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New results on helix surfaces in the Minkowski 3-space

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ABSTRACT

In this paper, we characterize and classify helix surfaces with principal direction relatived to a space-like and light-like, constant direction in the Minkowski 3-space.

Keywords: *Minkowski space; Space-like surfaces; constant angle surfaces.*

AMS Subject Classification (2010): *Primary: 53B25 ; Secondary: 53A35; 53C50.*

1. Introduction

It is well known that, a helix is a curve whose tangent lines make a constant angle with a fixed vector. After the question 'Are there any surface making a constant angle with some fixed vector direction?' was introduced in [5], the concept of constant angle surfaces, also called as helix surfaces, have been studied by some geometers. Specially, if one can take the position vector of the surface instead of the fixed vector, in that case, the surface is called as constant slope surface studied in [10, 11, 15].

The applications of constant angle surfaces in the theory of liquid crystals and layered fluids were firstly considered in [1], where the study of surfaces about the Hamilton-Jacobi equation, correlating the surface and the direction field, were used. Further, Munteanu and Nistor gave another approach to concern surfaces in Euclidean spaces for which the unit normal makes a constant angle with a fixed direction in [16]. Moreover, the study of constant angle surfaces was extended to the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, [5] and [6], respectively. When the ambient space is the Minkowski space $\mathbb{E}^3_1$, some classification results on such surfaces were obtained in [9, 13, 14]. Further results on higher dimensional Euclidean space were given in [3], where a local construction of constant angle hypersurfaces are given. We want to note that some geometers call constant angle hypersurfaces as helix hypersurfaces (See, for example, [3]).

One of common geometrical properties of constant angle surfaces is the following. If we denote the projection of the fixed direction k on the tangent plane of the surface by U^T, then U^T is a principal direction of the surface with the corresponding principal curvature 0. On the other hand, one another recent natural problem appearing in the context of constant angle surfaces is to study those surfaces for which U^T remains a principal direction but the corresponding principal curvature is different from zero. This problem was studied in $\mathbb{S}^2 \times \mathbb{R}$ [4] and $\mathbb{H}^2 \times \mathbb{R}$ [7]. Further, this problem has been recently studied in Euclidean spaces and semi-Euclidean spaces, (see in [8, 9, 12, 17, 18]) where T is replaced by a constant direction k.

In the present paper, we would like to move the study of constant angle hypersurfaces in Minkowski 3-space obtained in [13, 14] in which they obtained partial classification of these surfaces. This paper is organized as follows. In Sect. 2, we introduce the notation that we will use and give a brief summary of basic definitions in theory of submanifolds of semi-Euclidean spaces. In Sect. 3, we obtain some new characterizations and classifications of helix surfaces with a principal direction relative to a space-like and light-like, constant direction in the Minkowski 3-space.

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2. Helix Hypersurfaces in Minkowski spaces

In this section, we would like to give some basic equations and facts on hypersurfaces in Minkowski spaces, before we consider on some geometrical properties of hypersurfaces in Minkowski 3-spaces, $\mathbb{E}^3_1$ endowed with a canonical principal direction.

2.1. *Basic facts and definitions*

First, we would like to give a brief summary of basic definitions, facts and equations in the theory of submanifolds of pseudo-Euclidean space (see for detail, [2, 19]).

Let $\mathbb{E}^m_1$ denote the Minkowski m-space with the canonical Lorentzian metric tensor given by

$$\tilde{g} = \langle \cdot, \cdot \rangle = \sum_{i=1}^{m-1} dx_i^2 - dx_m^2,$$

where $x_1, x_2, \ldots, x_m$ are rectangular coordinates of the points of $\mathbb{E}^m_1$. We denote the Levi-Civita connection of $\mathbb{E}^m_1$ by $\widetilde{\nabla}$.

The causality of a vector in a Minkowski space is defined as following. A non-zero vector v in $\mathbb{E}^m_1$ is said to be space-like, time-like and light-like (null) regarding to $\langle v, v\rangle > 0$, $\langle v, v\rangle < 0$ and $\langle v, v\rangle = 0$, respectively. Note that v is said to be causal if it is not space-like.

