INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 11 NO. 2 PAGE 111–119 (2018)

The Cauchy-Length Formula and Holditch Theorem in the Generalized Complex Plane $\mathbb{C}_\mathbf{p}$

Tülay Erişir* **Mehmet Ali Güngör**

(Communicated by H.Hilmi Hacısalihoğlu)

ABSTRACT

In this paper, firstly, we calculate Cauchy-length formula for the one-parameter planar motion in generalized complex plane $\mathbb{C}_\mathbf{p}$ which is generalization of the complex, dual and hyperbolic planes. Then, we give the length of the enveloping trajectories of lines $\mathbb{C}_\mathbf{p}$. In addition, we prove the Holditch theorem for the non-linear three points with the aid of the length of the enveloping trajectories in $\mathbb{C}_\mathbf{p}$. So, the Holditch theorem for the linear three points which is given by Erişir et al. in $\mathbb{C}_\mathbf{p}$ is generalized for trajectories drawn by the non-linear three points in generalized complex plane $\mathbb{C}_\mathbf{p}$.

Keywords: *Generalized complex plane; Cauchy-length formulas; Holditch theorem.*

AMS Subject Classification (2010): *Primary:53A17 ; Secondary: 53A40.*

1. Introduction

Holditch theorem which published in 1858 by Hamnet Holditch [10] emphasized that if we allowed that a chord with fixed length rotated throughout a curve which is closed and convex, then any point on the chord which has a distance p from the end point and a distance q from the other end point draw a closed curve and the area between these curve is πpq and constant in the plane geometry. In other words, the area is independent of the choice of point on the chord. In the history of mathematics, this theorem is one of Clifford Pickover's 250 milestones by means of this feature, [4].

Since the area mentioned in this theorem is independent of the selection of the curve and this theorem is open to the technical applications, it has attracted a lot of attention and been studied often. Additionally, many scientists generalized this theorem with various methods and different perspectives.

Steiner first mentioned the Steiner formula in 1840, [17]. Then, Blaschke and Müller gave the Holditch theorem giving the relationship between the areas of trajectories for the linear three points in the Euclidean plane, [3]. In addition, Hering took non-linear three points and proved the Holditch theorem, [9].

Pottman considered an infinite convex curve instead of taking an oval and calculated the area and volume of the Holditch crescent. Moreover, Pottman gave Holditch theorem for open motions in Euclidean plane, [13, 14].

In Euclidean plane, the length of the enveloping trajectories of lines and Cauchy formula were given by Blaschke and Müller, [3]. In similar way, Cauchy-length formulas in Lorentzian plane was given by Yüce and Kuruoğlu. Moreover, they proved the length of the enveloping trajectories of non-null lines and gave the Holditch theorem under the planar Lorentzian motion, [20].

The task of ordinary numbers in Euclidean geometry is undertaken the generalized complex numbers in Cayley-Klein geometry, [18, 19]. The Cayley-Klein plane geometries including Euclidean, Galilean, Minkowskian and Bolyai-Lobachevsikan first introduced by F. Klein in 1871 and A. Cayley, [11, 12]. Then,

Received : *30-01-2018,* ***Accepted :*** *01-08-2018*
*** Corresponding author***

This research supported by Sakarya University BAPK (No: 2014-50-02-023).

Yaglom distinguished these geometries according to measuring length and angles (parabolic, elliptic or hyperbolic), [19].

Gürses and Yüce gave the one-parameter planar motion in Affine Cayley-Klein planes and p$-$ complex plane $\mathbb{C}_J = \left\{x + Jy \ : \ x, y \in \mathbb{R}, \quad J^2 = \mathrm{p}, \ \ \mathrm{p} \in \{-1, 0, 1\}\right\} \subset \mathbb{C}_{\mathrm{p}}$ in the generalized complex plane $\mathbb{C}_{\mathrm{p}}$, [1, 2]. Moreover, Erişir et al. calculated the Steiner area formula and proved Holditch theorem in the generalized complex plane $\mathbb{C}_{\mathrm{p}}$, [5]. Then, they calculated the polar moment of inertia of trajectories under the one-parameter planar motion and Holditch-type theorem in $\mathbb{C}_{\mathrm{p}}$, [6].

In this paper, we study on the one-parameter planar motion in the generalized complex plane $\mathbb{C}_{\mathrm{p}}$ which is the generalization of the complex, dual and hyperbolic planes (which are isomorphous to Euclidean, Galilean and Lorentzian number planes, respectively). We first consider the fixed points X and Y and the fixed point Z non-linear with the points X and Y on the generalized moving complex plane in $\mathbb{C}_{\mathrm{p}}$. Then, we calculate the Cauchy-length formula giving the length of the enveloping curve of the line XY formed by the points X and Y. In addition, we give the relationship between the areas of the trajectories drawn by the points X, Y and Z. Finally, we prove the Holditch theorem for the non-linear three points by making some special choosing by means of this relationship during the one-parameter planar motion in the generalized complex plane $\mathbb{C}_{\mathrm{p}}$.

