}(t)) + \sum_{p=0}^1 \sum_{q=0}^p R_{pq}(t) y^{(p)}(t) y^{(q)}(t) = g(t) \quad (1.1)$$

or precisely

$$\begin{aligned} & P_{00}(t)y(t - \tau_{00}(t)) + P_{10}(t)y'(t - \tau_{10}(t)) + P_{01}(t)y(t - \tau_{01}(t)) + P_{11}(t)y'(t - \tau_{11}(t)) \\ & + P_{20}(t)y''(t - \tau_{20}(t)) + P_{21}(t)y''(t - \tau_{21}(t)) + R_{00}(t)y^2(t) + R_{10}(t)y'(t)y(t) \\ & + R_{11}(t)(y'(t))^2 = g(t) \end{aligned}$$

with the initial conditions $y(a) = \lambda_1$ and $y'(a) = \lambda_2$.

Here, $P_{kj}(t)$, $R_{pq}(t)$, $g(t)$, and the variable delays $\tau_{kj}(t)$ are given continuous functions defined on $0 \leq a \leq t \leq b$, where $\tau_{kj}(t) \geq 0$.

We propose a new matrix technique, developed by Sezer et.al. [9, 15, 20, 32], to solve Eq. (1.1) with the initial conditions, in the finite Lucas series of the form

$$y(t) \cong y_N(t) = \sum_{n=0}^N a_n L_n(t), \quad a \leq t \leq b \quad (1.2)$$

where a_n , $n = 0, 1, \dots, N$ are unknown coefficients and $L_n(t)$, $n = 0, 1, \dots, N$; $N \geq m$ are the Lucas polynomials [26]. These polynomials are constructed from the recurrence relation

$$\begin{aligned} L_0(t) &= 2, \quad L_1(t) = t, \\ L_{n+2}(t) &= tL_{n+1}(t) + L_n(t), \quad n \geq 0 \end{aligned}$$

The Binet and power form representations of Lucas polynomials can be seen in [2, 3].

2. Operational matrix relations

In this section, we derive the operational matrix relations of Eq. (1.1) and (1.2). For this purpose, we write the series in Eq. (1.2) as a matrix equation as follows

$$y(t) \cong y_N(t) = \mathbf{L}(t)\mathbf{A} = \mathbf{T}(t)\mathbf{M}\mathbf{A} \quad (2.1)$$

where

$$\begin{aligned} \mathbf{L}(t) &= [L_0(t) \quad L_1(t) \quad \cdots \quad L_N(t)], \quad \mathbf{T}(t) = [1 \quad t \quad \cdots \quad t^N], \\ \mathbf{A} &= [a_0 \quad a_1 \quad \cdots \quad a_N]^T \end{aligned}$$

and \mathbf{M} has the form

$$\mathbf{M}^T = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & 0 & \cdots & 0 \\ \frac{2}{1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & 0 & \frac{2}{2} \begin{pmatrix} 2 \\ 0 \end{pmatrix} & & 0 \\ 0 & \frac{3}{2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(n-1)}{\binom{n-1}{2}} \begin{pmatrix} \frac{n-1}{2} \\ \frac{n-1}{2} \end{pmatrix} & 0 & \frac{n-1}{\binom{n+1}{2}} \begin{pmatrix} \frac{n+1}{2} \\ \frac{n-3}{2} \end{pmatrix} & \cdots & 0 \\ 0 & \frac{n}{\binom{n+1}{2}} \begin{pmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{pmatrix} & 0 & \cdots & \frac{n}{n} \begin{pmatrix} n \\ 0 \end{pmatrix} \end{bmatrix}$$