Now, let M be an oriented hypersurface in $\mathbb{E}^{n+1}_1$ by considering the case $m = n+1$. We denote by N and ∇, the unit normal vector field and Levi-Civita connection of M, respectively. Note that Gauss and Weingarten formulas are given by

$$\begin{aligned}\widetilde{\nabla}_X Y =& \nabla_X Y + h\,(X, Y)\,,\\ \widetilde{\nabla}_X N =& -S(X),\end{aligned}$$

respectively, whenever X, Y are tangent to M, where h and S are the second fundamental form and the shape operator (or Weingarten map) of M. The surface M is said to be space-like (resp. time-like) if the induced metric $g = \widetilde{g}|_M$ of M is Riemannian (resp. Lorentzian). This is equivalent to being time-like (resp. space-like) of N at each point of M.

The Codazzi equation is given by

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z) \tag{2.1}$$

for any vector fields X, Y, Z tangent to M, where $\bar{\nabla} h$ is defined by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

If M is space-like, then its shape operator S is diagonalizable, i.e., there exists a local orthonormal frame field $\{e_1, e_2\}$ of the tangent bundle of M such that $Se_i = k_i e_i, \quad i = 1, 2, \ldots, n$. In this case, the vector field e_i and the smooth function k_i are called as a principal direction and a principal curvature of M, respectively.

Now, let M be a surface in the Minkowski 3-space. Then, its mean curvature and Gaussian curvature are defined by $H = \operatorname{trace} S$ and $K = \det S$, respectively. M is said to be flat if K vanishes identically. On the other hand, if $H = 0$ and the surface M is space-like, then it is called maximal while a time-like surface with identically vanishing mean curvature is said to be a minimal surface.

Before we proceed to the next subsection, we would like to notice the notion of angle in the Minkowski 3-space (see for example, [8, 11]):

Definition 2.1. Let v and w be space-like vectors in $\mathbb{E}^3_1$ that span a space-like vector subspace. Then, we have $|\langle v, w\rangle| \leq \|v\|\,\|w\|$ and hence, there is a unique real number $\theta \in [0, \pi/2]$ such that

$$|\langle v, w\rangle| = \|v\|\,\|w\| \cos\theta.$$

The real number θ is called the Lorentzian space-like angle between v and w.

Definition 2.2. Let v and w be space-like vectors in $\mathbb{E}^3_1$ that span a time-like vector subspace. Then, we have $|\langle v, w\rangle| > \|v\|\,\|w\|$ and hence, there is a unique positive real number θ such that

$$|\langle v, w\rangle| = \|v\|\,\|w\| \cosh\theta.$$

The real number θ is called the Lorentzian time-like angle between v and w.

2.2. A characterization of helix surfaces

First, we would like to recall the following definition (see for example, [9, 13, 14, 16]).

Definition 2.3. Let M be a non-degenerated hypersurface in $\mathbb{E}_1^{n+1}$ and ζ a vector field in $\mathbb{E}_1^{n+1}$. M is said to be a helix hypersurface relative to ζ if its tangential component is a principal direction corresponding with principal curvature being zero. In particular if $\zeta = k$ for a fixed direction k in $\mathbb{E}_t^{n+1}$, we will say that M is a helix-hypersurface.

As we mentioned before, a surface M in $\mathbb{E}^3$ was said to be a constant angle surface (or helix surface), if its unit normal vector field makes a constant angle with a fixed vector, [16] (see also [5, 6, 9]). Later, in [9, 13, 14], this definition is extended to surfaces in Minkowski spaces with obvious restrictions on the causality of the fixed vector and the normal vector because of the definition of 'angle' in the Minkowski space (see, Definition 2.1,Definition 2.2).

Remark 2.1. In fact, the shape operator of any helix hypersurface is singular (see [3]).

Remark 2.2. In fact, if the ambient space is pseudo-Euclidean, then a constant angle surface is a surface with a principal direction corresponding principal curvature $k_1 = 0$ (see [13, 14, 16]).

Note that Remark 2.2 can be consider as a result of Remark 2.1.

Let M be a hypersurface and k be a fixed direction in a Minkowski space $\mathbb{E}_1^{n+1}$. The fixed vector k can be expressed as

$$k = U + \langle N, N\rangle \langle k, N\rangle N \tag{2.2}$$

for a tangent vector U. We would like to give the following new characterization of helix hypersurfaces.