2. Preliminaries

The generalized complex number system is defined as a two-parameter family of complex number system and involves in the ordinary, dual and double numbers as follows:

$$Z = x + iy \qquad \text{where} \qquad i^2 = iq + \mathrm{p}, \qquad (q, p, x, y \in \mathbb{R}).$$

Here, if $\mathrm{p} + q^2/4 < 0$, the generalized complex numbers are isomorphic to ordinary numbers, if we take $\mathrm{p} + q^2/4 = 0$, the generalized complex numbers are isomorphic to dual numbers and if we choose $\mathrm{p} + q^2/4 > 0$, the generalized complex numbers are isomorphic to double numbers, [8, 15, 18]. If $q = 0$ is taken, this number system is defined as

$$\mathbb{C}_{\mathrm{p}} = \left\{x + iy \ : \ x, y \in \mathbb{R}, \quad i^2 = \mathrm{p} \in \mathbb{R}\right\}.$$

In this paper, we study on this number system $\mathbb{C}_p$. Now, we give some operations on this system.

For $Z_1 = (x_1 + iy_1), \ Z_2 = (x_2 + iy_2) \in \mathbb{C}_{\mathrm{p}}$, the addition and subtraction on this generalized complex plane are

$$Z_1 \pm Z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = x_1 \pm x_2 + i(y_1 \pm y_2).$$

In addition, the product is written as

$$M^{\mathrm{p}}(Z_1, Z_2) = (x_1x_2 + \mathrm{p}y_1y_2) + i(x_1y_2 + x_2y_1).$$

The product is defined as

Ordinary Product: $(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$ for $\mathrm{p} = -1$,

Study Product: $(x_1x_2) + i(x_1y_2 + x_2y_1)$ for $\mathrm{p} = 0$,

Clifford Product: $(x_1x_2 + y_1y_2) + i(x_1y_2 + x_2y_1)$ for $\mathrm{p} = 1$,

for the special values of the number p, [8, 15, 18]. In addition, the p$-$magnitude of $Z = x + iy \in \mathbb{C}_{\mathrm{p}}$ is

$$|Z|_{\mathrm{p}} = \sqrt{\left|M^{\mathrm{p}}\left(Z, \bar{Z}\right)\right|} = \sqrt{|x^2 - \mathrm{p}y^2|}. \tag{2.1}$$

Moreover, the scalar product on $\mathbb{C}_{\mathrm{p}}$ is

$$\langle \mathbf{z}_1, \mathbf{z}_2 \rangle_{\mathrm{p}} = \mathrm{Re}\left(M^{\mathrm{p}}\left(\mathbf{z}_1, \bar{\mathbf{z}}_2\right)\right) = \mathrm{Re}\left(M^{\mathrm{p}}\left(\bar{\mathbf{z}}_1, \mathbf{z}_2\right)\right) = x_1y_1 - \mathrm{p}x_2y_2$$

for $\mathbf{z}_1 = x_1 + iy_1, \quad \mathbf{z}_2 = x_2 + iy_2$ where "$-$" is an ordinary complex conjugate, [8].

The unit circle in $\mathbb{C}_{\mathrm{p}}$ is called the geometric location of points at a unit distance from a fixed point and in the form of $|Z|_{\mathrm{p}} = 1$. Thus, we can give the special cases of p in $\mathbb{C}_{\mathrm{p}}$ as follows.

Now, if we take $\mathrm{p} < 0$, we obtain $x^2 + |\mathrm{p}|\, y^2 = 1$ with the aid of definition of the unit circle in $\mathbb{C}_{\mathrm{p}}$. Thus, in case of $\mathrm{p} < 0$ the generalized complex number system matches the elliptical complex number system. Specially, if we take $\mathrm{p} = -1$, the unit circle in $\mathbb{C}_{\mathrm{p}}$ corresponds to the Euclidean unit circle. So, the plane $\mathbb{C}_{-1}$ matches the Euclidean plane, [8].

Now, let us consider $\mathrm{p} = 0$. So, with the aid of definition of the unit circle in $\mathbb{C}_\mathrm{p}$, we obtain $x^2 = 1$ $(x = \pm 1)$. In addition, the generalized number system matches up with the parabolic complex number system and the plane $\mathbb{C}_0$ matches up with Galilean plane, (Figure 1), [8].

In other respects, we can give two different definitions of the circle in Euclidean plane. The first of these is "The set of points fixed distance r from a fixed point M is a circle" and the second is "The set of end points of the line segment AB making the fixed directed angle α is a circle". These indicate the same set in Euclidean plane. However, these two definitions match up with different sets in Galilean plane. The first is occurred the lines $x = \pm 1$ (is called Galilean **circle**). At that, any curvature can not be mentioned, (Figure 1). The second is occurred Galilean **cycle** and this cycle is equal to the Euclidean parabola $\left(y = ax^2 + 2bx + c\right)$.

Now, let us consider $\mathrm{p} > 0$. So, we obtain the hyperbolas $\left|x^2 - \mathrm{p}y^2\right| = 1$. In addition, these hyperbolas have asymptote $\left|x^2 - \mathrm{p}y^2\right| = 1$ with asymptote $y = \pm x/\sqrt{\mathrm{p}}$. In this case, the generalized number system is equal to the hyperbolic complex number system. Especially, if we take $\mathrm{p} = 1$, the plane $\mathbb{C}_1$ corresponds to the Lorentzian plane, (Figure 1), [8].

