Connections On The Coframe Bundle

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ABSTRACT

In this paper, we determine the horizontal and complete lifts of the linear connection from a smooth manifold to its coframe bundle. The torsion tensor and the curvature tensor of the horizontal lift $^H\nabla$ are found. We also studied geodesic curves corresponding to the horizontal and complete lifts of the linear connection in the coframe bundle.

1. Introduction

Let M be a smooth manifold, T^*M its cotangent bundle and F^*M its coframe bundle. Various questions of the differential geometry of the cotangent bundle have been studied by many authors (see, for example, [2], [3], [6],[7], [11], [12],[14], [16],[17]). On the other hand, the some problems concerning differential geometry of F^*M has been investigated in [4] by authors of the present paper (see also,[13]).

One of the basic differential-geometric structures on a smooth manifold is a linear connection. In this paper, we define the horizontal and complete lifts of linear connection in the coframe bundle based on the methods used in [1], [5], [6], [17].

In 2 we briefly describe the definitions and results that are needed later, after which the horizontal lift of the symmetric linear connection is constructed in 3. The torsion tensor and the curvature tensor of the horizontal lift of the symmetric linear connection are determined in 4. In 5 we consider the question of the geodesic curve of the horizontal lift of the linear connection. The complete lift of the symmetric linear connection is constructed in 6. In 7 we study the properties of the geodesic curve of the complete lift of the linear connection.

2. Preliminaries

In this section, we shall fix our notations and recall, for later use, the definitions and some properties of the complete and horizontal lifts of the vector fields to the coframe bundle.

Manifolds, tensor fields and linear connections, under consideration are all assumed to be differentiable and of class C^∞. Indices $i, j, k, ..., \alpha, \beta, \gamma, ...$ have range in $\{1, 2, ..., n\}$ and indices $A, B, C, ...$ run from 1 to $n + n^2$. We put $h_\alpha = \alpha \cdot n + h$. Summation over repeated indices is always implied.

Let M be an $n-$ dimensional differentiable manifold of class C^∞. Coordinate systems in M are denoted by (U, x^i), where U is the coordinate neighborhood and x^i are the coordinate functions . We denote by $\Im_s^r(M)$ the set of all differentiable tensor fields of type (r, s) on M.

Let T_x^*M be the cotangent space at a point $x \in M$, $(X^\alpha) = (X^1, ..., X^n)$ a linear coframe at x and F^*M the coframe bundle over M, that is, the set of all coframes at all points of M([4]). Let $\pi : F^*M \to M$ be the canonical projection of F^*M onto M; for the coordinate system (U, x^i) in M we put $F^*U = \pi^{-1}(U)$. A coframe (X^α) at x can be expressed uniquely in the form $X^\alpha = X_i^\alpha(dx^i)_x$. The induced coordinate system in F^*U is $\left\{F^*U, (x^i, X_i^\alpha)\right\}$; we shall denote $\frac{\partial}{\partial x^i}$ by ∂_i and $\frac{\partial}{\partial X_i^\alpha}$ by ∂_{i_α}. The matrix $[X_i^\alpha]$ is non-singular and its inverse will be written as $[X_\alpha^i]$. We denote by ∇ the linear connection on M with components Γ_{ij}^k.

3. Horizontal lifts of linear connections

Let ∇ be the symmetric linear connection on M and Γ^k_{ij} its components. The horizontal distribution H_∇ and the vertical distribution V on $F^*(U) = \pi^{-1}(U)$ are spanned by vectors D_i and D_{i_α}, $i = 1, ..., n, \alpha = 1, ..., n$ defined in induced local coordinates $(\pi^{-1}(U), x^i, X^\alpha_i)$ by

$$D_i = \frac{\partial}{\partial x^i} + \Gamma^m_{ij} X^\alpha_m \frac{\partial}{\partial X^\alpha_j}, D_{i_\alpha} = \frac{\partial}{\partial X^\alpha_i}.$$

The set of $\{D_i, D_{i_\alpha}\}$ $i, \alpha = 1, ..., n$ is called the frame adapted to the affine connection ∇. The dual coframes on $\pi^{-1}(U)$ with respect to the frame $\{D_i, D_{i_\alpha}\}$ consists of the 1-forms $\{\eta^j, \eta^{j_\beta}\}$ $j, \beta = 1, ..., n$ defined in local coordinates by formulas

$$\eta^j = dx^j, \ \eta^{j_\beta} = dX^\beta_j - \Gamma^m_{ij} X^\beta_m dx^i.$$

If a vector field X and a 1-form ω on M are of the form $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_j dx^j$ in local chart (U, x^i), respectively. Then the horizontal lift $^H X$ of X and the vertical lifts $^{V_\alpha} \omega$, $\alpha = 1, ..., n$ of ω to the coframe bundle $F^* M$ in the induced local chart $(\pi^{-1}(U), x^i, X^\alpha_i)$ are of the form:

$$^H X = X^i D_i, {}^{V_\alpha} \omega = \sum_j \omega_j \delta^\beta_\alpha D_{j_\beta}$$

with respect to the adapted frame $\{D_i, D_{i_\alpha}\}$.

Definition 3.1. *A horizontal lift of the symmetric linear connection ∇ on M to the coframe bundle $F^* M$ is the linear connection $^H \nabla$ defined by:*

$$^H \nabla_{^H X} {}^H Y = {}^H (\nabla_X Y), \qquad {}^H \nabla_{^H X} {}^{V_\alpha} \omega = {}^{V_\alpha} (\nabla_X \omega),$$
$$^H \nabla_{V_\alpha \theta} {}^H Y = 0, \qquad {}^H \nabla_{V_\alpha \theta} {}^{V_\beta} \omega = 0 \tag{3.1}$$

for any $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im(M)$.

We note that the horizontal lifts of the linear connection from a manifold to its cotangent bundle and the bundle of linear frames are constructed in the [1], [5], [6], [17].