Proposition 2.1. *Let M be an oriented hypersurface in the Minkowski space $\mathbb{E}_1^{n+1}$ and k be a fixed vector on the tangent plane to the surface. Consider a unit tangent vector field e_1 along U. Then, M is a helix hypersurface if and only if a curve α is a geodesic of M whenever it is an integral curve of e_1.*

Proof. We will consider three cases seperately subject to causality of U.

Case I. Let e_1 be time-like. Thus, we have

$$k = -\langle k, e_1\rangle e_1 + \langle k, N\rangle N.$$

Since $\widetilde{\nabla}_{e_1} k = 0$, this equation yields

$$0 = -\langle k, e_1\rangle \widetilde{\nabla}_{e_1} e_1 - \langle k, Se_1\rangle N - \langle k, N\rangle Se_1.$$

The tangential part of this equation yields $Se_1 = 0$ if and only if $\nabla_{e_1} e_1 = 0$ which is equivalent to being geodesic of all integral curves of e_1.

Case II. Let e_1 be space-like. Thus, we have

$$k = \langle k, e_1\rangle e_1 + \varepsilon\langle k, N\rangle N, \tag{2.3}$$

where ε is either 1 or -1 regarding to being time-like or space-like of M, respectively.

Similar to Case I, we obtain $Se_1 = 0$ if and only if $\nabla_{e_1} e_1 = 0$.

Case III. Let e_1 be light-like. In this case, k can be decompose as

$$k = \phi(e_1 - N), \tag{2.4}$$

for a non-constant function ϕ.

Similar to the other case, we obtain $Se_1 = 0$ if and only if $\nabla_{e_1} e_1 = 0$.

□

3. New classifications of Helix Surfaces in $\mathbb{E}_1^3$

In this section, we want to give a new classification of helix surfaces in $\mathbb{E}_1^3$. We would like to note that the complete classification of constant angle surfaces which relative to a time-like constant direction $k = (0, 0, 1)$ was obtained in [14].

3.1. Helix surfaces relative to a space-like, constant direction.

In this subsection, we consider space-like helix surfaces which relative to a space-like, constant direction k. In this case, up to a linear isometry of $\mathbb{E}^3_1$, we may assume that $k = (1, 0, 0)$.

First, we will assume that M is a space-like surface which relative to $k = (1, 0, 0)$. In this case, N is time-like and (2.2) becomes

$$k = \cosh\theta e_1 + \sinh\theta N \tag{3.1}$$

where θ is a smooth function. Let e_2 be a unit tangent vector field satisfying $\langle e_1, e_2\rangle = 0$. Considering (3.1), we obtain the following lemma by a simple computation.

Lemma 3.1. *The Levi-Civita connection ∇ of M is given by*

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = 0, \tag{3.2a}$$
$$\nabla_{e_2} e_1 = \tanh\theta k_2 e_2, \qquad \nabla_{e_2} e_2 = -\tanh\theta k_2 e_1, \tag{3.2b}$$

and the matrix representation shape operator S of M with respect to $\{e_1, e_2\}$ is

$$S = \begin{pmatrix} 0 & 0 \\ 0 & k_2 \end{pmatrix} \tag{3.3}$$

for a function k_2 satisfying

$$e_1(k_2) + \tanh\theta {k_2}^2 = 0. \tag{3.4}$$

Proof. By considering (3.1), one can get

$$0 = X(\cosh\theta)e_1 + \cosh\theta\nabla_X e_1 + \cosh\theta h(e_1, X) - \sinh\theta SX + X(\sinh\theta)N \tag{3.5}$$

whenever X is tangent to M. (3.5) for $X = e_1$ gives

$$\begin{aligned} \nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = 0, \\ e_1(\theta) = k_1 = 0, \end{aligned} \tag{3.6}$$

while (3.5) for $X = e_2$ is giving

$$\nabla_{e_2} e_1 \;=\; \tanh\theta k_2 e_2, \quad \nabla_{e_2} e_2 = -\tanh\theta k_2 e_1,$$

where e_2 is the other principal direction of M with the corresponding principal curvature k_2. Thus, we have (3.2), (3.3) and so the second fundamental form of M becomes

$$h(e_1, e_1) = 0, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = -k_2 N. \tag{3.7}$$

By considering the Codazzi equation, we obtain (3.4). □

Now, we would like to prove the following lemma.

