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A Note on Nearly Sasakian and Nearly Cosymplictic Structures of 5-Dimensional Spheres

Sharief Deshmukh* and Ibrahim Al-Dayel

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ABSTRACT

In this paper, we show that with the nearly Sasakian structure (φ, ξ, η, g) on the 5-dimensional sphere $S^5(2)$ of constant curvature 2 (cf. [2]), there are naturally associated two additional structures $(\varphi_1, \xi, \eta, g)$, $(\varphi_2, \xi, \eta, g)$ on $S^5(2)$, where $S^5(2)(\varphi_1, \xi, \eta, g)$ is homothetic to a Sasakian manifold and $S^5(2)(\varphi_2, \xi, \eta, g)$ is a nearly cosymplectic manifold. Similarly, we show that on the unit sphere S^5, which is known to have a nearly cosymplectic structure (ψ_1, ξ, η, g) (cf. [2]), there are two additional structures (ψ_2, ξ, η, g), (ψ_3, ξ, η, g) on S^5 such that $S^5(\psi_2, \xi, \eta, g)$ is a Sasakian manifold and $S^5(\psi_3, \xi, \eta, g)$ is a nearly cosymplectic manifold and the last nearly cosymplectic structure is independent of the nearly cosymplectic structure (ψ_1, ξ, η, g), in the sense that these three structures satisfy $\psi_1\psi_2 = -\psi_2\psi_1 = \psi_3$, $\psi_2\psi_3 = -\psi_3\psi_2 = \psi_1$ and $\psi_3\psi_1 = -\psi_1\psi_3 = \psi_2$.

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1. Introduction

Recall that, as an odd dimensional analogue of the nearly Kaehler structure on the unit sphere S^6, which is not Kaehler [4], in [2,3], nearly Sasakian structure was introduced on the 5-dimensional sphere $S^5(2)$ of constant curvature 2 as totally umbilical hypersurface of S^6, which is not a Sasakian structure. However, the nearly Kaehler structure on S^6 is not related to any Kaehler structure as there are no known Kaehler structures on the six dimensional sphere S^6. Though, in case of the 5-dimensional sphere $S^5(2)$, which is a nearly Sasakian manifold, it does not admit any Sasakian structure owing to the fact that $S^5(2)$ has constant sectional curvature 2 and a Sasakian structure requires sectional curvatures of the plane sections containing Reeb vector field ξ to be the constant 1. However, in this paper, it is shown that the nearly structure (φ, ξ, η, g) on $S^5(2)$ naturally induces two more structures $(\varphi_1, \xi, \eta, g)$, $(\varphi_2, \xi, \eta, g)$ on $S^5(2)$ such that $S^5(2)(\varphi_1, \xi, \eta, g)$ is homothetic to a Sasakian manifold and $S^5(2)(\varphi_1, \xi, \eta, g)$ is a nearly cosymplectic manifold. Thus the link that nearly nearly Kaehler structure of S^6 is not related to any Kaehler structure is broken in case of the nearly Sasakian structure of $S^5(2)$.

Similarly, it is known that the unit sphere S^5 as totally geodesic hypersurface of the nearly Kaehler 6-sphere S^6 inherits a nearly cosymplectic structure (ψ_1, ξ, η, g) [2], which is not cosymplectic. In this paper, we show that there are two additional structures (ψ_i, ξ, η, g), $i = 2, 3$ associated to the nearly cosymplectic structure (ψ_1, ξ, η, g) on S^5 of which (ψ_2, ξ, η, g) is Sasakian and (ψ_3, ξ, η, g) is nearly cosymplectic structure and these structures satisfy

$$\psi_1\psi_2 = -\psi_2\psi_1 = \psi_3, \quad \psi_2\psi_3 = -\psi_3\psi_2 = \psi_1, \quad \psi_3\psi_1 = -\psi_1\psi_3 = \psi_2,$$

that is, the new nearly cosymplectic structure (ψ_3, ξ, η, g) on S^5 is independent of the original nearly cosymplectic structure (ψ_1, ξ, η, g) of S^5.