The components of the horizontal lift $^H \nabla$ of ∇ on M with components Γ^k_{ij} in the natural frame $\left\{ \frac{\partial}{\partial x^i} \right\}$, are defined in the adapted frame $\{D_i, D_{i_\alpha}\} = \{D_I\}$ by decomposition

$$^H \nabla_{D_I} D_J = {}^H \Gamma^K_{IJ} D_K. \tag{3.2}$$

From (3.1) and (3.2) we obtain:

$$1) \ ^H \nabla_{V_\alpha \theta} {}^{V_\beta} \omega = 0,$$
$$^H \nabla_{\sum_i \theta_i \delta^\gamma_\alpha D_{i_\gamma}} \left(\sum_j \omega_j \delta^\tau_\beta D_{j_\tau} \right) = 0,$$
$$\sum_{i,j} \theta_i \omega_j \delta^\gamma_\alpha \delta^\tau_\beta {}^H \nabla_{D_{i_\gamma}} D_{j_\tau} + \sum_i \theta_i \delta^\gamma_\alpha D_{i_\gamma} ({}^{V_\beta} \omega) = 0,$$
$$\sum_{i,j} \theta_i \omega_j \delta^\gamma_\alpha \delta^\tau_\beta {}^H \Gamma^L_{i_\gamma j_\tau} D_L = 0,$$
$$\sum_{i,j} \theta_i \omega_j \delta^\gamma_\alpha \delta^\tau_\beta ({}^H \Gamma^l_{i_\gamma j_\tau} D_l + {}^H \Gamma^{l_\sigma}_{i_\gamma j_\tau} D_{l_\sigma}) = 0,$$
$$^H \Gamma^l_{i_\alpha j_\beta} D_l + {}^H \Gamma^{l_\sigma}_{i_\alpha j_\beta} D_{l_\sigma} = 0,$$

consequently, $\ ^H \Gamma^l_{i_\alpha j_\beta} = 0, \ ^H \Gamma^{l_\sigma}_{i_\alpha j_\beta} = 0.$

$$2) \ ^H \nabla_{V_\alpha \theta} {}^H Y = 0,$$
$$^H \nabla_{\sum_i \theta_i \delta^\gamma_\alpha D_{i_\gamma}} (Y^j D_j) = 0,$$
$$\sum_i \theta_i \delta^\gamma_\alpha Y^j {}^H \nabla_{D_{i_\gamma}} D_j + \sum_i \theta_i \delta^\gamma_\alpha D_{i_\gamma} (Y^j) D_j = 0,$$
$$\sum_i \theta_i \delta^\gamma_\alpha Y^j {}^H \Gamma^L_{i_\gamma j} D_L = \sum_i \theta_i \delta^\gamma_\alpha Y^j ({}^H \Gamma^l_{i_\gamma j} D_l + \Gamma^{l_\sigma}_{i_\gamma j} D_\sigma) = 0,$$

from which we find: $^H\Gamma^l_{i_\alpha j} = 0, \quad ^H\Gamma^{l_\sigma}_{i_\alpha j} = 0.$

$$3)^H\nabla_{^HX}{}^HY = {}^H(\nabla_X Y), {}^H\nabla_{X^i D_i}(Y^j D_j) = (\nabla_X Y)^k D_k,$$

$$X^i Y^{j\,H}\nabla_{D_i} D_j + X^i D_i(Y^j)D_j = X^m(\partial_m Y^k + \Gamma^k_{mr} Y^r)D_k,$$

$$X^i Y^{j\,H}\Gamma^L_{ij} D_L + X^i \partial_i Y^j D_j + X^i \Gamma^s_{ih} X^\alpha_s \frac{\partial}{\partial X^\alpha_h}(Y^j)D_j = X^m \partial_m Y^k D_k + X^m \Gamma^k_{mr} Y^r D_k,$$

$$X^i Y^{j\,H}\Gamma^l_{ij} D_l + X^i Y^{j\,H}\Gamma^{l_\sigma}_{ij} D_{l_\sigma} = X^m \Gamma^k_{mr} Y^r D_k,$$

consequently, $^H\Gamma^l_{ij} = \Gamma^l_{ij}, \quad ^H\Gamma^{l_\sigma}_{ij} = 0.$

$$4)^H\nabla_{^HX}{}^{V_\alpha}\omega = {}^{V_\alpha}(\nabla_X \omega),$$

$$^H\nabla_{X^i D_i}\left(\sum_j \omega_j \delta^\tau_\alpha D_{j_\tau}\right) = \sum_k (\nabla_X \omega)\delta^\sigma_\alpha D_{k_\sigma},$$

$$X^i \sum_j \omega_j \delta^{\tau\,H}_\alpha \nabla_{D_i} D_{j_\tau} + X^i \delta^\tau_\alpha D_i\left(\sum_j \omega_j\right)D_{j_\tau} = \sum_k X^m (\partial_m \omega_k - \Gamma^r_{mk}\omega_r)\delta^\sigma_\alpha D_{k_\sigma},$$

$$X^i \sum_j \omega_j \delta^{\tau\,H}_\alpha \Gamma^L_{ij_\tau} D_L + X^i \delta^\tau_\alpha\left(\left(\sum_j \partial_i \omega_j\right) + \Gamma^m_{ih} X^\gamma_m D_{h_\gamma}\left(\sum_j \omega_j\right)\right)D_{j_\tau}$$

$$= \sum_k X^m \partial_m \omega_k \delta^\sigma_\alpha D_{k_\sigma} - \sum_k X^m \Gamma^r_{mk}\omega_r \delta^\sigma_\alpha D_{k_\sigma},$$

$$X^i \sum_j \omega_j \delta^{\tau\,H}_\alpha \Gamma^l_{ij_\tau} D_l + X^i \sum_j \omega_j \delta^{\tau\,H}_\alpha \Gamma^{l_\sigma}_{ij_\tau} D_{l_\sigma} + X^i \delta^\tau_\alpha \sum_j \partial_i \omega_j D_{j_\tau}$$

$$+ X^i \delta^\tau_\alpha \Gamma^m_{ih} X^\gamma_m D_{h_\gamma}\left(\sum_j \omega_j\right)D_{j_\tau} = \sum_k X^m \partial_m \omega_k \delta^\sigma_\alpha D_{k_\sigma} - \sum_k X^m \Gamma^r_{mk}\omega_r \delta^\sigma_\alpha D_{k_\sigma},$$

$$X^i \sum_j \omega_j \delta^{\tau\,H}_\alpha \Gamma^l_{ij_\tau} D_l + X^i \sum_j \omega_j \delta^{\tau\,H}_\alpha \Gamma^{l_\sigma}_{ij_\tau} D_{l_\sigma} = -\sum_k X^m \Gamma^r_{mk}\omega_r \delta^\sigma_\alpha D_{k_\sigma},$$

from which we obtain that
$^H\Gamma^l_{ij_\beta} = 0$ and $^H\Gamma^{l_\sigma}_{ij_\beta} = -\Gamma^j_{il}\delta^\sigma_\beta.$
Thus, the following theorem holds.