Lemma 3.2. *There exists a local coordinate system (s, t) defined in a neighborhood $\mathcal{N}_p$ of p such that the induced metric of M is*

$$g = ds^2 + m^2 dt^2 \tag{3.8}$$

for a function m satisfying

$$e_1(m) - \tanh\theta k_2 m = 0. \tag{3.9}$$

Furthermore, the vector fields e_1, e_2 described as above become $e_1 = \partial_s$, $e_2 = \frac{1}{m}\partial_t$ in $\mathcal{N}_p$.

Proof. Because of (3.2), we have $[e_1, e_2] = -\tanh\theta k_2 e_2$. Thus, if m is a non-vanishing smooth function on M satisfying (3.9), then we have $[e_1, m e_2] = 0$. Therefore, there exists a local coordinate system (s, t) such that $e_1 = \partial_s$ and $e_2 = \frac{1}{m}\partial_t$. Thus, the induced metric of M is as given in (3.8). □

Now, we are ready to obtain the classification theorem.

Theorem 3.1. *Let M be an oriented space-like surface in $\mathbb{E}^3_1$. Then, M is a helix surface endowed with a principal direction relative to a space-like constant direction if and only if it is congruent to the surface given by one of the followings:*

(i) A surface is an open part of the plane which is parallel to $Oyz-$plane.
(ii A flat surface is given by

$$x(s,t) = s\cosh\theta\Big(1,0,0\Big) + s\sinh\theta\Big(0,\sinh t,\cosh t\Big) + \gamma(t) \tag{3.10a}$$

where γ is the $\mathbb{E}^3_1$-valued function given by

$$\gamma(t) = \sinh\theta\left(0, \int^t \phi(\tau)\cosh\tau d\tau, \int^t \phi(\tau)\sinh\tau d\tau\right). \tag{3.10b}$$

for a function $\phi \in C^\infty(M)$ and the angle θ is a non-zero constant;
(iii) A surface can be parametrized by

$$x(s,t) = s\cosh\theta\Big(1,0,0\Big) + s\sinh\theta\Big(0,\sinh t_0,\cosh t_0\Big) + \gamma_0(t) \tag{3.11}$$

where $\gamma_0(t) = t\sinh\theta\,(0,\cosh t_0,\sinh t_0)$ and t_0 is a non-zero real constant.

Proof. Let M be a space-like helix surface relatived to a space-like fixed direction k. If the angle θ is vanishing identically, then k is always normal to M. This gives Item (i).
Now, we would like to consider the angle θ being non-zero constant. In order to proof the necessary condition, we assume that M is a space-like helix surface endowed with a principal direction relative to $k=(1,0,0)$ with the isometric immersion $x: M \to \mathbb{E}^3_1$. Let $\{e_1,e_2;N\}$ be the local orthonormal frame field described before Lemma 3.1, k_1,k_2 be the principal curvatures of M and (s,t) be a local coordinate system given in Lemma 3.2.

Considering Lemma 3.2, we have

$$e_1 = x_s. \tag{3.12}$$

Note that (3.4) and (3.9) become

$$(k_2)_s + {k_2}^2\tanh\theta = 0, \tag{3.13}$$
$$m_s - m\tanh\theta k_2 = 0, \tag{3.14}$$

respectively. By combining (3.13) and (3.14), one could directly obtain the function m as given

$$m(s,t) = \Psi(t)\Big(\sinh\theta s + \phi(t)\Big), \tag{3.15a}$$

or

$$m(s,t) = m(t), \tag{3.15b}$$

for some smooth functions Ψ,ϕ depending only the parameter t.

Case 1. m satisfies (3.15a). In this case, the Levi-Civita connections of M given in (3.2) become

$$\begin{aligned}\nabla_{\partial_s}\partial_s &= 0, \quad \nabla_{\partial_s}\partial_t = \frac{m_s}{m}\partial_t,\\ \nabla_{\partial_t}\partial_t &= -mm_s\partial_s + \frac{m_t}{m}\partial_t.\end{aligned}$$

By combining these equations with (3.7) and considering in Gauss formula, we obtain

$$x_{ss} = 0, \tag{3.16}$$
$$x_{st} = \frac{1}{s+\phi(t)}x_t, \tag{3.17}$$
$$x_{tt} = -\tanh^2\theta\Psi^2(t)\Big(\frac{1}{s+\phi(t)}\Big)x_s + \Big(\frac{\Psi'(t)}{\Psi(t)} + \frac{\phi'(t)}{s+\phi(t)}\Big)x_t - \tanh\theta\Psi^2(t)(s+\phi(t))N. \tag{3.18}$$