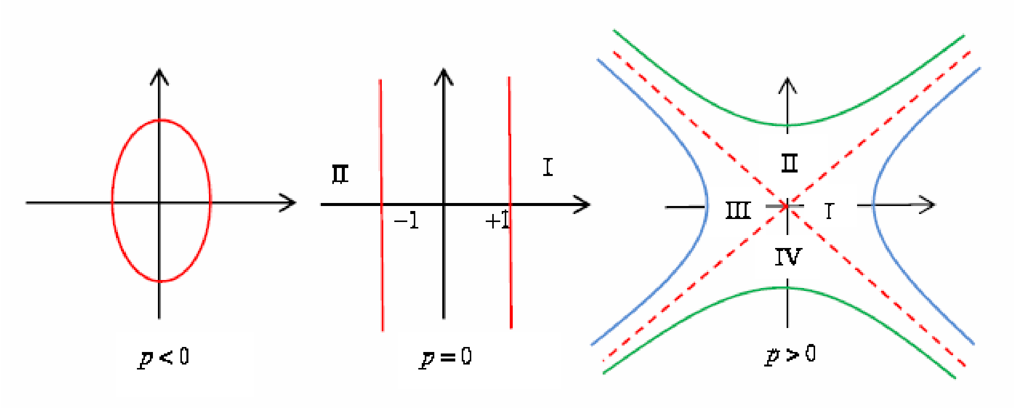

Figure 1. Unit Circles in $\mathbb{C}_\mathrm{p}$.

So, we can give the following definition.

Definition 2.1. Let us consider a circle in the generalized complex plane $\mathbb{C}_\mathrm{p}$. This circle has the center $M(a, b)$ and the radius r. So, the equation of this circle is

$$\left|(x-a)^2 - \mathrm{p}(y-b)^2\right| = r^2$$

where $i^2 = \mathrm{p} \in \mathbb{R}$, [8].

Let $Z = x + iy$ be a number in $\mathbb{C}_p$ and this number be expressed with a ray OT as in Figure 2. In addition, the intersection point of the ray OT and the unit circle in $\mathbb{C}_\mathrm{p}$. Thus, the angle θ_p showed in Figure 2 is written by

$$\theta_\mathrm{p} = \begin{cases} \frac{1}{\sqrt{|\mathrm{p}|}}\tan^{-1}\left(\sigma\sqrt{|\mathrm{p}|}\right), & \mathrm{p} < 0 \\ \sigma, & \mathrm{p} = 0 \\ \frac{1}{\sqrt{\mathrm{p}}}\tan^{-1}\left(\sigma\sqrt{\mathrm{p}}\right), & \mathrm{p} > 0 \ (branch\ I, III) \end{cases}$$

where $\sigma \equiv y/x$ and $z = x + iy$.

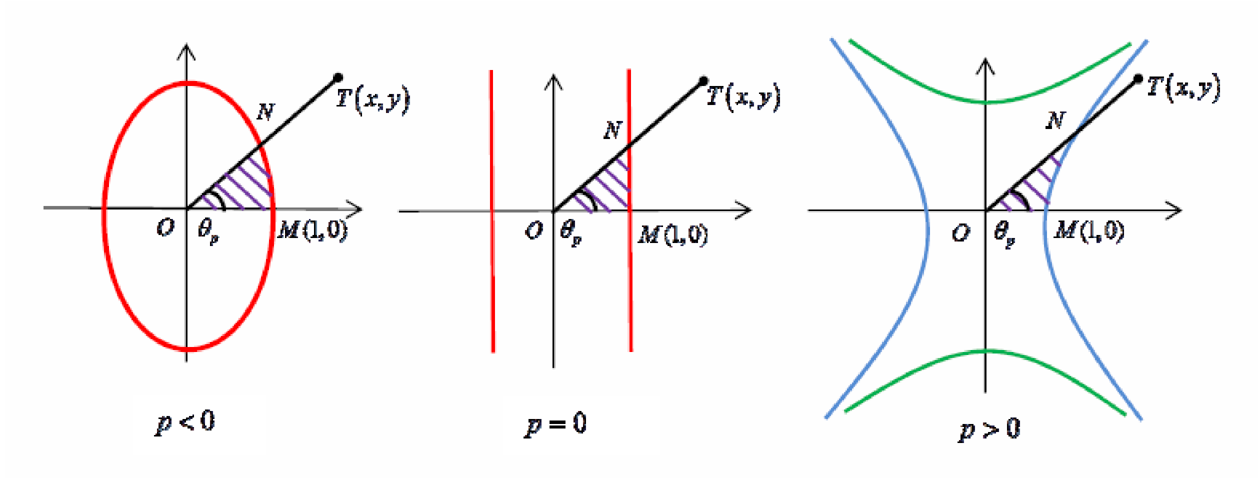

Figure 2. Elliptic, parabolic and hyperbolic angles.