Theorem 3.1. *The horizontal lift $^H\nabla$ of the symmetric linear connection ∇ given on M, to the coframe bundle F^*M have the components*

$$^H\Gamma^l_{ij} = \Gamma^l_{ij}, \quad ^H\Gamma^{l_\sigma}_{ij} = {}^H\Gamma^l_{ij_\beta} = 0, {}^H\Gamma^{l_\sigma}_{ij_\beta} = -\Gamma^j_{il}\delta^\sigma_\beta,$$

$$^H\Gamma^l_{i_\alpha j} = 0, \quad ^H\Gamma^{l_\sigma}_{i_\alpha j} = 0, {}^H\Gamma^l_{i_\alpha j_\beta} = 0, \quad ^H\Gamma^{l_\sigma}_{i_\alpha j_\beta} = 0 \tag{3.3}$$

in the adapted frame $\{D_i, D_{i_\alpha}\}$.

We have the relations:

$$\{D_i, D_{i_\alpha}\} = \left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial X^\beta_j}\right)\begin{pmatrix} \delta^j_i & 0 \\ \Gamma^m_{ji} X^\beta_m & \delta^i_j \delta^\beta_\alpha \end{pmatrix}, \tag{3.4}$$

i.e.

$$D_I = B^J_I \partial_J,$$

where $\left(B^J_I\right)$ −the matrix of transformation (3.4) and its inverse matrix in the form:

$$\left(B^I_K\right) = \begin{pmatrix} \delta^i_k & 0 \\ -\Gamma^m_{ik} X^\alpha_m & \delta^k_i \delta^\alpha_\gamma \end{pmatrix}. \tag{3.5}$$

We denote the components of the linear connection $^H\nabla$ in the natural frame $\partial_I = \{\partial_i, \partial_{i_\alpha}\}$ by $^H\bar\Gamma^K_{IJ}$, i.e.

$$^H\nabla_{\partial_I}\partial_J = {}^H\bar\Gamma^K_{IJ}\partial_K.$$

Then

$$^{H}\bar{\Gamma}_{IJ}^{K} = B_{L}^{K}\,^{H}\Gamma_{PS}^{L}\tilde{B}_{I}^{P}\tilde{B}_{J}^{S} - \left(D_{P}B_{S}^{K}\right)\tilde{B}_{I}^{P}\tilde{B}_{J}^{S}. \tag{3.6}$$

From (3.6), by using (3.3), (3.4) and (3.5) we obtain:

$$^{H}\bar{\Gamma}_{ij}^{k} = B_{l}^{k}\,^{H}\Gamma_{ps}^{l}\tilde{B}_{i}^{p}\tilde{B}_{j}^{s} + B_{l_{\tau}}^{k}\,^{H}\Gamma_{ps_{\sigma}}^{l_{\tau}}\tilde{B}_{i}^{p}\tilde{B}_{j}^{s_{\sigma}} = \delta_{l}^{k}\Gamma_{ps}^{l}\delta_{i}^{p}\delta_{j}^{s} = \Gamma_{ps}^{l};$$

$$^{H}\bar{\Gamma}_{i_{\alpha}j}^{k} = B_{l}^{k}\,^{H}\Gamma_{ps}^{l}\tilde{B}_{i_{\alpha}}^{p}\tilde{B}_{j}^{s} + B_{l_{\tau}}^{k}\,^{H}\Gamma_{ps_{\sigma}}^{l_{\tau}}\tilde{B}_{i_{\alpha}}^{p}\tilde{B}_{j}^{s_{\sigma}} = 0;$$

$$^{H}\bar{\Gamma}_{ij_{\beta}}^{k} = B_{l}^{k}\,^{H}\Gamma_{ps}^{l}\tilde{B}_{i}^{p}\tilde{B}_{j_{\beta}}^{s} + B_{l_{\tau}}^{k}\,^{H}\Gamma_{ps_{\sigma}}^{l_{\tau}}\tilde{B}_{i}^{p}\tilde{B}_{j_{\beta}}^{s_{\sigma}} = 0;$$

$$^{H}\bar{\Gamma}_{i_{\alpha}j_{\beta}}^{k} = B_{l}^{k}\,^{H}\Gamma_{ps}^{l}\tilde{B}_{i_{\alpha}}^{p}\tilde{B}_{j_{\beta}}^{s} + B_{l_{\tau}}^{k}\,^{H}\Gamma_{ps_{\sigma}}^{l_{\tau}}\tilde{B}_{i_{\alpha}}^{p}\tilde{B}_{j_{\beta}}^{s_{\sigma}} = 0;$$

$$^{H}\bar{\Gamma}_{i_{\alpha}j}^{k_{\gamma}} = B_{l}^{k_{\gamma}}\,^{H}\Gamma_{ps}^{l}\tilde{B}_{i_{\alpha}}^{p}\tilde{B}_{j}^{s} + B_{l_{\tau}}^{k_{\gamma}}\,^{H}\Gamma_{ps_{\sigma}}^{l_{\tau}}\tilde{B}_{i_{\alpha}}^{p}\tilde{B}_{j}^{s_{\sigma}} - \left(D_{p}B_{s}^{k_{\gamma}}\right)\tilde{B}_{i_{\alpha}}^{p}\tilde{B}_{j}^{s}$$
$$- \left(D_{p_{\nu}}B_{s}^{k_{\gamma}}\right)\tilde{B}_{i_{\alpha}}^{p_{\nu}}\tilde{B}_{j}^{s} - \left(D_{p_{\nu}}B_{s_{\sigma}}^{k_{\gamma}}\right)\tilde{B}_{i_{\alpha}}^{p_{\nu}}\tilde{B}_{j}^{s_{\sigma}} = -\delta_{m}^{p}\Gamma_{sk}^{m}\delta_{\nu}^{\gamma}\delta_{p}^{i}\delta_{\alpha}^{\nu} = -\Gamma_{jk}^{i}\delta_{\alpha}^{\gamma};$$