On the other hand, from the decomposition (3.1), we have $\langle x_s,k\rangle = \cosh\theta$ and $\langle x_t,k\rangle = 0$. By considering these equations, we can assume that x has the form of

$$x(s,t) = (s\cosh\theta, x_2(s,t), x_3(s,t)) + \gamma(t) \tag{3.19}$$

for a $\mathbb{E}^3_1$-valued smooth function $\gamma = (0, \gamma_2, \gamma_3)$. On the other hand, by considering $\langle x_s, x_s \rangle = 1$ in (3.19), we obtain

$$x(s,t) = s\cosh\theta\Big(1,0,0\Big) + s\sinh\theta\Big(0, \sinh\varphi(t), \cosh\varphi(t)\Big) + \gamma(t) \tag{3.20}$$

for a smooth function $\varphi = \varphi(t)$. Note that (3.20) implies

$$\begin{aligned} x_s &= \cosh\theta\Big(1,0,0\Big) + \sinh\theta\Big(0, \sinh\varphi(t), \cosh\varphi(t)\Big), \\ x_t &= s\varphi'(t)\sinh\theta\Big(0, \cosh\varphi(t), \sinh\varphi(t)\Big) + \Big(0, \gamma_2'(t), \gamma_3'(t)\Big). \end{aligned} \tag{3.21}$$

Because of $\langle x_s, x_t \rangle = 0$, we have

$$(0, \gamma_2', \gamma_3') = h(t)(0, \cosh\varphi(t), \sinh\varphi(t)) \tag{3.22}$$

for a smooth function $h = h(t)$. Therefore, (3.21) turns into

$$x_t = \Big(s\varphi'(t)\sinh\theta + h(t)\Big)\Big(0, \cosh\varphi(t), \sinh\varphi(t)\Big).$$

By combining this equation with $\langle x_t, x_t \rangle = m^2$ and using (3.15a), we obtain $\varphi'(t) = \Psi(t)$ and $h(t) = \Psi(t)\phi(t)$. So these equations consider in (3.22), we get (3.10b) with an appropriate choice of the parameter t. Thus, we have the Item (ii) of the theorem.

Case 2. m is given as (3.15b). Here, we can take $m(s,t) = 1$ by re-defining t properly. In this case, the induced metric of M becomes $g = ds^2 + dt^2$, the Levi Civita connection of M satisfies

$$\nabla_{\partial_s}\partial_s = 0, \quad \nabla_{\partial_s}\partial_t = 0, \quad \nabla_{\partial_t}\partial_t = 0 \tag{3.23}$$

and (3.3) becomes $S = 0$. A straightforward computation yields that M is congruent to the surface given in (3.11). Hence, the proof for the necessary condition is obtained.

The proof of sufficient condition follows from a direct computation. □

Example 3.1. We take different choices of the function $\phi(t)$ in (3.10b).

1. Let $\phi(t) = 1$. Then the parametrization of surface is

$$x(s,t) = s\cosh\theta\Big(1,0,0\Big) + \sinh\theta\Big(0, \sinh t(s+1), \cosh t(s+1)\Big). \tag{3.24}$$

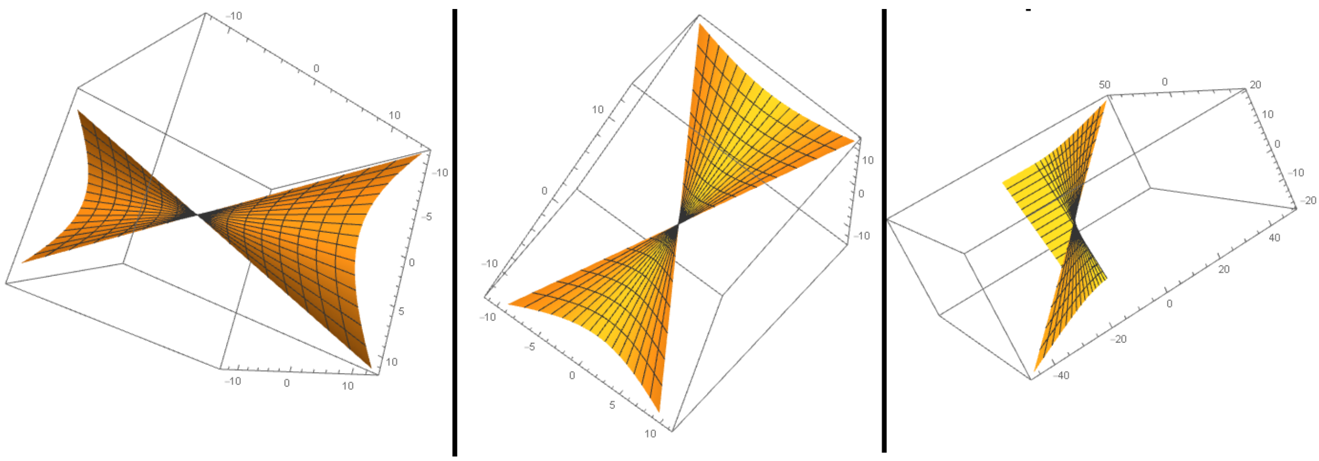
2. Let $\phi(t) = \dfrac{1}{\cosh t}$. Then the parametrization of surface is