In addition, the angular measure is written as follows

$$\theta_p = \sum_{n=0}^{\infty} \frac{p^n}{2n+1}\sigma^{2n+1}, \qquad |\sigma|\sqrt{|p|} < 1$$

as a power series .

Let the point L be the orthogonal projection of the point N on the radius OM. In addition, the line QM is the tangent of unit circle at the point M in $\mathbb{C}_p$, (Figure 3). So, we take the p−trigonometric functions (the p−cosine ($\cos p$) and p−sine ($\sin p$) can be written as

$$\cos p\theta_p = \begin{cases} \cos\left(\theta_p\sqrt{|p|}\right), & p<0 \\ 1, & p=0 \quad (branch\ \ I) \\ \cosh\left(\theta_p\sqrt{p}\right), & p>0 \quad (branch\ \ I) \end{cases}$$
$$\sin p\theta_p = \begin{cases} \frac{1}{\sqrt{|p|}}\sin\left(\theta_p\sqrt{|p|}\right), & p<0 \\ \theta_p, & p=0 \quad (branch\ \ I) \\ \frac{1}{\sqrt{p}}\sinh\left(\theta_p\sqrt{p}\right), & p>0 \quad (branch\ \ I) \end{cases}$$

and $\frac{\overline{QM}}{\overline{OM}} = \frac{\overline{NL}}{\overline{OL}}$ gives

$$\tan p\theta_p = \frac{\sin p\theta_p}{\cos p\theta_p}.$$

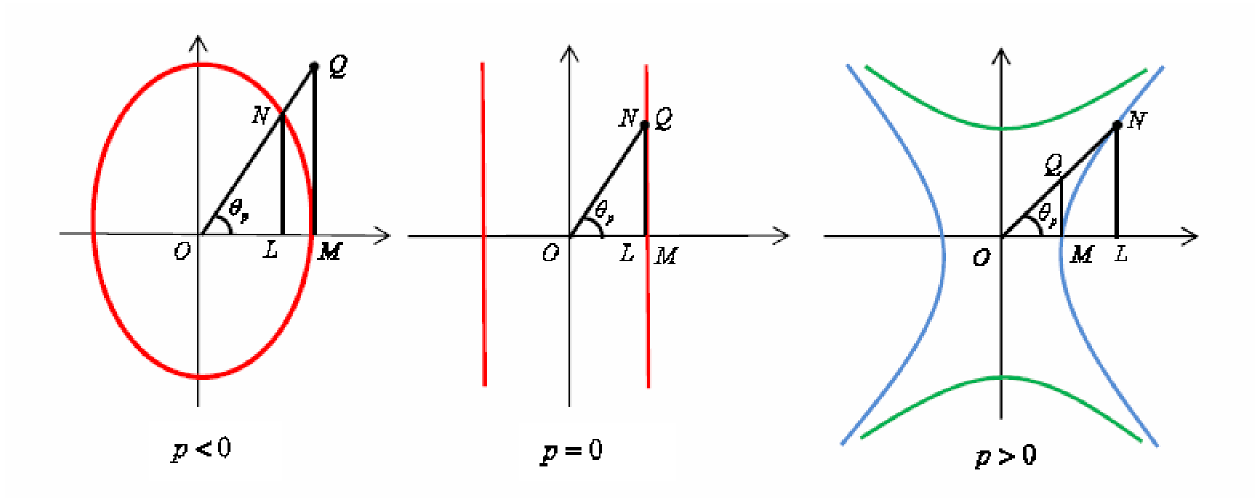

Figure 3. Geometric definitions of $\cos p$, $\sin p$ and $\tan p$.

In addition, we can write the p-trigonometric functions in $\mathbb{C}_\mathrm{p}$ for the other branches. For the parabolic trigonometric functions we hold

$$\cos \mathrm{p}_{II}\theta_\mathrm{p} = -\cos \mathrm{p}_I\theta_\mathrm{p} \quad \text{and} \quad \sin \mathrm{p}_{II}\theta_\mathrm{p} = -\sin \mathrm{p}_I\theta_\mathrm{p}.$$

Also, for the hyperbolic trigonometric functions we have the following equations;

$$\begin{array}{lll} \cos \mathrm{p}_{II}\theta_\mathrm{p} = \frac{i}{\sqrt{\mathrm{p}}}\cos \mathrm{p}_I\theta_p, & \cos \mathrm{p}_{III}\theta_\mathrm{p} = -\cos \mathrm{p}_I\theta_\mathrm{p}, & \cos \mathrm{p}_{IV}\theta_\mathrm{p} = -\frac{i}{\sqrt{\mathrm{p}}}\cos \mathrm{p}_I\theta_\mathrm{p}, \\ \sin \mathrm{p}_{II}\theta_\mathrm{p} = \frac{\sqrt{\mathrm{p}}}{i}\sin \mathrm{p}_I\theta_\mathrm{p}, & \sin \mathrm{p}_{III}\theta_\mathrm{p} = -\sin \mathrm{p}_I\theta_\mathrm{p}, & \sin \mathrm{p}_{IV}\theta_\mathrm{p} = -\frac{\sqrt{\mathrm{p}}}{i}\sin \mathrm{p}_I\theta_\mathrm{p} \end{array}$$

where the subscripts give the number of branches.