$$^{H}\bar{\Gamma}_{ij_{\beta}}^{k_{\gamma}} = B_{l}^{k_{\gamma}}\,^{H}\Gamma_{ps}^{l}\tilde{B}_{i}^{p}\tilde{B}_{j_{\beta}}^{s} + B_{l_{\tau}}^{k_{\gamma}}\,^{H}\Gamma_{ps_{\sigma}}^{l_{\tau}}\tilde{B}_{i}^{p}\tilde{B}_{j_{\beta}}^{s_{\sigma}}$$
$$= \delta_{k}^{l}\delta_{\tau}^{\gamma}\left(-\Gamma_{pl}^{s}\delta_{\sigma}^{\tau}\right)\delta_{i}^{p}\delta_{s}^{j}\delta_{\beta}^{\sigma} = -\Gamma_{ik}^{j}\delta_{\beta}^{\gamma};$$

$$^{H}\bar{\Gamma}_{i_{\alpha}j_{\beta}}^{k_{\gamma}} = B_{l}^{k_{\gamma}}\,^{H}\Gamma_{ps}^{l}\tilde{B}_{i_{\alpha}}^{p}\tilde{B}_{j_{\beta}}^{s} + B_{l_{\tau}}^{k_{\gamma}}\,^{H}\Gamma_{ps_{\sigma}}^{l_{\tau}}\tilde{B}_{i_{\alpha}}^{p}\tilde{B}_{j_{\beta}}^{s_{\sigma}}$$
$$- \left(D_{p_{\nu}}B_{s_{\sigma}}^{k_{\gamma}}\right)\tilde{B}_{i_{\alpha}}^{p_{\nu}}\tilde{B}_{j_{\beta}}^{s_{\sigma}} = 0;$$

$$^{H}\bar{\Gamma}_{ij}^{k_{\gamma}} = B_{l}^{k_{\gamma}}\,^{H}\Gamma_{ps}^{l}\tilde{B}_{i}^{p}\tilde{B}_{j}^{s} + B_{l_{\tau}}^{k_{\gamma}}\,^{H}\Gamma_{ps_{\sigma}}^{l_{\tau}}\tilde{B}_{i}^{p}\tilde{B}_{j}^{s_{\sigma}} - \left(D_{P}B_{S}^{k_{\gamma}}\right)\tilde{B}_{i_{\alpha}}^{P}\tilde{B}_{j}^{S}$$
$$= \Gamma_{kl}^{m}X_{m}^{\gamma}\Gamma_{ps}^{l}\delta_{i}^{p}\delta_{j}^{s} + \delta_{k}^{l}\delta_{\tau}^{\gamma}\left(-\Gamma_{pl}^{s}\delta_{\sigma}^{\tau}\right)\delta_{i}^{p}\left(-\Gamma_{sj}^{r}X_{r}^{\sigma}\right) - \left(D_{p}B_{s}^{k_{\gamma}}\right)\tilde{B}_{i}^{p}\tilde{B}_{j}^{s}$$
$$- \left(D_{p}B_{s_{\sigma}}^{k_{\gamma}}\right)\tilde{B}_{i}^{p}\tilde{B}_{j}^{s_{\sigma}} - \left(D_{p_{\nu}}B_{s}^{k_{\gamma}}\right)\tilde{B}_{i}^{p_{\nu}}\tilde{B}_{j}^{s} = X_{m}^{\gamma}\Gamma_{kl}^{m}\Gamma_{ij}^{l} + X_{r}^{\gamma}\Gamma_{ik}^{s}\Gamma_{sj}^{r}$$
$$- \left(\partial_{p} + \Gamma_{pr}^{l}X_{l}^{\alpha}\frac{\partial}{\partial X_{r}^{\alpha}}\right)\left(\Gamma_{ks}^{m}X_{m}^{\gamma}\right)\delta_{i}^{p}\delta_{j}^{s} - \frac{\partial}{\partial X_{p}^{\nu}}\left(\Gamma_{ks}^{m}X_{m}^{\gamma}\right)\left(-\Gamma_{pi}^{r}X_{r}^{\nu}\delta_{j}^{s}\right)$$
$$= X_{m}^{\gamma}\Gamma_{kl}^{m}\Gamma_{ij}^{l} + X_{r}^{\gamma}\Gamma_{ik}^{s}\Gamma_{sj}^{r} - \left(\partial_{p}\Gamma_{ks}^{m}\right)X_{m}^{\gamma}\delta_{i}^{p}\delta_{j}^{s} - \Gamma_{pr}^{l}X_{l}^{\alpha}\Gamma_{ks}^{m}\delta_{\alpha}^{\gamma}\delta_{m}^{r}\delta_{i}^{p}\delta_{j}^{s}$$
$$- \Gamma_{ks}^{m}X_{r}^{\nu}\Gamma_{pi}^{r}\delta_{\nu}^{\gamma}\delta_{m}^{p}\delta_{j}^{s} = X_{m}^{\gamma}\Gamma_{kl}^{m}\Gamma_{ij}^{l} + X_{r}^{\gamma}\Gamma_{ik}^{s}\Gamma_{sj}^{r} - X_{m}^{\gamma}\partial_{i}\Gamma_{kj}^{m}$$
$$- X_{l}^{\gamma}\Gamma_{im}^{l}\Gamma_{kj}^{m} + X_{r}^{\gamma}\Gamma_{kj}^{p}\Gamma_{pi}^{r} = X_{m}^{\gamma}\left(\Gamma_{kl}^{m}\Gamma_{ij}^{l} + \Gamma_{ik}^{l}\Gamma_{lj}^{m} - \partial_{i}\Gamma_{kj}^{m}\right).$$

Thus we have

Theorem 3.2. *The horizontal lift $^{H}\nabla$ of the symmetric linear connection ∇ given on M, to the coframe bundle $F^{*}M$ have the components*

$$^{H}\bar{\Gamma}_{ij}^{k} = \Gamma_{ij}^{k}, \quad ^{H}\bar{\Gamma}_{ij}^{k_{\gamma}} = X_{m}^{\gamma}\left(\Gamma_{kl}^{m}\Gamma_{ij}^{l} + \Gamma_{ik}^{l}\Gamma_{lj}^{m} - \partial_{i}\Gamma_{kj}^{m}\right),$$

$$^{H}\bar{\Gamma}_{i_{\alpha}j}^{k_{\gamma}} = -\Gamma_{jk}^{i}\delta_{\alpha}^{\gamma}, ^{H}\bar{\Gamma}_{ij_{\beta}}^{k_{\gamma}} = -\Gamma_{ik}^{j}\delta_{\beta}^{\gamma}, ^{H}\bar{\Gamma}_{i_{\alpha}j}^{k} = 0, \tag{3.7}$$

$$^{H}\bar{\Gamma}_{i_{\alpha}j}^{k_{\gamma}} = 0, ^{H}\bar{\Gamma}_{i_{\alpha}j_{\beta}}^{k} = 0, \quad ^{H}\bar{\Gamma}_{i_{\alpha}j_{\beta}}^{k_{\gamma}} = 0$$

in the natural frame $\left(\frac{\partial}{\partial x^{j}}, \frac{\partial}{\partial X_{j}^{\beta}}\right)$.