$$x(s,t) = s\cosh\theta\Big(1,0,0\Big) + s\sinh\theta\Big(0, \sinh t, \cosh t\Big) + \sinh\theta\left(0, t, \ln(\cosh t)\right). \tag{3.25}$$

3. Let $\phi(t) = \dfrac{1}{\sinh t}$. Then the parametrization of surface is

$$x(s,t) = s\cosh\theta\Big(1,0,0\Big) + s\sinh\theta\Big(0, \sinh t, \cosh t\Big) + \sinh\theta\left(0, \ln(\sinh t), t\right). \tag{3.26}$$

Figure 1. Surfaces given by (3.24)-(3.26) (All pictures are realized by using Mathematica.).



3.2. Helix surfaces relative to a light-like, constant direction.

In this subsection we will consider helix hypersurfaces with the fixed vector $k = (1, 0, 1)$ which is light-like.

Theorem 3.2. *Let M be an oriented surface in $\mathbb{E}^3_1$ with diagonalizable shape operator. Then, M is a helix hypersurface with a principal direction relative to a light-like, constant direction if and only if it is congruent to the surface given by*

$$x(s,t) = s\Big(\frac{c}{2}(1-\varepsilon-\varepsilon t^2) + \frac{\varepsilon}{c}, t, \frac{c}{2}(1-\varepsilon-\varepsilon t^2)\Big) + \int_0^t a(\tau)\,(-c\varepsilon\tau, 1, -c\varepsilon\tau)\,d\tau \tag{3.27}$$

where a is a smooth function depending on t, c is a constant and $\varepsilon \in \{-1, 1\}$. Moreover, the tangential vector field $e_1 = \dfrac{(1,0,1)^T}{\|(1,0,1)^T\|}$ is a principal direction of the surfaces given by (3.27).

Proof. Let N be the unit normal vector field of M associated with its orientation and $x : M \to \mathbb{E}^3_1$ an isometric immersion. In order to prove necessary condition, assume that M is a helix surface with a principal direction relative to a light-like, constant direction k. Up to isometries of $\mathbb{E}^3_1$, we may assume $k = (1, 0, 1)$. We put $\varepsilon = -\langle N, N\rangle$ and

$$e_1 = \frac{(1,0,1)^T}{\|(1,0,1)^T\|}.$$

Then, we have

$$(1, 0, 1) = \varepsilon(e_1 - N). \tag{3.28}$$

Note that we have $\langle e_1, e_1\rangle = \varepsilon$.

Because of the assumption, e_1 is a principal direction of M with corresponding principal curvature k_1. By a simple computation considering (3.28) we obtain

$$0 = \varepsilon\nabla_X e_1 + \varepsilon h(e_1, X) + \varepsilon SX, \tag{3.29}$$

whenever X is tangent to M. Note that (3.29) for $X = e_1$ gives

$$\nabla_{e_1} e_1 = 0, \tag{3.30a}$$
$$\nabla_{e_1} e_2 = 0, \tag{3.30b}$$

while (3.29) for $X = e_2$ is giving

$$\nabla_{e_2} e_1 = -k_2 e_2, \tag{3.30c}$$
$$\nabla_{e_2} e_2 = \varepsilon k_2 e_1, \tag{3.30d}$$

where e_2 is the other principal direction of M with corresponding principal curvature k_2 and $\langle e_2, e_2\rangle = 1$. In addition, the second fundamental form of M becomes

$$h(e_1, e_1) = 0, \quad h(e_1, e_2) = 0, \qquad h(e_2, e_2) = -\varepsilon k_2 N. \tag{3.31}$$

Therefore, the Codazzi equation gives

$$e_1(k_2) = k_2^2 \quad \text{and} \quad e_2(k_1) = 0. \tag{3.32}$$

Let $p \in M$. First, we would like to prove the following claim.