Moreover, the Maclaurin expansion is written by

$$\cos \mathrm{p}\theta_\mathrm{p} = \sum_{n=0}^{\infty} \frac{\mathrm{p}^n}{(2n)!}\theta_\mathrm{p}^{2n} \text{ and } \sin \mathrm{p}\theta_\mathrm{p} = \sum_{n=0}^{\infty} \frac{\mathrm{p}^n}{(2n+1)!}\theta_\mathrm{p}^{2n+1}$$

for branch I. With the aid of the Maclaurin expansion, the generalized Euler Formula can be given by

$$e^{i\theta_\mathrm{p}} = \cos \mathrm{p}\theta_\mathrm{p} + i\sin \mathrm{p}\theta_\mathrm{p}$$

where $i^2 = \mathrm{p}$ in $\mathbb{C}_\mathrm{p}$.

In addition, the polar and exponential forms of the generalized complex number z is

$$z = r_\mathrm{p}(cos\mathrm{p}\theta_\mathrm{p} + i\sin \mathrm{p}\theta_\mathrm{p}) = r_\mathrm{p}e^{i\theta_\mathrm{p}}$$

where θ_p and $r_\mathrm{p} = \|\mathbf{z}\|_\mathrm{p}$ are $\mathrm{p}-$argument and $\mathrm{p}-$magnitude of generalized complex number z, respectively, [8].

The $\mathrm{p}-$rotation matrix obtained by $e^{i\theta_\mathrm{p}}$ is

$$A(\theta_\mathrm{p}) = \begin{bmatrix} \cos \mathrm{p}\theta_\mathrm{p} & p\sin \mathrm{p}\theta_\mathrm{p} \\ \sin \mathrm{p}\theta_\mathrm{p} & \cos \mathrm{p}\theta_\mathrm{p} \end{bmatrix}.$$

Moreover, the derivatives of the $\mathrm{p}-$trigonometric functions $\cos \mathrm{p}$ and $\sin \mathrm{p}$ can be written by

$$\frac{d}{d\alpha}(\cos \mathrm{p}\alpha) = p\sin \mathrm{p}\alpha, \qquad \frac{d}{d\alpha}(\sin \mathrm{p}\alpha) = \cos \mathrm{p}\alpha,$$

[8].

Throughout this study, we consider one-parameter planar motion $\mathbb{K}_\mathrm{p}/\mathbb{K}'_\mathrm{p}$ in generalized complex plane $\mathbb{C}_\mathrm{p}$. Moreover, we study in the branch I of $\mathbb{C}_\mathrm{p}$.

3. The Cauchy-Length Formula in the Generalized Complex Plane

Now, we calculate the Cauchy-length formula giving the length of the enveloping curve of the line g in $\mathbb{C}_\mathrm{p}$.

Let $\mathbb{C}_\mathrm{p}$ be the generalized complex plane and g be a line in the branch I in $\mathbb{C}_\mathrm{p}$. So, the Hesse form of this line g in $\mathbb{C}_\mathrm{p}$ is determined by the equation

$$h = x_1 \cos \mathrm{p}\psi_\mathrm{p} - \mathrm{p}x_2 \sin \mathrm{p}\psi_\mathrm{p} \tag{3.1}$$

where (h, ψ_p) is the Hesse coordinates in $\mathbb{C}_\mathrm{p}$. In addition, $h = h(\psi_\mathrm{p})$ is the distance to the origin O from the right line and the point $X\,(x_1, x_2)$ is the contact point of the line g with the envelope curve (g).

We assume that $h = h(\psi_\mathrm{p})$ is a continuously differentiable support function of the line g. For each the value ψ_p, there is a line in the generalized complex plane $\mathbb{C}_\mathrm{p}$. Thus, the line bundle in terms of ψ_p in the generalized complex plane can be written as

$$h\left(\psi_\mathrm{p}\right) = x_1 \cos \mathrm{p}\psi_\mathrm{p} - \mathrm{p}x_2 \sin \mathrm{p}\psi_\mathrm{p}. \tag{3.2}$$

From the equation (3.2), by notating with "." the derivation with respect to ψ_p, the parametric representation of the envelope line g is

$$\frac{dh\left(\psi_\mathrm{p}\right)}{d\psi_\mathrm{p}} = \mathrm{p}x_1 \sin \mathrm{p}\psi_\mathrm{p} - \mathrm{p}x_2 \cos \mathrm{p}\psi_\mathrm{p}. \tag{3.3}$$