4. Torsion and curvature tensors of the horizontal lifts

In the next two theorems, we determine the torsion and curvature tensors of $^{H}\nabla$.

Theorem 4.1. *Let $\tilde{T}$ be the torsion tensor of the horizontal lift $^{H}\nabla$ of the symmetric linear connection ∇ in M to the coframe bundle $F^{*}M$. Then $\tilde{T}$ is the skew-symmetric tensor field of type $(1, 2)$ in $F^{*}M$ determined by*

$$\tilde{T}(^{V_{\alpha}}\theta, {}^{V_{\beta}}\omega) = 0, \quad \tilde{T}(^{H}X, {}^{V_{\alpha}}\omega) = 0, \quad \tilde{T}(^{H}X, {}^{H}Y) = -\gamma(R(X, Y)) \tag{4.1}$$

for any $X, Y \in \Im_{0}^{1}(M)$ and $\theta, \omega \in \Im_{1}^{0}(M)$, where R is the curvature tensor of ∇.

Proof. We shall prove (4.1) by using components $^H\bar{\Gamma}^K_{IJ}$ of $^H\nabla$ given by (3.7). Since local components $\tilde{T}^K_{IJ}$ of $\tilde{T}$ are given by

$$\tilde{T}^K_{IJ} = {}^H\bar{\Gamma}^K_{IJ} - {}^H\bar{\Gamma}^K_{JI}$$

with respect to the induced coordinates, we have from (3.7):

$$\tilde{T}^k_{ij} = 0, \tilde{T}^k_{i_\alpha j} = 0, \tilde{T}^k_{ij_\beta} = 0, \tilde{T}^k_{i_\alpha j_\beta} = 0,$$
$$\tilde{T}^{k_\gamma}_{ij} = {}^H\bar{\Gamma}^{k_\gamma}_{ij} - {}^H\bar{\Gamma}^{k_\gamma}_{ji} = X^\gamma_m(\Gamma^m_{kl}\Gamma^l_{ij} + \Gamma^l_{ik}\Gamma^m_{lj} - \partial_i\Gamma^m_{kj}) - X^\gamma_m(\Gamma^m_{kl}\Gamma^l_{ji} +$$
$$+ \Gamma^l_{jk}\Gamma^m_{li} - \partial_j\Gamma^m_{ki}) = -X^\gamma_m(\partial_i\Gamma^m_{kj} - \partial_j\Gamma^m_{ki} + \Gamma^l_{ik}\Gamma^m_{lj} - \Gamma^l_{jk}\Gamma^m_{li})$$
$$- X^\gamma_m R^m_{ijk}, \tilde{T}^{k_\gamma}_{i_\alpha j} = 0, \tilde{T}^{k_\gamma}_{ij_\beta} = 0, \tilde{T}^{k_\gamma}_{i_\alpha j_\beta} = 0,$$

with respect to the induced coordinates. Thus, (4.1) are satisfied. $\square$

Theorem 4.2. *Let $\tilde{R}$ be the curvature tensor of the horizontal lift $^H\nabla$ of the symmetric affine connection ∇ in M to the coframe bundle F^*M. Then*

$$\tilde{R}({}^{V_\alpha}\theta, {}^{V_\beta}\omega) = 0, \quad \tilde{R}({}^HX, {}^{V_\alpha}\omega) = 0,$$
$$\tilde{R}({}^HX, {}^HY){}^{V_\gamma}\phi = -{}^{V_\gamma}(\phi(R(X,Y))),$$
$$\tilde{R}({}^HX, {}^HY){}^HZ = {}^H((R(X,Y)Z), \tag{4.2}$$

for any $X, Y, Z \in \Im^1_0(M)$ and $\theta, \omega, \phi \in \Im^0_1(M)$.

Proof. We shall prove this result by using components $^H\bar{\Gamma}^K_{IJ}$ of $^H\nabla$ given by (3.7). Since components $\tilde{R}^K_{IJL}$ of curvature tensor $\tilde{R}$ are given by

$$\tilde{R}^K_{IJL} = \partial_I {}^H\bar{\Gamma}^K_{JL} - \partial_J {}^H\bar{\Gamma}^K_{IL} + {}^H\bar{\Gamma}^K_{IP} {}^H\bar{\Gamma}^P_{JL} - {}^H\bar{\Gamma}^K_{JP} {}^H\bar{\Gamma}^P_{IL}$$

with respect to the induced coordinates, we have from (4.2)

$$\tilde{R}^k_{ijl} = R^k_{ijl},$$
$$\tilde{R}^{k_\gamma}_{ijl} = X^\gamma_m(\Gamma^m_{kr}R^r_{ijl} + \Gamma^m_{lr}R^r_{ijk}), \tag{4.3}$$
$$\tilde{R}^{k_\gamma}_{ijl_\tau} = -R^l_{ijk}\delta^\gamma_\tau,$$

the remaining components being zero with respect to the induced coordinates. Thus we have (4.2) from (4.3). $\square$

Using the results of Theorems 4.1 and 4.2, we have

Theorem 4.3. *The horizontal lift $^H\nabla$ of the symmetric linear connection ∇ on M to the coframe bundle F^*M is torsionless linear connection iff $R = 0$. A curvature tensor $\tilde{R}$ of the horizontal lift $^H\nabla$ on F^*M vanishes iff the curvature tensor R of ∇ on M vanishes.*

5. Geodesics of horizontal lifts

Different problems of geodesics has been very well investigated (see for example [8-10]).