Claim 1. There exists a neighborhood $\mathcal{N}_p$ of p on which the induced metric of M becomes

$$g = \varepsilon ds^2 + b^2(t)(a(t) + s)^2 dt^2 \tag{3.33}$$

for some smooth functions depending only on t a, b such that $e_1 = \partial_s$, $e_2 = \dfrac{1}{b(t)(a(t)+s)}\partial_t$.

Proof of Claim 1. Note that we have $[e_1, e_2] = k_2 e_2$ because of (3.30b) and (3.30c). Therefore, if G is a non-vanishing smooth function on M, then we have $[e_1, Ge_2] = 0$ such that

$$e_1(G) = -k_2 G. \tag{3.34}$$

Therefore, there exists a local coordinate system (s,t) such that $e_1 = \partial_s$ and $e_2 = \frac{1}{G}\partial_t$. Thus, the induced metric of M is

$$g = \varepsilon ds^2 + G^2 dt^2. \tag{3.35}$$

Thus, the first equation in (3.32) and (3.34) become

$$(k_2)_s = k_2^2 \tag{3.36}$$

and

$$G_s = -k_2 G \tag{3.37}$$

respectively. Now, getting derivative of (3.37) and considering (3.36) implies

$$G(s,t) = b(t)(s + a(t)). \tag{3.38}$$

for some smooth functions a, b. Therefore, (3.35) becomes (3.33).

Hence, the proof of the Claim 1 is completed. ■

Now, let s, t be local coordinates described in the Claim 1. Note that we have

$$e_1 = x_s. \tag{3.39}$$

Moreover, (3.30), (3.31) and (3.33) yield

$$x_{ss} = 0, \tag{3.40a}$$

$$x_{st} = \frac{1}{s + a(t)} x_t. \tag{3.40b}$$

Considering the decomposition (3.28), we have $\langle x_s, k\rangle = \varepsilon$ and $\langle x_t, k\rangle = 0$. Thus, we can assume that x has the form given by

$$x(s,t) = \Big(x_3(s,t) + \frac{\varepsilon}{c}s, x_2(s,t), x_3(s,t)\Big). \tag{3.41}$$

On the other hand, by considering (3.40a) in the last form yields

$$x(s,t) = s\Big(\beta_1(t) + \frac{\varepsilon}{c}, \beta_2(t), \beta_1(t)\Big) + \gamma(t) \tag{3.42}$$

for some smooth functions β_1, β_2 and a $\mathbb{E}^3_1$-valued smooth function $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_1(t))$. Now, by considering $\langle x_s, x_s\rangle = \varepsilon$ in (3.42), we obtain

$$x(s,t) = s\Big(\frac{c}{2} - \frac{\varepsilon c}{2} - \frac{\varepsilon c}{2}\beta_2^2(t) + \frac{\varepsilon}{2}, \beta_2(t), \frac{c}{2} - \frac{\varepsilon c}{2} - \frac{\varepsilon c}{2}\beta_2^2(t)\Big) + \gamma(t) \tag{3.43}$$

We may assume $\beta_2(t) = t$ without loss of generality, so (3.43) becomes

$$x(s,t) = s\Big(\frac{c}{2}(1 - \varepsilon - \varepsilon t^2) + \frac{\varepsilon}{c}, t, \frac{c}{2}(1 - \varepsilon - \varepsilon t^2)\Big) + \gamma(t) \tag{3.44}$$

Since (3.40b), we get directly the function γ as

$$\gamma(t) = \int_0^t a(\tau)\,(-c\varepsilon\tau, 1, -c\varepsilon\tau)\,d\tau. \tag{3.45}$$

Considering the obtained function γ in (3.45), we obtain (3.27). Hence, the proof of Theorem 3.2 for the necessary condition is obtained. The proof of sufficient condition follows from a direct computation. □

Example 3.2. We take different choices of the function $a(t)$ in (3.27).