If this equation system formed by the equations (3.2) and (3.3) is solved according to x_1 and x_2, the following equations

$$\begin{aligned} x_1 &= h\cos \mathrm{p}\psi_{\mathrm{p}} - \dot{h}\sin \mathrm{p}\psi_{\mathrm{p}}, \\ \mathrm{p}x_2 &= \mathrm{p}h\sin \mathrm{p}\psi_{\mathrm{p}} - \dot{h}\cos \mathrm{p}\psi_{\mathrm{p}} \end{aligned} \tag{3.4}$$

can be obtained. So, from the equation (3.4), we have

$$\begin{aligned} \dot{x}_1 &= \left(\mathrm{p}h - \ddot{h}\right)\sin \mathrm{p}\psi_{\mathrm{p}}, \\ \mathrm{p}\dot{x}_2 &= \left(\mathrm{p}h - \ddot{h}\right)\cos \mathrm{p}\psi_{\mathrm{p}}. \end{aligned} \tag{3.5}$$

In addition, with the aid of the equation (3.5), the arc-element ds can be obtained that

$$ds = \frac{1}{\sqrt{|\mathrm{p}|}}\left|\mathrm{p}h - \ddot{h}\right| d\psi_{\mathrm{p}}.$$

So, we obtain that the length of the enveloping curve g is

$$L = \frac{1}{\sqrt{|\mathrm{p}|}}\int_{t_0}^{t_1}\left|\mathrm{p}h - \ddot{h}\right| d\psi_{\mathrm{p}}. \tag{3.6}$$

This formula is called as the **Cauchy-length formula** in the generalized complex plane $\mathbb{C}_{\mathrm{p}}$.

Similarly, we calculate the length of the enveloping curve (g) according to the fixed generalized complex plane $\mathbb{K}'_{\mathrm{p}}$. So, we can write the Hesse form of the line g according to the fixed generalized complex plane $\mathbb{K}'_{\mathrm{p}}$ as

$$h' = x'_1\cos \mathrm{p}\psi'_{\mathrm{p}} - \mathrm{p}x'_2\sin \mathrm{p}\psi'_{\mathrm{p}} \tag{3.7}$$

where h' is the distance to the origin O' from the right line g.

In the other hand, if θ_{p} is the rotation angle of the one-parameter planar motion in the generalized complex plane $\mathbb{C}_{\mathrm{p}}$, there is a relationship as

$$\psi'_{\mathrm{p}} = \theta_{\mathrm{p}} + \psi_{\mathrm{p}} \tag{3.8}$$

between the angles ψ'_{p}, θ_{p} and ψ_{p}. So, from the equations (3.7) and (3.8) we can write

$$h' = h - u_1\cos \mathrm{p}\psi_{\mathrm{p}} + \mathrm{p}u_2\sin \mathrm{p}\psi_{\mathrm{p}}. \tag{3.9}$$

Since the line g is fixed on the moving generalized complex plane $\mathbb{K}_{\mathrm{p}}$, ψ_{p} is fixed. Thus, the equivalent $d\psi'_{\mathrm{p}} = d\theta_{\mathrm{p}}$ can be written from the equation (3.8). From this, with the aid of the equations (3.6) and (3.9) we obtain that

$$L' = \frac{1}{\sqrt{|\mathrm{p}|}}\int_{t_0}^{t_1}\left|\mathrm{p}h' - \ddot{h}'\right| d\theta_{\mathrm{p}} \tag{3.10}$$

or

$$L' = \frac{1}{\sqrt{|\mathrm{p}|}}\left|\mathrm{p}h\delta_{\mathrm{p}} - A\cos \mathrm{p}\psi_{\mathrm{p}} + \mathrm{p}B\sin \mathrm{p}\psi_{\mathrm{p}}\right| \tag{3.11}$$

where $A = \int_{t_0}^{t_1}(\mathrm{p}u_1 - \ddot{u}_1)d\theta_{\mathrm{p}}$ and $B = \int_{t_0}^{t_1}(\mathrm{p}u_2 - \ddot{u}_2)d\theta_{\mathrm{p}}$.

Now, we express the length of enveloping curve g in the equation (3.11) in geometrically. So, if we make the necessary arrangements in the equation (3.11), we obtain that

$$L' = \sqrt{|\mathrm{p}|}\left(\int_{t_0}^{t_1}\bar{q}d\theta_{\mathrm{p}} + L_Q^g\right).$$

where $L_Q^g = q_2\cos \mathrm{p}\psi_{\mathrm{p}} - q_1\sin \mathrm{p}\psi_{\mathrm{p}}\big|_{t_0}^{t_1}$ is the length of orthogonal projection of the line segment Q_1Q_2 of the moving pole curve (Q) on the line g. Moreover, $\bar{q} = h - q_1\cos \mathrm{p}\psi_{\mathrm{p}} + \mathrm{p}q_2\sin \mathrm{p}\psi_{\mathrm{p}}$ is distance of the pole point Q

to the line g in the generalized complex plane.

Now, we assume that the enveloping trajectories of all the fixed lines g of the generalized moving plane $\mathbb{K}_{\mathrm{p}}$ have the same length $L' = c$. If we consider the equation (3.11) we can give the following theorem.