when N is odd, and

$$\mathbf{M}^T = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & 0 & \cdots & 0 \\ \frac{2}{1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & 0 & \frac{2}{2} \begin{pmatrix} 2 \\ 0 \end{pmatrix} & & 0 \\ 0 & \frac{3}{2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{n-1}{\binom{n}{2}} \begin{pmatrix} \frac{n}{2} \\ \frac{n-2}{2} \end{pmatrix} & 0 & \cdots & 0 \\ \frac{n}{\binom{n}{2}} \begin{pmatrix} \frac{n}{2} \\ \frac{n}{2} \end{pmatrix} & 0 & \frac{n}{\binom{n+2}{2}} \begin{pmatrix} \frac{n+2}{2} \\ \frac{n-2}{2} \end{pmatrix} & \cdots & \frac{n}{n} \begin{pmatrix} n \\ 0 \end{pmatrix} \end{bmatrix}$$

when N is even.

One can write the relation between $\mathbf{T}(t)$ and its derivatives $\mathbf{T}'(t)$ and $\mathbf{T}''(t)$ as follows

$$\mathbf{T}'(t) = \mathbf{T}(t)\mathbf{B} \text{ and } \mathbf{T}''(t) = \mathbf{T}(t)\mathbf{B}^2 \tag{2.2}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \text{ and } \mathbf{B}^0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

In a similar way, the approximate solution and its derivatives can be expressed by using Eq. (2.1) and (2.2) as

$$\begin{aligned} y(t) &\cong y_N(t) = \mathbf{L}(t)\mathbf{A} = \mathbf{T}(t)\mathbf{M}\mathbf{A} \\ y'(t) &\cong y'_N(t) = \mathbf{T}'(t)\mathbf{M}\mathbf{A} = \mathbf{T}(t)\mathbf{B}\mathbf{M}\mathbf{A} \\ y''(t) &\cong y''_N(t) = \mathbf{T}''(t)\mathbf{M}\mathbf{A} = \mathbf{T}(t)\mathbf{B}^2\mathbf{M}\mathbf{A}. \end{aligned} \tag{2.3}$$

Replacing t by $t - \tau_{kj}(t)$ in each equation in (2.3) yields the recurrence relation,

$$\begin{aligned}
 y(t - \tau_{kj}(t)) &\cong y_N(t - \tau_{kj}(t)) = \mathbf{T}(t - \tau_{kj}(t))\mathbf{MA} = \mathbf{T}(t)\mathbf{S}(-\tau_{kj}(t))\mathbf{MA} \\
 y'(t - \tau_{kj}(t)) &\cong y'_N(t - \tau_{kj}(t)) = \mathbf{T}(t - \tau_{kj}(t))\mathbf{BMA} = \mathbf{T}(t)\mathbf{S}(-\tau_{kj}(t))\mathbf{BMA} \\
 y''(t - \tau_{kj}(t)) &\cong y''_N(t - \tau_{kj}(t)) = \mathbf{T}(t - \tau_{kj}(t))\mathbf{B}^2\mathbf{MA} = \mathbf{T}(t)\mathbf{S}(-\tau_{kj}(t))\mathbf{B}^2\mathbf{MA}.
 \end{aligned} \tag{2.4}$$

Note that, $\mathbf{T}(t - \tau_{kj}(t)) = \mathbf{T}(t)\mathbf{S}(-\tau_{kj}(t))$ and

$$\mathbf{S}(-\tau_{kj}(t)) = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} (-\tau_{kj}(t))^0 & \begin{pmatrix} 1 \\ 0 \end{pmatrix} (-\tau_{kj}(t))^1 & \cdots & \begin{pmatrix} N \\ 0 \end{pmatrix} (-\tau_{kj}(t))^N \\ 0 & \begin{pmatrix} 1 \\ 1 \end{pmatrix} (-\tau_{kj}(t))^0 & \cdots & \begin{pmatrix} N \\ 1 \end{pmatrix} (-\tau_{kj}(t))^{N-1} \\ 0 & 0 & \cdots & \begin{pmatrix} N \\ 2 \end{pmatrix} (-\tau_{kj}(t))^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \begin{pmatrix} N \\ N \end{pmatrix} (-\tau_{kj}(t))^0 \end{bmatrix}.$$