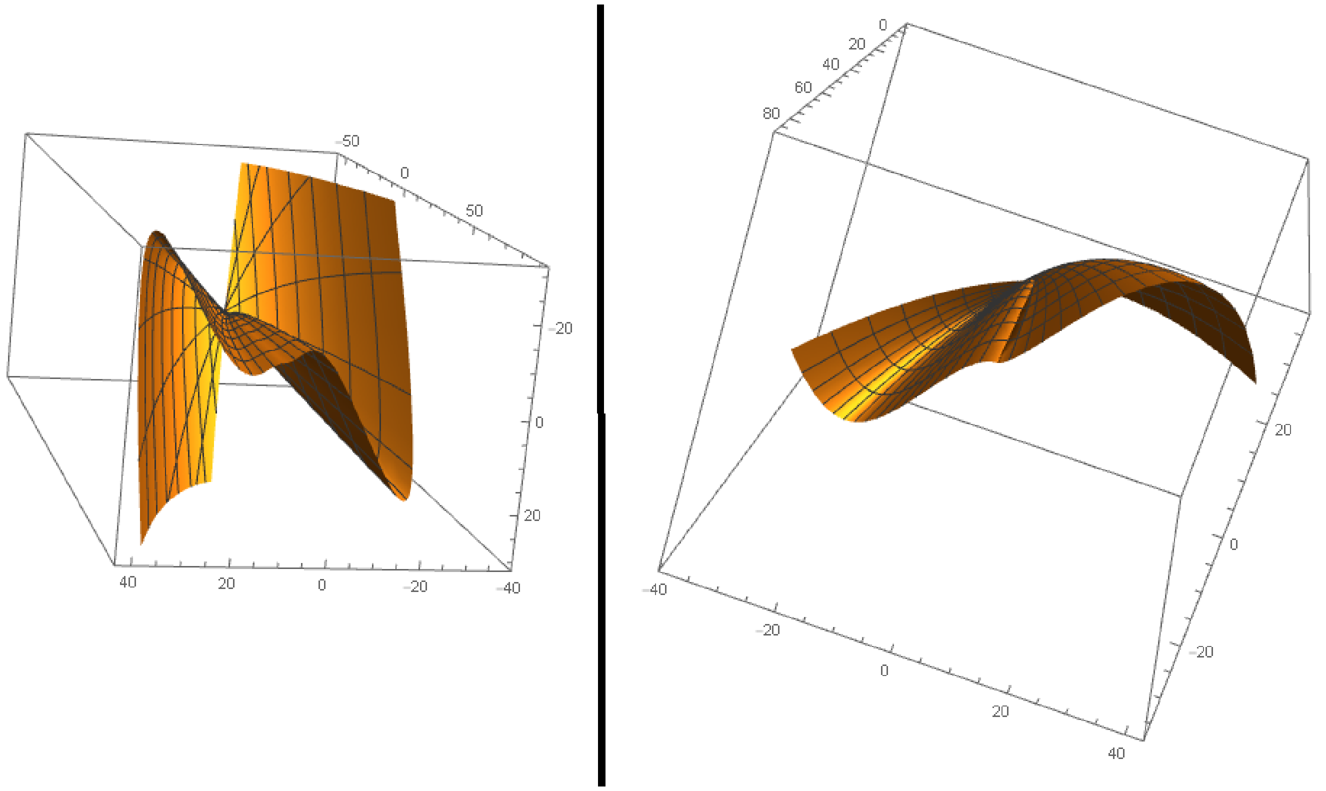
1. Let $a(t) = \sin t$. Then the parametrization of surface is

$$x(s,t) = \Big(st^2 - \frac{s}{2} + 2\sin t - 2t\cos t, st + 1 + \cos t, st^2 + 2\sin t - 2t\cos t\Big) \tag{3.46}$$

2. Let $a(t) = \ln t$. Then the parametrization of surface is

$$x(s,t) = \Big(st^2 - \frac{s}{2} + t^2 \ln t - \frac{t^2}{2}, st + t\ln t - t, st^2 + t^2 \ln t - \frac{t^2}{2}\Big) \tag{3.47}$$

Figure 2. Surfaces given by (3.46),(3.47) (All pictures are realized by using Mathematica.).



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