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*** Corresponding author***

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2. Preliminaries

Let M be a real hypersurface of the nearly Kaehler 6-sphere $(S^6, J, \overline{g})$ with unit normal vector field N. Then the hypersurface M admits an almost contact metric structure (φ, ξ, η, g), where g is the induced metric and

$$JX = \varphi X + \eta(X)N, \quad J\xi = N, \quad X, Y \in \mathfrak{X}(M),$$

where η is the smooth 1-form dual to the unit vector field ξ, φX is the tangential component of JX and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on the hypersurface M. The almost contact metric structure (φ, ξ, η, g) satisfies

$$\varphi^2 = -I + \eta \otimes \xi, \varphi(\xi) = 0, \eta \circ \varphi = 0,$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

In [3], it is shown that the 5-dimensional sphere $S^5(2)$ of constant curvature 2 as totally umbilical hypersurface of the nearly Kaehler 6-sphere $(S^6, J, \overline{g})$ is a nearly nearly Sasakian manifold, that is, it admits an almost contact metric structure (φ, ξ, η, g) that satisfies

$$(\nabla\varphi)(X, Y) + (\nabla\varphi)(Y, X) = \eta(Y)X + \eta(X)Y - 2g(X, Y)\xi, \quad X \in \mathfrak{X}(S^5(2)), \tag{2.1}$$

where $(\nabla\varphi)(X, Y) = \nabla_X \varphi Y - \varphi(\nabla_X Y)$, ∇ is covariant derivative operator with respect to the induced metric g (see also [2], [6], [7]).

Also, in ([2], [3]), it is shown that the unit 5-sphere S^5 as totally geodesic hypersurface of the nearly Kaehler 6-sphere $(S^6, J, \overline{g})$ is a nearly cosymplectic manifold, that is, it admits an almost contact metric structure (φ, ξ, η, g) that satisfies

$$(\nabla\varphi)(X, Y) + (\nabla\varphi)(Y, X) = 0, \quad X \in \mathfrak{X}(S^5). \tag{2.2}$$

An almost contact metric manifold $M(\varphi, \xi, \eta, g)$ is said to be a Sasakian manifold if the following holds

$$(\nabla\varphi)(X, Y) = \eta(Y)X - g(X, Y)\xi, \quad X \in \mathfrak{X}(M). \tag{2.3}$$

It follows from above equation that a Sasakian manifold is a nearly Sasakian manifold but the converse is not true. In particular, the nearly Sasakian manifold $S^5(2)(\varphi, \xi, \eta, g)$ is not a Sasakian manifold as a Sasakian manifold requires sectional curvatures of the plane sections containing the Reeb vector field ξ to be constant 1.

A smooth vector field ξ on a Riemannian manifold (M, g) is said to be a Killing vector field if its flow consists of isometries of the Riemannian manifold (M, g) or equivalently

$$\pounds_\xi g = 0,$$

where $\pounds_\xi$ is the Lie derivative with respect to ξ. If η is a smooth 1-form dual to the Killing vector field ξ and if we define a skew-symmetric (1,1) tensor field ψ on M by $d\eta(X, Y) = 2g(\psi X, Y)$, $X, Y \in \mathfrak{X}(M)$, then using Koszul's formula (cf [1]), we get

$$\nabla_X \xi = \psi X, \quad X \in \mathfrak{X}(M). \tag{2.4}$$

The above equation, immediately gives the following expression

$$R(X, Y)\xi = (\nabla\psi)(X, Y) - (\nabla\psi)(Y, X), \quad X, Y \in \mathfrak{X}(M), \tag{2.5}$$

where R is the curvature tensor of the Riemannian manifold (M, g). Also, as the smooth 2-form $\Omega(X, Y) = g(\psi X, Y)$ is closed, we have

$$g((\nabla\psi)(X, Y), Z) + g((\nabla\psi)(Y, Z), X) + g((\nabla\psi)(Z, X), Y) = 0,$$

which together with skew-symmetry of ψ and the equation (2.5), gives

$$(\nabla\psi)(X, Y) = R(X, \xi)Y, \quad X, Y \in \mathfrak{X}(M). \tag{2.6}$$

3. Nearly Sasakian manifold $S^5(2)(\varphi,\xi,\eta,g)$

In this section, we investigate the existence of other structures on the nearly Sasakian manifold $S^5(2)(\varphi,\xi,\eta,g)$. Recall that the Reeb vector field ξ on the nearly Sasakian manifold $S^5(2)(\varphi,\xi,\eta,g)$, is Killing [2,3] and hence there exists a skew-symmetric tensor field φ_1 on $S^5(2)$ satisfying

$$\nabla_X\xi = \varphi_1 X, \quad X\in\mathfrak{X}(S^5(2)) \tag{3.1}$$

and by equation (2.6), we have

$$(\nabla\varphi_1)(X,Y) = R(X,\xi)Y = 2\left(g(Y,\xi)X - g(X,Y)\xi\right), \quad X,Y\in\mathfrak{X}(S^5(2)). \tag{3.2}$$