In addition, we can obtain the matrix forms of $(y^{(0)}(t))^2, y^{(1)}(t)y^{(0)}(t)$ and $(y^{(1)}(t))^2$ which appears in the nonlinear part of Eq. (1.1), by using Eq. (2.3) as

$$\begin{aligned}
 (y^{(0)}(t))^2 &= \mathbf{T}(t)\mathbf{M}\overline{\mathbf{T}(t)} \overline{\mathbf{M}} \overline{\mathbf{A}} \\
 y^{(1)}(t) y^{(0)}(t) &= \mathbf{T}(t) \mathbf{B}\mathbf{M}\overline{\mathbf{T}(t)} \overline{\mathbf{M}} \overline{\mathbf{A}} \\
 (y^{(1)}(t))^2 &= \mathbf{T}(t)\mathbf{B}\mathbf{M}\overline{\mathbf{T}(t)} \overline{\mathbf{B}} \overline{\mathbf{M}} \overline{\mathbf{A}}
 \end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
 \overline{\mathbf{T}(t)} &= \text{diag} [\mathbf{T}(t)]_{(N+1) \times (N+1)^2}, \quad \overline{\mathbf{M}} = \text{diag} [\mathbf{M}]_{(N+1)^2 \times (N+1)^2}, \\
 \overline{\mathbf{B}} &= \text{diag} [\mathbf{B}]_{(N+1)^2 \times (N+1)^2}, \quad \overline{\mathbf{A}} = [a_0\mathbf{A} \quad a_1\mathbf{A} \quad \cdots \quad a_N\mathbf{A}]^T
 \end{aligned}$$

Substituting the collocation points ($t_i = a + (b - a)i/N, i = 0, 1, \dots, N$) into Eq. (1.1), gives the system of equations

$$\sum_{k=0}^2 \sum_{j=0}^1 P_{kj}(t_i) y^{(k)}(t_i - \tau_{kj}(t_i)) + \sum_{p=0}^1 \sum_{q=0}^p R_{pq}(t_i) y^{(p)}(t_i) y^{(q)}(t_i) = g(t_i),$$

which can be expressed with the aid of Eqs. (2.4) and (2.5) as

$$\sum_{k=0}^2 \sum_{j=0}^1 \mathbf{P}_{kj} \overline{\mathbf{T}} \mathbf{S}_{kj} \mathbf{B}^k \mathbf{MA} + \sum_{p=0}^1 \sum_{q=0}^p \mathbf{R}_{pq} \mathbf{T}_{pq} \overline{\mathbf{A}} = \mathbf{G} \tag{2.6}$$

where

$$\mathbf{P}_{kj} = \text{diag} [P_{kj}(t_0) \quad P_{kj}(t_1) \quad \cdots \quad P_{kj}(t_N)],$$

$$\overline{\mathbf{T}} = \begin{bmatrix} T(t_0) \\ T(t_1) \\ \vdots \\ T(t_N) \end{bmatrix} = \begin{bmatrix} 1 & t_0 & \cdots & t_0^N \\ 1 & t_1 & \cdots & t_1^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_N & \cdots & t_N^N \end{bmatrix},$$

$$\mathbf{S}_{kj} = \begin{bmatrix} S(-\tau_{kj}(t_0)) \\ S(-\tau_{kj}(t_1)) \\ \vdots \\ S(-\tau_{kj}(t_N)) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g(t_0) \\ g(t_1) \\ \vdots \\ g(t_N) \end{bmatrix},$$

$$\mathbf{R}_{pq} = \text{diag} [R_{pq}(t_0) \quad R_{pq}(t_1) \quad \cdots \quad R_{pq}(t_N)],$$

$$\mathbf{T}_{pq} = \begin{bmatrix} T(t_0)\mathbf{B}^p\overline{\mathbf{M}}T(t_0)\overline{B}^q\overline{\mathbf{M}} \\ T(t_1)\mathbf{B}^p\overline{\mathbf{M}}T(t_1)\overline{B}^q\overline{\mathbf{M}} \\ \vdots \\ T(t_N)\mathbf{B}^p\overline{\mathbf{M}}T(t_N)\overline{B}^q\overline{\mathbf{M}} \end{bmatrix}; \quad p, q = 0, 1.$$