Let $\tilde{C}$ be a geodesic curve on the coframe bundle F^*M with respect to the horizontal lift $^H\nabla$ of the symmetric linear connection ∇ on the M. In induced coordinates $(\pi^{-1}(U), x^i, X^\alpha_i)$ the equation of the geodesic curve $\tilde{C} : I \to F^*M, \tilde{C} : t \to \tilde{C}(t) = (x^i(t), X^\alpha_i(t)) = (x^I(t))$ are of the form:

$$\frac{d^2x^K}{dt^2} + {}^H\bar{\Gamma}^K_{IJ}\frac{dx^I}{dt}\frac{dx^J}{dt} = 0, \quad I, J.K = 1, 2, ..., n + n^2. \tag{5.1}$$

Using the formulas (3.7) for $^H\bar{\Gamma}^K_{IJ}$, from (5.1) we get:

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0,$$

$$\frac{d^2 X_k^\gamma}{dt^2} + (\Gamma_{kl}^m \Gamma_{ij}^l + \Gamma_{ik}^l \Gamma_{lj}^m - \partial_i \Gamma_{kj}^m) X_m^\gamma \frac{dx^i}{dt}\frac{dx^j}{dt} - 2\Gamma_{kj}^i \frac{dX_i^\gamma}{dt}\frac{dx^j}{dt} = 0. \tag{5.2}$$

Now we consider the covariant differentiation of $X_k^\gamma(t)$:

$$\frac{\delta}{dt}(X_k^\gamma(t)) = \frac{dX_k^\gamma}{dt} - \Gamma_{li}^m X_m^\gamma \frac{dx^l}{dt}. \tag{5.3}$$

Assuming that,

$$\frac{d^2 x^k}{dt^2} + \Gamma_{ij}^k \frac{dx^i}{dt}\frac{dx^j}{dt} = 0,$$

and taking into account the symmetry of the connection ∇ on M, from (5.3) we obtain:

$$\begin{aligned}
\frac{\delta^2 X_k^\gamma}{dt^2} &= \frac{\delta}{dt}\left(\frac{\delta X_k^\gamma}{dt}\right) = \frac{\delta}{dt}\left(\frac{dX_k^\gamma}{dt} - \Gamma_{lk}^m X_m^\gamma \frac{dx^l}{dt}\right) \\
&= \frac{d}{dt}\left(\frac{dX_k^\gamma}{dt} - \Gamma_{lk}^m X_m^\gamma \frac{dx^l}{dt}\right) - \Gamma_{ik}^r\left(\frac{dX_r^\gamma}{dt} - \Gamma_{jr}^m X_m^\gamma \frac{dx^j}{dt}\right)\frac{dx^i}{dt} \\
&= \frac{d^2 X_k^\gamma}{dt^2} + (\Gamma_{kl}^m \Gamma_{ij}^l + \Gamma_{ik}^l \Gamma_{lj}^m - \partial_i \Gamma_{kj}^m) X_m^\gamma \frac{dx^i}{dt}\frac{dx^j}{dt} - 2\Gamma_{kj}^i \frac{dX_i^\gamma}{dt}\frac{dx^j}{dt}.
\end{aligned} \tag{5.4}$$

Comparing expression (5.4) and the second equality of (5.2) , we have

Theorem 5.1. *A geodesic curve on the coframe bundle F^*M with respect to the horizontal lift $^H\nabla$ of the symmetric linear connection ∇ on M has in induced coordinates $(\pi^{-1}(U), x^i, X_i^\alpha)$ in F^*M the equations of the form:*

$$\frac{d^2 x^k}{dt^2} + \Gamma_{ij}^k \frac{dx^i}{dt}\frac{dx^j}{dt} = 0, \quad \frac{\delta^2 X_k^\gamma}{dt^2} = 0.$$

Theorem 5.2. *A curve $\tilde{C}$ on the coframe bundle F^*M is a geodesic curve with respect to the horizontal lift $^H\nabla$ of the symmetric linear connection ∇, if the projection $C = \pi(\tilde{C})$ on M is a geodesic curve with respect to ∇ on M and the second covariant differentiation of each covector $X^\gamma(t) = X_k^\gamma(t)\, dx^k\big|_{C(t)}, \gamma = 1, 2, ..., n$ of the coframe u along C vanishes, where $\pi : F^*M \to M$ is the natural projection.*

6. Complete lifts of linear connections

Let ∇ is the torsionless linear connection on the differentiable manifold M, i.e. $\Gamma_{ij}^k = \Gamma_{ji}^k$. We determine the linear connection $\hat{\nabla}$ on the coframe bundle F^*M by following manner:

$$\hat{\Gamma}_{IJ}^K = {}^H\bar{\Gamma}_{IJ}^K - L_{IJ}^K, \tag{6.1}$$

where L_{IJ}^K—the (1,2)-tensor field on F^*M with unique non zero components

$$L_{ij}^{k\gamma} = X_m^\gamma R_{jki}^m \tag{6.2}$$

in the induced natural frame $\left\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial X_j^\beta}\right\}$.

By using (3.7), (6.1) and (6.2), we obtain the non zero components of the linear connection $\hat{\nabla}$ in the induced natural frame $\left\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial X_j^\beta}\right\}$:

$$\begin{aligned}
&\hat{\Gamma}_{ij}^k = \Gamma_{ij}^k, \hat{\Gamma}_{i_\alpha j}^{k\gamma} = -\Gamma_{jk}^i \delta_\alpha^\gamma, \hat{\Gamma}_{ij_\beta}^{k\gamma} = -\Gamma_{ik}^j \delta_\beta^\gamma, \\
&\hat{\Gamma}_{ij}^{k\gamma} = X_m^\gamma(\Gamma_{kl}^m \Gamma_{ij}^l + \Gamma_{ik}^l \Gamma_{lj}^m - \partial_i \Gamma_{kj}^m) - X_m^\gamma(\partial_j \Gamma_{ki}^m - \partial_k \Gamma_{ji}^m + \Gamma_{ki}^l \Gamma_{jl}^m - \Gamma_{ji}^l \Gamma_{kl}^m) \\
&\quad = X_m^\gamma(\partial_k \Gamma_{ji}^m - \partial_j \Gamma_{ki}^m - \partial_i \Gamma_{kj}^m + 2\Gamma_{ij}^l \Gamma_{kl}^m).
\end{aligned} \tag{6.3}$$

We have

Theorem 6.1. *Covariant differentiation with respect to the linear connection $\hat{\nabla}$ has the following property:*

$$\hat{\nabla}_{C_X}{}^C Y = {}^C(\nabla_X Y) + \gamma(\tilde{L})$$

for $X, Y \in \Im_0^1(M)$, where $\gamma(\tilde{L})-$the vertical vector field on the coframe bundle F^*M such that

$$\gamma(\tilde{L}) = \begin{pmatrix} 0 \\ X_l^{\gamma}\left[(\nabla_k X^i \nabla_i Y^l + \nabla_k Y^i \nabla_i X^l) + X^i Y^m(R_{kim}^l + R_{kmi}^l)\right] \end{pmatrix}$$

and R is the curvature tensor of ∇.