Taking $Y=\xi$, we have

$$\varphi_1^2 X = 2\left(-X+\eta(X)\xi\right), \tag{3.3}$$

and as ξ is a unit vector field, the equation (3.1) gives $\varphi_1(\xi)=\nabla_\xi\xi=0$. Moreover, equations (3.1) and (3.2), give

$$\nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi = (\nabla\varphi_1)(X,Y) = 2\left(g(Y,\xi)X - g(X,Y)\xi\right),$$

which by a result in [5], implies that $S^5(2)(\varphi_1,\xi,\eta,g)$ is homothetic to a Sasakian manifold. Indeed, this new Sasakian structure $(\varphi_1^{'},\xi^{'},\eta^{'},g^{'})$ on $S^5(2)$ is given by

$$\varphi_1=\sqrt{2}\varphi_1^{'},\ \xi=\sqrt{2}\xi^{'},\ \eta=\frac{1}{\sqrt{2}}\eta^{'},\ g=\frac{1}{2}g^{'}.$$

Now, define a $(1,1)$ tensor field φ_2 on the nearly Sasakian manifold $S^5(2)(\varphi,\xi,\eta,g)$ by

$$\varphi_2(X) = (\nabla\varphi)(\xi,X), \quad X\in\mathfrak{X}(S^5(2)),$$

then by equation (2.1), it follows that $\varphi_2(\xi)=0$ and that φ_2 is a skew-symmetric tensor. First, we investigate the relations between these three operators φ, φ_1 and φ_2 in the following:

Lemma 3.1. *The operators φ_1 and φ_2 on the nearly Sasakian manifold $S^5(2)(\varphi,\xi,\eta,g)$ satisfy*

$$\begin{aligned}\varphi_1 X &= \varphi\varphi_1\varphi X + 2\varphi X, \quad \frac{1}{2}\left(\varphi\left(\varphi_1 X\right)+\varphi_1(\varphi X)\right) = -X+\eta(X)\xi,\\ \varphi_2 X &= \frac{1}{2}\left(\varphi\varphi_1 X - \varphi_1\varphi X\right), \quad X,Y\in X\left(S^5(2)\right).\end{aligned}$$

Proof. Taking $Y=\xi$ in equation (2.1), we get

$$-\varphi\left(\nabla_X\xi\right) + (\nabla\varphi)(X,Y) = X - \eta(X)\xi,$$

which gives,

$$\varphi_1 X = \varphi X - \varphi\varphi_2 X, \tag{3.4}$$

that is,

$$\varphi\varphi_1 X = -X + \eta(X)\xi + \varphi_2 X, \tag{3.5}$$

where we used $\varphi_2(\xi)=0$. Now, replacing X by φX in (3.4), we conclude

$$\varphi_1\varphi X = -X + \eta(X)\xi - \varphi\varphi_2\varphi X. \tag{3.6}$$

We have

$$\begin{aligned}\varphi\varphi_2\varphi X &= \varphi\left(\nabla\varphi\right)(\xi,\varphi X) = \varphi\left[-\nabla_\xi X + \eta\left(\nabla_\xi X\right)\xi - \varphi\left(\nabla_\xi\varphi X\right)\right]\\ &= \varphi\left[-\nabla_\xi X - \varphi\left(\left(\nabla\varphi\right)(\xi,X) + \varphi\left(\nabla_\xi X\right)\right)\right]\\ &= \varphi\left[-\nabla_\xi X - \varphi\varphi_2 X + -\nabla_\xi X + \eta\left(\nabla_\xi X\right)\xi\right]\\ &= \varphi_2 X,\end{aligned}$$

consequently, equation (3.6) takes the form

$$\varphi_1\varphi X = -X + \eta(X)\xi - \varphi_2 X. \tag{3.7}$$

Then equations (3.5) and (3.7), give

$$\varphi_2 X = \frac{1}{2}\left(\varphi\varphi_1 X) - \varphi_1\varphi X)\right) \text{ and } \frac{1}{2}\left(\varphi\left(\varphi_1 X\right) + \varphi_1(\varphi X))\right) = -X + \eta(X)\xi, \tag{3.8}$$

which on substituting $\varphi_2 X$ in equation (3.4) proves the Lemma. □

Now, we are in position to prove the following:

Theorem 3.1. *There are two structures* $(\varphi_i, \xi, \eta, g)$, $i = 1, 2$ *associated to the nearly Sasakian structure* (φ, ξ, η, g) *on the* 5*-sphere* $S^5(2)$ *such that* $S^5(2)(\varphi_1, \xi, \eta, g)$ *is homothetic to a Sasakian manifold and* $S^5(2)(\varphi_2, \xi, \eta, g)$ *is a nearly cosymplectic manifold.*

Proof. It remains to prove that $(\varphi_2, \xi, \eta, g)$ is a nearly cosymplectic structure on $S^5(2)$. We use equation (2.1), to compute

$$\begin{aligned} \varphi_2^2 X &= (\nabla\varphi)(\xi, (\nabla\varphi)(\xi, X)) \\ &= -(\nabla\varphi)((\nabla\varphi)(\xi, X), \xi) + (\nabla\varphi)(\xi, X), \end{aligned}$$

where we used $\eta((\nabla\varphi)(\xi, X)) = g((\nabla\varphi)(\xi, X), \xi) = -g(X, (\nabla\varphi)(\xi, \xi)) = 0$. Thus, again on using equation (2.1),

$$\begin{aligned} \varphi_2^2 X &= -(\nabla\varphi)\left(-(\nabla\varphi)(X, \xi) - \eta(X)\xi + X, \xi\right) - (\nabla\varphi)(X, \xi) - \eta(X)\xi + X \\ &= -(\nabla\varphi)\varphi(\nabla_X\xi), \xi) - 2(\nabla\varphi)(X, \xi) - \eta(X)\xi + X \\ &= \varphi\left(\nabla_{\varphi(\nabla_X\xi)}\xi\right) + 2\varphi(\nabla_X\xi) - \eta(X)\xi + X \\ &= \varphi\varphi_1\varphi\varphi_1 X + 2\varphi\varphi_1 X - \eta(X)\xi + X \\ &= (\varphi\varphi_1\varphi + 2\varphi)\varphi_1 X - \eta(X)\xi + X, \end{aligned}$$

which by Lemma 3.1, gives

$$\varphi_2^2 X = \varphi_1^2 X - \eta(X)\xi + X = -X + \eta(X)\xi,$$

where we used equation (3.3). Also, as φ_2 is skew symmetric, we have

$$g(\varphi_2 X, \varphi_2 Y) = -g(\varphi_2^2 X, Y) = g(X, Y) - \eta(X)\eta(Y).$$

Hence, $(\varphi_2, \xi, \eta, g)$ is an almost contact metric structure on $S^5(2)$. Finally, on using equations (2.1), (3.1) and Lemma 2.1, we have

$$\begin{aligned} (\nabla\varphi_2)(X, Y) &= \nabla_X(\nabla\varphi)(\xi, Y) - (\nabla\varphi)(\xi, \nabla_X Y) \\ &= \nabla_X\left(-(\nabla\varphi)(Y, \xi) - \eta(Y)\xi + Y\right) + (\nabla\varphi)(\nabla_X Y, \xi) + \eta(\nabla_X Y)\xi - \nabla_X Y \\ &= \nabla_X(\varphi\varphi_1 Y) - X(\eta(Y))\xi - \eta(Y)\varphi_1 X - \varphi\varphi_1(\nabla_X Y) + \eta(\nabla_X Y)\xi \\ &= \nabla_X(-\varphi_1\varphi Y - 2Y + 2\eta(Y)\xi) - X(\eta(Y))\xi - \eta(Y)\varphi_1 X + \varphi_1\varphi(\nabla_X Y) \\ &\quad +2\nabla_X Y - 2\eta(\nabla_X Y)\xi + \eta(\nabla_X Y)\xi \\ &= -\left[(\nabla\varphi_1)(X, \varphi Y) + \varphi_1(\nabla\varphi)(X, Y) + \varphi_1\varphi\nabla_X Y\right] + X(\eta(Y))\xi \\ &\quad +\eta(Y)\varphi_1 X - \eta(\nabla_X Y)\xi + \varphi_1\varphi(\nabla_X Y) \\ &= -(\nabla\varphi_1)(X, \varphi Y) - \varphi_1(\nabla\varphi)(X, Y) + g(Y, \varphi_1 X)\xi + \eta(Y)\varphi_1 X. \end{aligned}$$