The fundamental matrix equation (2.6) can be briefly expressed in the form

$$\mathbf{W}\mathbf{A} + \mathbf{Z}\overline{\mathbf{A}} = \mathbf{G} \tag{2.7}$$

where

$$\mathbf{W} = \sum_{k=0}^2 \sum_{j=0}^1 \mathbf{P}_{kj} \overline{\mathbf{T}}\mathbf{S}_{kj}\mathbf{B}^k\mathbf{M} = [w_{ij}]; \quad i, j = 0, 1, \dots, N$$

$$\mathbf{Z} = \sum_{p=0}^1 \sum_{q=0}^p \mathbf{R}_{pq}\mathbf{T}_{pq} = [z_{mn}]; \quad m = 0, 1, \dots, N, \quad n = 0, 1, \dots, (N + 1)^2$$

$$\mathbf{G} = [g(t_0) \quad g(t_1) \quad \cdots \quad g(t_N)]^T.$$

Also we can write the matrix equation (2.7) in the augmented form as

$$[\mathbf{W}; \mathbf{Z}; \mathbf{G}] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & z_{00} & z_{01} & \cdots & z_{0(N+1)^2} & ; & g(t_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & z_{10} & z_{11} & \cdots & z_{1(N+1)^2} & ; & g(t_1) \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ w_{N0} & w_{N1} & \cdots & w_{NN} & ; & z_{N0} & z_{N1} & \cdots & z_{N(N+1)^2} & ; & g(t_N) \end{bmatrix}. \tag{2.8}$$

Now, let us write the initial conditions $y(a) = \lambda_1$ and $y'(a) = \lambda_2$ in the matrix form by using the relations in Eq. (2.3)

$$\mathbf{T}(a)\mathbf{M}\mathbf{A} = \lambda_1 \text{ and } \mathbf{T}(a)\mathbf{B}\mathbf{M}\mathbf{A} = \lambda_2$$

or briefly

$$\mathbf{U}_1\mathbf{A} + \mathbf{O}^*\overline{\mathbf{A}} = \lambda_1 \text{ and } \mathbf{U}_2\mathbf{A} + \mathbf{O}^*\overline{\mathbf{A}} = \lambda_2, \tag{2.9}$$

where

$$\mathbf{U}_1 = [u_{00} \quad u_{01} \quad \cdots \quad u_{0N}] = \mathbf{T}(a)\mathbf{M},$$

$$\mathbf{U}_2 = [u_{10} \quad u_{11} \quad \cdots \quad u_{1N}] = \mathbf{T}(a)\mathbf{B}\mathbf{M},$$

$$\mathbf{O}^* = [0 \quad 0 \quad \cdots \quad 0].$$

In order to find the unknown Lucas coefficients a_n , ($n = 0, 1, \dots, N$), related to the approximate solution Eq. (1.2), we replace the row matrices in Eq. (2.9) by any rows of the augmented matrix in Eq. (2.8). Consequently, we obtain a new augmented matrix $\left[\widetilde{\mathbf{W}} \ ; \ \widetilde{\mathbf{Z}} \ ; \ \widetilde{\mathbf{G}} \right]$ related to the matrix equation $\widetilde{\mathbf{W}}\mathbf{A} + \widetilde{\mathbf{Z}}\overline{\mathbf{A}} = \widetilde{\mathbf{G}}$.

We solve this nonlinear algebraic system using NSolve routine in Mathematica, and obtain the unknown coefficients. Then, substitute them in Eq. (2.1) to obtain the approximate solution. A detailed theoretical convergence and error analysis of Lucas expansion of a function is given in [2,3]. Thus, rate of convergence is investigated numerically in this study.