Proof. Consider the vector $X, Y \in \Im_0^1(M)$. The complete lift ${}^C X$ of X from the manifold M to the coframe bundle F^*M is given by [4]

$$^C X^i = X^i, \, ^C X^{i_\alpha} = -X_l^\alpha \partial_i X^l \tag{6.4}$$

with respect to the induced natural frame $\left\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial X_j^\beta}\right\}$.

1) If $K = k$, then by using (6.3) and (6.4), we obtain:

$$\left(\hat{\nabla}_{C_X}{}^C Y\right)^k = {}^C X^I(\partial_I Y^k + \hat{\Gamma}_{IJ}^k {}^C Y^J)$$

$$= X^i(\partial_i Y^k + \Gamma_{ij}^k Y^j) = (\nabla_X Y)^k = {}^C(\nabla_X Y)^k;$$

2) In the case $K = k_\gamma$ by the same way we have

$$\left(\hat{\nabla}_{C_X}{}^C Y\right)^{k_\gamma} = {}^C X^I(\partial_I Y^{k_\gamma} + \hat{\Gamma}_{IJ}^{k_\gamma} {}^C Y^J) = {}^C X^i \partial_i {}^C Y^{k_\gamma}$$

$$+ {}^C X^{i_\alpha} \partial_{i_\alpha} {}^C Y^{k_\gamma} + {}^C X^i \hat{\Gamma}_{ij}^{k_\gamma} {}^C Y^j + {}^C X^{i_\alpha} \hat{\Gamma}_{i_\alpha j}^{k_\gamma} {}^C Y^j + {}^C X^i \hat{\Gamma}_{ij_\beta}^{k_\gamma} {}^C Y^{j_\beta}$$

$$= {}^C X^i \partial_i(-X_m^\gamma \partial_k Y^m) + (-X_l^\alpha \partial_i X^l)\partial_{i_\alpha}(-X_m^\gamma \partial_k Y^m)$$

$$+ X^i X_m^\gamma(\partial_k \Gamma_{ji}^m - \partial_j \Gamma_{ki}^m - \partial_i \Gamma_{kj}^m + 2\Gamma_{ij}^l \Gamma_{kl}^m)Y^j$$

$$+ (-X_l^\alpha \partial_i X^l)(-\Gamma_{jk}^i \delta_\alpha^\gamma)Y^j + X^i(-\Gamma_{ik}^j \delta_\beta^\gamma)(-X_l^\beta \partial_k Y^l)$$

$$= -X^i X_m^\gamma \partial_i \partial_k Y^m + X_l^\alpha \partial_i X^l \delta_m^i \delta_\alpha^\gamma \partial_k Y^m + X^i X_m^\gamma \partial_k \Gamma_{ji}^m Y^j$$

$$- X^i X_m^\gamma \partial_j \Gamma_{ki}^m Y^j - X^i X_m^\gamma \partial_i \Gamma_{kj}^m Y^j + 2X^i X_m^\gamma \Gamma_{ij}^l \Gamma_{kl}^m Y^j$$

$$+ X_l^\gamma \partial_i X^l \Gamma_{jk}^i Y^j + X^i \Gamma_{ik}^j X_l^\gamma \partial_j Y^l = -X^i X_m^\gamma \partial_i \partial_k Y^m$$

$$+ X_l^\gamma \partial_m X^l \partial_k Y^m + X^i X_m^\gamma \partial_k \Gamma_{ji}^m Y^j - X^i X_m^\gamma \partial_j \Gamma_{ki}^m Y^j$$

$$- X^i X_m^\gamma \partial_i \Gamma_{kj}^m Y^j + 2X^i X_m^\gamma \Gamma_{ij}^l \Gamma_{kl}^m Y^j + X_l^\gamma \partial_i X^l \Gamma_{jk}^i Y^j$$

$$+ X^i \Gamma_{ik}^j X_l^\gamma \partial_j Y^l = -X_m^\gamma(\partial_k X^i \partial_i Y^m + X^i \partial_k \partial_i Y^m + \partial_k X^i \Gamma_{il}^m Y^l$$

$$+ X^i \partial_k \Gamma_{il}^m Y^l + X^i \Gamma_{il}^m \partial_k Y^l) + X_l^\gamma \partial_k X^i \partial_i Y^l + X_l^\gamma \partial_k X^i \Gamma_{im}^l Y^m$$

$$+ X_l^\gamma X^i \partial_k \Gamma_{im}^l Y^m + X_l^\gamma X^i \Gamma_{im}^l \partial_k Y^m + X_l^\gamma \partial_m X^l \partial_k Y^m$$

$$+ X^i X_m^\gamma \partial_k \Gamma_{ji}^m Y^j - X^i X_m^\gamma \partial_j \Gamma_{ki}^m Y^j - X^i X_m^\gamma \partial_i \Gamma_{kj}^m Y^j$$

$$+ 2X^i X_m^\gamma \Gamma_{ij}^l \Gamma_{kl}^m Y^j + X_l^\gamma \partial_i X^l \Gamma_{jk}^i Y^j + X^i \Gamma_{ik}^j X_l^\gamma \partial_j Y^l = {}^C(\nabla_X Y)^{k_\gamma}$$

$$+ X_l^\gamma(\partial_k X^i \partial_i Y^l + \partial_k X^i \Gamma_{im}^l Y^m + X^i \Gamma_{ik}^j \partial_j Y^l + \Gamma_{ki}^m X^i \Gamma_{mj}^l Y^j)$$

$$+ X_l^\gamma(\partial_m X^l \partial_k Y^m + X^i \Gamma_{im}^l \partial_k Y^m + \partial_i X^l \Gamma_{jk}^i Y^j + Y^j \Gamma_{kj}^m \Gamma_{mi}^l X^i)$$