Now, using equation (3.2), we conclude

$$(\nabla\varphi_2)(X, Y) = 2g(X, \varphi Y)\xi - \varphi_1(\nabla\varphi)(X, Y) + g(Y, \varphi_1 X)\xi + \eta(Y)\varphi_1 X,$$

which in view of equation (2.1), gives

$$\begin{aligned} (\nabla\varphi_2)(X, Y) + (\nabla\varphi_2)(Y, X) &= -\varphi_1(\eta(Y)X + \eta(X)Y - 2g(X, Y)\xi) \\ &\quad +\eta(Y)\varphi_1 X + +\eta(X)\varphi_1 Y \\ &= 0. \end{aligned}$$

Hence, $(\varphi_2, \xi, \eta, g)$ is a nearly cosymplectic structure on $S^5(2)$. □

4. Nearly cosymplictic manifold $S^5(\psi_1, \xi, \eta, g)$

Recall that the totally geodesic hypersphere S^5 of the nearly Kaehler 6-sphere $(S^6, J, \overline{g})$ admits an almost contact metric structure (ψ_1, ξ, η, g) satisfying $JX = \psi_1 X + \eta(X)N$, $X \in \mathfrak{X}(S^5)$, where $\psi_1 X$ is the tangential component of JX and N is the unit normal vector field to S^5, which is related to the Reeb vector field ξ by $J\xi = N$ and η is smooth 1-form dual to the unit vector field ξ. It follows that the almost contact metric structure (ψ_1, ξ, η, g) on S^5 is nearly cosymplectic, that is, it satisfies

$$(\nabla\psi_1)(X, Y) + (\nabla\psi_1)(Y, X) = 0, \quad X, Y \in \mathfrak{X}(S^5), \tag{4.1}$$

where $(\nabla\psi_1)(X, Y) = \nabla_X \psi_1 Y - \psi_1(\nabla_X Y)$ (cf. [2], [3]). Also, it is known that the Reeb vector field ξ on the nearly cosymplectic manifold $S^5(\psi_1, \xi, \eta, g)$ is Killing (cf. [2]) and consequently, there is a $(1, 1)$ skew-symmetric tensor field ψ_2 on the nearly cosymplectic manifold $S^5(\psi_1, \xi, \eta, g)$ defined by $d\eta(X, Y) = 2g(\psi_2 X, Y)$, $X, Y \in \mathfrak{X}(S^5)$, which by Koszul's formula satisfies

$$\nabla_X \xi = \psi_2 X, \quad X \in \mathfrak{X}(S^5) \tag{4.2}$$

and using the fact that the smooth 2-form $\Omega_2(X, Y) = g(\psi_2 X, Y)$ is closed, we arrive at

$$(\nabla\psi_2)(X, Y) = R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi. \tag{4.3}$$

Since, ξ is a unit Killing vector field and ψ_2 is skew-symmetric, equation (4.2) gives $\psi_2(\xi) = \nabla_\xi \xi = 0$. Taking $Y = \xi$ in equation (4.3) and using $\psi_2(\xi) = 0$, we get

$$\psi_2^2(X) = -X + \eta(X)\xi, \quad X \in \mathfrak{X}(S^5),$$

and it follows that $g(\psi_1(X), \psi_2(Y)) = g(X, Y) - \eta(X)\eta(Y)$. Hence, (ψ_2, ξ, η, g) is an almost contact metric structure on $S^5(\psi_1, \xi, \eta, g)$ and equation (4.3) confirms that (ψ_2, ξ, η, g) is a Sasakian structure.

Now, define an operator ψ_3 on the nearly cosymplectic manifold $S^5(\psi_1, \xi, \eta, g)$ by

$$\psi_3(X) = (\nabla\psi_1)(\xi, X), \quad X \in \mathfrak{X}(S^5),$$

which in view of equation (4.1) confirms that ψ_3 is a $(1, 1)$ skew-symmetric tensor field and it satisfies $\psi_3(\xi) = 0$. Also, equations (4.1) and (4.2), imply that

$$\psi_3(X) = -(\nabla\psi_1)(X, \xi) = \psi_1(\nabla_X \xi) = \psi_1\psi_2(X), \quad X \in \mathfrak{X}(S^5), \tag{4.4}$$

which gives

$$\psi_2(X) = -\psi_1\psi_3(X), \quad X \in \mathfrak{X}(S^5). \tag{4.5}$$

Replacing X by $\psi_1 X$ in equation (4.5), we get

$$\psi_2\psi_1(X) = -\psi_1\psi_3\psi_1(X), \quad X \in \mathfrak{X}(S^5). \tag{4.6}$$