3. Examples and discussion

In this section, we apply the method to three problems to demonstrate the validity and accuracy of the method. In the first problem, the application of the method yields the exact solution. In order to show the efficiency of the method for the next two problems, we compute the absolute errors for each collocation point t_i as follows

$$E_N(t_i) = |y(t_i) - y_N(t_i)|.$$

3.1. Example 1:

Consider the first order nonlinear differential equation with variable delay t^2 :

$$\begin{cases} y'(t) + ty(t - t^2) + ty^2(t) = 1 + t^2, & 0 \leq t \leq 1 \\ y(0) = 0 \end{cases}$$

The exact solution of the above problem is $y(t) = t$. We aim to show that the exact solution could be found using the present method.

First, we approximate the solution $y(t)$ by the Lucas polynomial $y_N(t) = \sum_{k=0}^N a_k L_k(t)$, and formulate the problem in the form of Eq. (1.1). Here,

$$\begin{cases} P_{10}(t) = 1, & P_{11}(t) = 0, & P_{00}(t) = t, & P_{01}(t) = 0, \\ \tau_{10}(t) = 0, & \tau_{11}(t) = 0, & \tau_{00}(t) = t^2, & \tau_{01}(t) = 0, \\ R_{00}(t) = t, & R_{10}(t) = 0, & R_{11}(t) = 0, \\ g(t) = 1 - t^2 \end{cases}$$

The collocation points are computed as $\left\{ t_0 = 0, t_1 = \frac{1}{2}, t_2 = 1 \right\}$ by taking $N = 2$. Then, we write the fundamental matrix equation of the given problem as

$$\underbrace{\left[\mathbf{P}_{10}\overline{\mathbf{T}}\mathbf{S}_{10}\mathbf{B}^1\mathbf{M} + \mathbf{P}_{00}\overline{\mathbf{T}}\mathbf{S}_{00}\mathbf{B}^0\mathbf{M} \right]}_{\mathbf{W}}\mathbf{A} + \underbrace{\left[\mathbf{R}_{00}\overline{\mathbf{T}}_{00} \right]}_{\mathbf{Z}}\overline{\mathbf{A}} = \mathbf{G}.$$

where

$$\begin{aligned}
 \mathbf{P}_{10} &= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \overline{\mathbf{T}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \\
 \mathbf{S}_{10} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B}^1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
 \mathbf{P}_{00} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{S}_{00} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{21} & \frac{1}{16} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{R}_{00} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
 \mathbf{T}_{00} &= \begin{bmatrix} 4 & 0 & 4 & 0 & 0 & 0 & 4 & 2 & 6 \\ 4 & 0 & 4 & 1 & \frac{1}{4} & \frac{9}{8} & \frac{9}{2} & \frac{9}{4} & \frac{27}{4} \\ 4 & 0 & 4 & 2 & \frac{1}{2} & \frac{9}{4} & 6 & 3 & 9 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 \\ \frac{5}{4} \\ 2 \end{bmatrix}.
 \end{aligned}$$

Now, we can calculate \mathbf{W} and \mathbf{Z}

$$\begin{aligned}
 \mathbf{W} &= \mathbf{P}_{10} \overline{\mathbf{T}} \mathbf{S}_{10} \mathbf{B}^1 \mathbf{M} + \mathbf{P}_{00} \overline{\mathbf{T}} \mathbf{S}_{00} \mathbf{B}^0 \mathbf{M} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1.125 & 2.03125 \\ 2 & 1 & 4 \end{bmatrix}, \\
 \mathbf{Z} &= \mathbf{R}_{00} \mathbf{T}_{00} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & \frac{1}{2} & \frac{1}{8} & \frac{9}{16} & \frac{9}{4} & \frac{9}{8} & \frac{27}{8} \\ 4 & 0 & 4 & 2 & \frac{1}{2} & \frac{9}{4} & 6 & 3 & 9 \end{bmatrix}.
 \end{aligned}$$