$$+ X_l^\gamma(-Y^j \Gamma_{kj}^m \Gamma_{mi}^l X^i - \Gamma_{ki}^m X^i \Gamma_{mj}^l Y^j + 2X^i \Gamma_{ij}^m \Gamma_{km}^l Y^j$$

$$+ X^i \partial_k \Gamma_{im}^l Y^m + X^i \partial_k \Gamma_{ji}^l Y^j - X^i \partial_j \Gamma_{ki}^l Y^j - X^i \partial_i \Gamma_{kj}^l Y^j)$$

$$= {}^C(\nabla_X Y)^{k_\gamma} + X_l^\gamma(\nabla_k X^i \nabla_i Y^l + \nabla_k Y^i \nabla_i X^l)$$

$$+ X_l^\gamma X^i Y^m\left[(\partial_k \Gamma_{im}^l - \partial_i \Gamma_{km}^l + \Gamma_{im}^r \Gamma_{kr}^l - \Gamma_{km}^r \Gamma_{ri}^l)\right.$$

$$+ (\partial_k \Gamma_{mi}^l - \partial_m \Gamma_{ki}^l + \Gamma_{mi}^r \Gamma_{kr}^l - \Gamma_{ki}^r \Gamma_{rm}^l)\right] = {}^C(\nabla_X Y)^{k_\gamma}$$

$$+ X_l^\gamma\left[(\nabla_k X^i \nabla_i Y^l + \nabla_k Y^i \nabla_i X^l) + X^i Y^m(R_{kim}^l + R_{kmi}^l)\right].$$

Thus, we have shown that

$$\hat{\nabla}_{{}^C X}{}^C Y = {}^C(\nabla_X Y) + \gamma(\tilde{L}), \tag{6.5}$$

where $\gamma(\tilde{L})$ is the vertical vector field in the form

$$\gamma(\tilde{L}) = \begin{pmatrix} 0 \\ X_l^\gamma \left[(\nabla_k X^{\,i}\, \nabla_i Y^l + \nabla_k Y^{\,i} \nabla_i X^l) + X^i Y^m (R_{kim}^l + R_{kmi}^l) \right] \end{pmatrix}$$

defined on the coframe bundle $F^* M$. Therefore, the proof is complete. $\square$

The complete lift of the symmetric linear connection in the cotangent bundle ${}^C T(M)$ satisfies a relation analogous to (6.5) (see, [18, p.269]). Therefore, the connection $\hat{\nabla}$ defined by the formula (6.1) and satisfying the relation (6.5) is called the complete lift of the symmetric affine connection ∇ to the coframe bundle $F^* M$ and denoted by ${}^C \nabla$.

Remark 6.1. The complete lift of the symmetric linear connection in the cotangent bundle is defined as the Riemannian connection of the Riemannian extension of the Riemannian metric in the cotangent bundle of a Riemannian manifold (see, [6], [18, p.268]). On the other hand, in article [13], considering the coframe bundle $F^* M$ of a Riemannian manifold M, a so-called g-lift of the Riemannian metric g in the coframe bundle $F^* M$ is found that is analogous to the Riemannian extension in the cotangent bundle ${}^C T(M)$:

$$\tilde{g} = \begin{pmatrix} \sum_{\beta=1}^n X_m^\beta (-2\bar{\Gamma}_{kl}^m) & \delta_k^l \sum_{\beta=1}^n \delta_\sigma^\beta \\ \delta_l^k \sum_{\beta=1}^n \delta_\gamma^\beta & 0 \end{pmatrix}. \tag{6.6}$$

In view of the fact that the $g-$ lift (6.6) is degenerate, it is impossible to determine the complete lift ${}^C \nabla$ of symmetric connection ∇ given on Riemannian manifold M in the coframe bundle $F^* M$ as Riemannian connection of the $g-$lift $\tilde{g}$.

7. Geodesics of the complete lifts

Geodesics of complete lifts of connections in tangent, cotangent and tensor bundles has been investigated in [15], [6] and [18, p.273], respectively. In the present section we study geodesics of the complete lifts of connections in the coframe bundle.

Let $\tilde{C}$ be a geodesic curve on the coframe bundle $F^* M$ with respect to the complete lift ${}^C \nabla$ of the symmetric linear connection ∇ given on the M. In induced coordinates $(\pi^{-1}(U), x^i, X_i^\alpha)$ the equation of the geodesic curve $\tilde{C} : I \to F^* M$, $\tilde{C} : t \to \tilde{C}(t) = (x^i(t), X_i^\alpha(t)) = (x^I(t))$ are of the form:

$$\frac{d^2 x^K}{dt^2} + {}^C \Gamma_{IJ}^K \frac{dx^I}{dt} \frac{dx^J}{dt} = 0, \quad I, J.K = 1, 2, ..., n + n^2. \tag{7.1}$$

Using the formulas (6.3) for ${}^C \Gamma_{IJ}^K$, from (7.1) we get:

$$\frac{d^2 x^k}{dt^2} + \Gamma_{ij}^k \frac{dx^i}{dt} \frac{dx^j}{dt} = 0,$$
$$\frac{d^2 X_k^\gamma}{dt^2} + (\partial_k \Gamma_{ji}^m - \partial_j \Gamma_{ki}^m - \partial_i \Gamma_{kj}^m + 2\Gamma_{ij}^l \Gamma_{kl}^m) X_m^\gamma \frac{dx^i}{dt} \frac{dx^j}{dt} - 2\Gamma_{kj}^i \frac{dX_i^\gamma}{dt} \frac{dx^j}{dt} = 0. \tag{7.2}$$

Taking into account expression (5.4), the second of the equations (7.2) is written in the form:

$$\frac{\delta}{dt} \left(\frac{\delta X_k^\gamma}{dt} \right) + R_{kji}^m \frac{dx^i}{dt} \frac{dx^j}{dt} = 0. \tag{7.3}$$

From here follows

Theorem 7.1. *Let ∇ be a torsion-free linear connection on a differentiable manifold M, and let $\tilde{C}(t) = (C(t),\ X_k^\gamma(t))$ be a curve on the coframe bundle $F^* M$. In order for the curve $\tilde{C}(t)$ to be a geodesic line for the connection ${}^C \nabla$, it is necessary and sufficient that the following conditions be satisfied:*

1. *The curve $C(t)$ is a geodesic line for the linear connection ∇;*

2. *The each covector field $X_k^\gamma(t)$ satisfies the relation (7.3) along the curve $C(t)$.*

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