Also, we have

$$\begin{aligned}
\psi_1\psi_3\psi_1(X) &= \psi_1\left[(\nabla\psi_1)(\xi, \psi_1 X)\right] = \psi_1\left[\nabla_\xi(-X + \eta(X)\xi) - \psi_1(\nabla_\xi \psi_1 X)\right] \\
&= \psi_1\left[-\nabla_\xi X + \xi(\eta(X))\xi - \psi_1\left((\nabla\psi_1)(\xi, X) + \psi_1(\nabla_\xi X)\right)\right] \\
&= \psi_1\left[-\nabla_\xi X + \xi(\eta(X))\xi - \psi_1\psi_3 X + \nabla_\xi X\right] = \psi_3 X,
\end{aligned}$$

which in view of equation (4.6) gives

$$\psi_2\psi_1(X) = -\psi_3 X, \quad X \in \mathfrak{X}(S^5). \tag{4.7}$$

Thus, equations (4.4) and (4.7) lead to

$$\psi_3 = \psi_1\psi_2 = -\psi_2\psi_1. \tag{4.8}$$

Now, using above equation, we get

$$\psi_3^2 X = \psi_1\psi_2\psi_1\psi_2(X) = -\psi_1\psi_2^2\psi_1(X) = \psi_1^2(X) = -X + \eta(X)\xi,$$

and

$$g(\psi_3 X, \psi_3 Y) = g(X,Y) - \eta(X)\eta(Y).$$

Hence, (ψ_3, ξ, η, g) is an almost contact metric structure on $S^5(\psi_1, \xi, \eta, g)$ and the three almost contact structures (ψ_i, ξ, η, g), $i = 1,2,3$ satisfy equations (4.5)-(4.8), consequently, we have the following

$$\psi_1\psi_2 = -\psi_2\psi_1 = \psi_3, \quad \psi_2\psi_3 = -\psi_3\psi_2 = \psi_1, \quad \psi_3\psi_1 = -\psi_1\psi_3 = \psi_2.$$

Thus, we have the following:

Theorem 4.1. *There are three (ψ_i, ξ, η, g), $i = 1,2,3$ almost contact metric structures on the unit sphere S^5 satisfying*

$$\psi_1\psi_2 = -\psi_2\psi_1 = \psi_3, \quad \psi_2\psi_3 = -\psi_3\psi_2 = \psi_1, \quad \psi_3\psi_1 = -\psi_1\psi_3 = \psi_2$$

such that $S^5(\psi_i, \xi, \eta, g)$, $i = 1,3$ are nearly cosymplectic manifolds and $S^5(\psi_2, \xi, \eta, g)$ is a Sasakian manifold.

Proof. It remains to prove that the almost contact metric structure (ψ_3, ξ, η, g) is a nearly cosymplectic structure on S^5. We use equations (4.8) and (4.3), to compute

$$\begin{aligned}(\nabla\psi_3)(X,Y) &= -(\nabla\psi_2\psi_1)(X,Y) = -\nabla_X\psi_2\psi_1 Y + \psi_2\psi_1\nabla_X Y \\ &= -[(\nabla\psi_2)(X,\psi_1 Y) + \psi_2(\nabla\psi_1)(X,Y) + \psi_2\psi_1\nabla_X Y] + \psi_2\psi_1\nabla_X Y \\ &= g(X,\psi_1 Y) - \psi_2(\nabla\psi_1)(X,Y),\end{aligned}$$

which in view of equation (4.1), gives

$$(\nabla\psi_3)(X,Y) + (\nabla\psi_3)(Y,X) = 0, \quad X,Y \in \mathfrak{X}(S^5).$$

Hence, (ψ_3, ξ, η, g) is a nearly cosymplectic structure on S^5. □

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Affiliations

Sharief Deshmukh
Address: Department of Mathematics, College of Science, King Saud University P.O. Box-2455 Riyadh-11451, Saudi Arabia.
e-mail: shariefd@ksu.edu.sa

ORCID ID : orcid.org/0000-0003-3700-8164

Ibrahim Al-Dayel
Address: Department of Mathematics, College of Science, Al-Imam Muhammad Ibn Saud Islamic University P.O. Box-65892, Riyadh-11566, Saudi Arabia.
e-mail: ibraheem96@live.com

ORCID ID : orcid.org/0000-0002-5901-2511