Hence the augmented matrix $[\mathbf{W}; \mathbf{Z}; \mathbf{G}]$ can be written as

$$\begin{bmatrix} 0 & 1 & 0 & ; & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1 \\ 1 & 1.125 & 2.03125 & ; & 2 & 0 & 2 & \frac{1}{2} & \frac{1}{8} & \frac{9}{16} & \frac{9}{4} & \frac{9}{8} & \frac{27}{8} & ; & \frac{5}{4} \\ 2 & 1 & 4 & ; & 4 & 0 & 4 & 2 & \frac{1}{2} & \frac{9}{4} & 6 & 3 & 9 & ; & 2 \end{bmatrix}.$$

The initial condition matrix is calculated as

$$\mathbf{U} = [2 \ 0 \ 2 \ ; \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ ; \ 0].$$

Replacing this row by the third row of the augmented matrix gives

$$\begin{bmatrix} 0 & 1 & 0 & ; & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1 \\ 1 & 1.125 & 2.03125 & ; & 2 & 0 & 2 & \frac{1}{2} & \frac{1}{8} & \frac{9}{16} & \frac{9}{4} & \frac{9}{8} & \frac{27}{8} & ; & \frac{5}{4} \\ 2 & 0 & 2 & ; & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 0 \end{bmatrix}.$$

Once we solve this system, we get the unknown coefficients as $\mathbf{A} = [0 \quad 1 \quad 0]^T$, and hence we obtain the analytical solution

$$y(t) = \mathbf{L}(t)\mathbf{A} = [2 \quad t \quad t^2 + 2] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = t.$$

The present method is said to be accurate and efficient.

3.2. Example 2:

The next example is a second order NDE with variable delays $\{t^2, -\frac{t}{2}\}$

$$\begin{cases} y''(t) + y'(t - t^2) - t^2 y(t + \frac{t}{2}) + (y'(t))^2 - y'(t)y(t) = e^t + e^{t-t^2} - t^2 e^{3t/2} \\ y(0) = y'(0) = 1, t \in [0, 1] \end{cases}$$

The analytical solution of this problem is $y(t) = e^t$. We solve the problem for several values of N . Table 1 shows the absolute errors for $N = 4, 5, 7$ and 9 . One can see that even $N = 4$ yields an accuracy up to four decimal places. Increasing N decreases the absolute error for each collocation point.

Table 1. Absolute errors for several values of N

t_i	$E_4(t_i)$	$E_5(t_i)$	$E_7(t_i)$	$E_9(t_i)$
0.2	$5.12e - 06$	$2.05e - 06$	$2.79e - 09$	$1.86e - 12$
0.4	$2.44e - 05$	$6.22e - 06$	$5.53e - 09$	$3.56e - 12$
0.6	$1.77e - 04$	$1.37e - 05$	$8.75e - 09$	$4.95e - 12$
0.8	$2.08e - 04$	$1.85e - 05$	$4.84e - 09$	$8.55e - 11$
1.0	$9.90e - 04$	$9.09e - 05$	$5.35e - 07$	$4.78e - 09$

Table 2 presents the convergence rate calculated by, [33]

$$R_N = \log \left[\frac{\|y(t) - y_N(t)\|}{\|y(t) - y_{N+1}(t)\|} \right] \frac{1}{\log 2}, t \in [a, b].$$

We observe a cubic convergence for several values of N at the point $t = 1$.

Table 2. Convergence rate of the present method at $t = 1$.

N	4	5	6	7	8
R_N	3.4446	3.5060	3.9020	3.7218	3.0843

Figure 1 presents the analytical and approximate solution when $N = 4$. One can see that the approximate solution agrees very well with the exact solution.