Theorem 3.1. *During the one-parameter planar motion $\mathbb{K}_p/\mathbb{K}'_p$ in the generalized complex plane $\mathbb{C}_p$, all the fixed lines with Hesse coordinates (h, ψ_p) of the generalized moving complex plane $\mathbb{K}_p$ whose enveloping trajectories have the same length $L' = c$ are tangent to the cycles with center $S_G = \left(\frac{A}{p\delta_p}, \frac{B}{p\delta_p}\right)$ and radius $\frac{c}{\sqrt{|p|}\delta_p}$ in the generalized moving plane $\mathbb{K}_p$.*

4. Holditch Theorem for the non-linear points in $\mathbb{C}_{\mathrm{p}}$

In this section, we give a new generalization of the Holditch theorem given by [5] by using non-linear points in the generalized complex plane $\mathbb{C}_{\mathrm{p}}$.

Let $\mathbb{K}_{\mathrm{p}}$ and $\mathbb{K}'_{\mathrm{p}}$ be moving and fixed generalized complex planes in $\mathbb{C}_{\mathrm{p}}$, respectively. Under the one-parameter planar motion, the non-linear fixed points X, Y and Z taken in the moving plane $\mathbb{K}_{\mathrm{p}}$ draw the trajectories k_X, k_Y and k_Z with areas F_X, F_Y and F_Z, respectively.

Especially, we take the non-linear points $X = (0,0)$, $Y = (a+b, 0)$ and $Z = (a, c)$ on the generalized moving plane $\mathbb{K}_{\mathrm{p}}$ in $\mathbb{C}_{\mathrm{p}}$. We know that the area of trajectory drawn by a point X in $\mathbb{C}_{\mathrm{p}}$ is calculated

$$F_X = F_0 + \frac{1}{2}\delta_{\mathrm{p}}\left({x_1}^2 - \mathrm{p}{x_2}^2 - 2x_1 s_1 + 2\mathrm{p}x_2 s_2\right)$$

from [5]. So, for the points $X = (0,0)$, $Y = (a+b, 0)$ and $Z = (a, c)$ the areas are written by

$$F_X = F_0 \quad \text{for} \quad X = (0,0) \tag{4.1}$$

$$F_Y = F_X + \frac{1}{2}\delta_{\mathrm{p}}\left((a+b)^2 - 2(a+b)s_1\right) \quad \text{for} \quad Y = (a+b, 0) \tag{4.2}$$

$$F_Z = F_X + \frac{1}{2}\delta_{\mathrm{p}}\left(a^2 - \mathrm{p}c^2 - 2as_1 + 2\mathrm{p}cs_2\right) \quad \text{for} \quad Z = (a, c)\,. \tag{4.3}$$

From the equation (4.2) we obtain that

$$s_1 = \frac{a+b}{2} + \frac{F_X - F_Y}{\delta_{\mathrm{p}}(a+b)}. \tag{4.4}$$

In addition, if we use the equations (4.3) and (4.4) we have

$$F_Z = \frac{aF_Y + bF_X}{a+b} - \frac{1}{2}\delta_{\mathrm{p}}\left(\mathrm{p}c^2 + ab\right) + \mathrm{p}\delta_{\mathrm{p}}cs_2.$$

In the other hand, we consider the one-parameter planar motion $S = S_G$ in $\mathbb{C}_{\mathrm{p}}$. So, we have

$$s_2 = \frac{B}{\mathrm{p}\delta_{\mathrm{p}}}.$$

Moreover, we consider that L' is called as L_{XY} if the points $X = (0,0)$, $Y = (a+b, 0)$ and $Z = (a, c)$ is written in equation (3.11).

Finally, we obtain

$$F_Z = \frac{aF_Y + bF_X}{a+b} - \frac{1}{2}\delta_{\mathrm{p}}\left(\mathrm{p}c^2 + ab\right) - \sqrt{|\mathrm{p}|}cL_{XY}.$$

Thus, we give the following theorem.

Theorem 4.1. Main Theorem: *During the one-parameter planar motion $\mathbb{K}_p/\mathbb{K}_p'$ with $S = S_G$ in the generalized complex plane, let the points X, Y and Z, non-linear with the points X and Y, be fixed on the moving plane $\mathbb{K}_p$. If the points $X = (0,0)$, $Y = (a+b,0) \in \mathbb{K}_p$ move along the trajectories k_X and k_Y with the areas F_X and F_Y, respectively, then the point $Z = (a,c) \in \mathbb{K}_p$ draws the trajectory with the area*

$$F_Z = \frac{aF_Y + bF_X}{a+b} - \frac{1}{2}\delta_p\left(pc^2 + ab\right) - \sqrt{|p|}cL_{XY}. \tag{4.5}$$

where L_{XY} is the length of the enveloping curve of (XY). So, the area of section between curves k_X, k_Y and k_Z depends on the distances of the point R to the end points X and Y, the distance of the point Z to the line $\overline{XY}$, the length of the enveloping curve (XY) and the rotation angle of the motion. This area is independent of the choice of curves.

So, the following corollary can be given.

Corollary 4.1. *We take that the points X, Y and Z are linear points during the one-parameter planar motion in $\mathbb{C}_p$. So, the points R and Z are coincident. Namely, $c = 0$. Thus, from the equation (4.5) we can obtain*

$$F_Z = \frac{aF_Y + bF_X}{a+b} - \frac{1}{2}\delta_p ab.$$

This formula is the area formula given for the linear points in [5]. Namely, the formula (4.5) is generalization of the formula in [5].

Note: If we choice $\mathrm{p} = 0$, the formula (4.5) is written by

$$F_Z = \frac{aF_Y + bF_X}{a+b} - \frac{1}{2}\delta_{\mathrm{p}} ab. \tag{4.6}$$

The formula (4.6) is also the area for the linear points in [5]. Thus, we can give the following corollary.

Corollary 4.2. *The area formula for non-linear three points for $p = 0$ is same the formula of area for linear three points in $\mathbb{C}_p$.*

The reason of this is the metric in the plane C_0. According to the metric in C_0 the distance between the points X and R is same the distance between the points X and Z. Similarly, this situation is valid for the points Y, R and Y, Z.

Finally, we can give the following corollary.

Corollary 4.3. *The relationship between L', the length of envelope curve (g), and L_{XY}, the length of the enveloping curve of (XY), with the aid of the areas F_X and F_Y is written by*

$$L' = \sqrt{|p|}\left(h\delta_p + \left(\frac{F_Y - F_X}{\delta_p\left(a+b\right)} - \frac{a+b}{2}\right)\delta_{\mathrm{p}}\cos p\psi_p - \sqrt{|p|}L_{XY}\sin p\psi_p\right).$$

References

[1] (Bayrak) Gürses, N., Yüce, S., One-Parameter Planar Motions in Generalized Complex Number Plane $\mathbb{C}_j$. *Adv. Appl. Clifford Algebr.* 4 (2015), no.25, 889-903.

[2] (Bayrak) Gürses, N., Yüce, S., One-Parameter Planar Motions in Affine Cayley-Klein Planes. *European Journal of Pure and Applied Mathematics* 7 (2014), no.3, 335-342.

[3] Blaschke W., Müller, H.R., *Ebene Kinematik.* Verlag Oldenbourg, München, 1956.

[4] A. P. Clifford, *The Math Book: 250 Milestones in the History of Mathematics.* Sterling, ISBN 978-1-4027-5796-9, 2009.

[5] Erişir, T., Güngör, M.A., Tosun, M., A New Generalization of the Steiner Formula and the Holditch Theorem. *Adv. Appl. Clifford Algebr.* 26 (2016), no.1, 97-113.

[6] Erişir, T., Güngör, M.A., Tosun, M., The Holditch-Type Theorem for the Polar Moment of Inertia of the Orbit Curve in the Generalized Complex Plane. *Adv. Appl. Clifford Algebr.* 26 (2016), no.4, 1179-1193.

[7] Hacısalihoğlu, H. Hilmi., On the Geometry of Motion of Lorentzian Plane. Proc. of Assiut First International Conference of Mathematics and Statistics, Part I, University of Assiut, Assiut, Egypt, (1990), 87-107.

[8] Harkin, A.A., Harkin, J.B., Geometry of Generalized Complex Numbers. *Math. Mag.* 77 (2004), no.2, 118-129.

[9] Hering, L., Sätze vom Holditch-Typ für ebene Kurven. *Elem. Math.* 38 (1983), 39-49.

[10] Holditch, H., Geometrical Theorem. *Q. J. Pure Appl. Math.* 2 (1858), 38.

[11] Klein, F., Über die sogenante nicht-Euklidische Geometrie. *Gesammelte Mathematische Abhandlungen* (1921), 254-305.
[12] Klein, F., Vorlesungen über nicht-Euklidische Geometrie. Springer, Berlin, 1928.
[13] Potmann, H., Holditch-Sicheln. *Arc. Math.* 44 (1985), 373-378.
[14] Potmann, H., Zum Satz von Holditch in der Euklidischen Ebene. *Elem. Math.* 41 (1986), 1-6.
[15] Sachs, H., Ebene Isotrope Geometrie. Fiedr. Vieweg-Sohn, 1987.
[16] Spivak, M., Calculus on Manifolds: A Modern Approach To Classical Theorems Of Advanced Calculus. W. A. Benjamin, New York, 1965.
[17] Steiner, J., Gesammelte Werke II. De Gruyter Verlag, Berlin, 1882.
[18] Yaglom, I.M., Complex Numbers in Geometry. Academic, Press, New York, 1968.
[19] Yaglom, I.M., A Simple non-Euclidean Geometry and its Physical Basis. Springer-Verlag, New-York, 1979.
[20] Yüce, S., Kuruoğlu, N., Cauchy Formulas for Enveloping Curves in the Lorentzian Plane and Lorentzian Kinematics. *Result. Math.* 54 (2009), 199-206.

Affiliations

TÜLAY ERIŞIR
ADDRESS: Erzincan Binali Yildirim University, Dept. of Mathematics, 24100, Erzincan-Turkey.
E-MAIL: tulay.erisir@erzincan.edu.tr

ORCID ID : orcid.org/0000-0001-6444-1460

MEHMET ALI GÜNGÖR
ADDRESS: Sakarya University, Dept. of Mathematics, 54050, Sakarya-Turkey.
E-MAIL: agungor@sakarya.edu.tr

ORCID ID : orcid.org/0000-0003-1863-3183