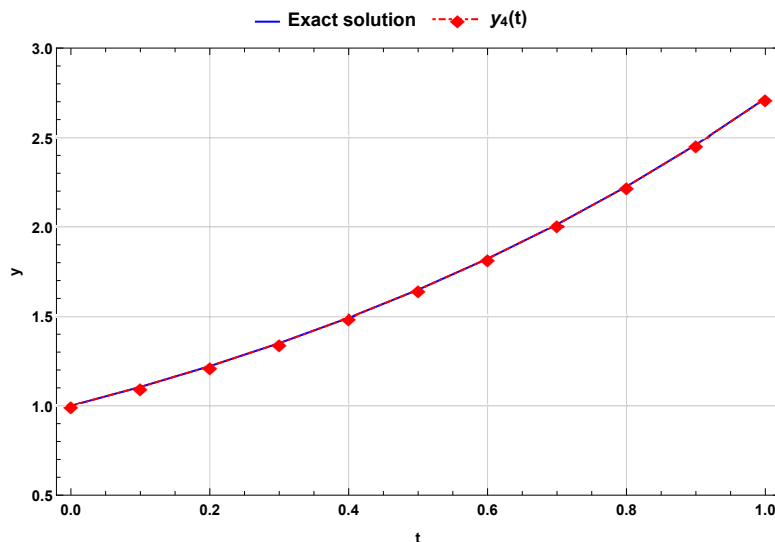


Figure 1. Exact solution and approximate solution for $N = 4$.

3.3. Example 3:

For the third example, we consider a second order NDE with variable delay $t - t^3/8$, [10]

$$\begin{cases} y''(t) + 2y(t) - y^2(t) + y(t^3/8) = \sin t - \sin^2 t + \sin(t^3/8), & 0 \leq t \leq 1 \\ y(0) = 0, y'(0) = 1 \end{cases}$$

The analytical solution of this problem is $y(t) = \sin t$. Figure 2 presents the the exact and approximate solutions for $N = 9, 10$ and 11 . One can see that the numerical solution with $N = 11$ agrees well with the exact solution.

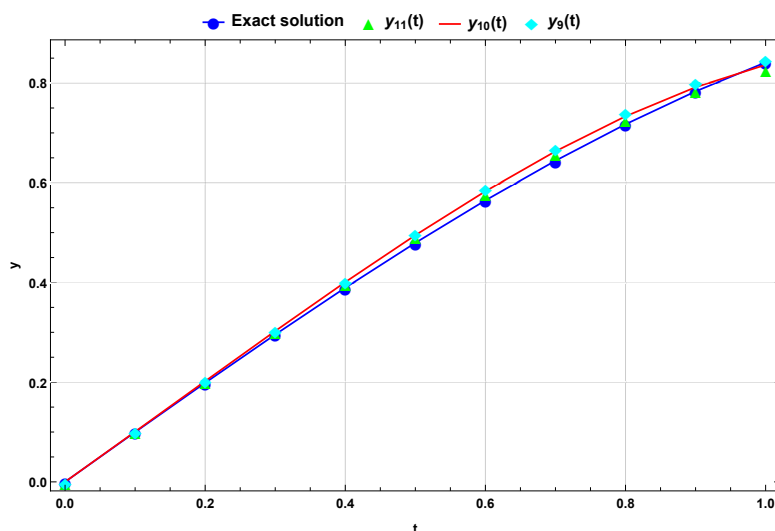


Figure 2. Comparison of analytical and numerical solutions for several values of N .

4. Conclusion

In this paper, nonlinear differential equations with variable delays are solved by Lucas polynomial approach. The main advantage of the method is to convert the nonlinear

equations to a system of nonlinear algebraic equations. The efficiency of the proposed method is tested on three problems. The results are presented in terms of absolute errors calculated at each collocation point. It is observed that the method enables high accuracy numerical solutions or even analytical solution. Thus, we can say that this is an effective and convenient approach to solve the indicated type of problems.

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