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Chen Inequalities for Submanifolds of Real Space Forms with a Ricci Quarter-Symmetric Metric Connection

Nergiz (Önen) Poyraz* Halil İbrahim Yoldaş

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ABSTRACT

In this paper, we establish some inequalities for submanifolds of real space forms endowed with a Ricci quarter-symmetric metric connection. Using these inequalities, we obtain the relation between Ricci curvature, scalar curvature and the mean curvature endowed with the Ricci quarter-symmetric metric connection.

Keywords: *Chen inequality; Ricci quarter-symmetric metric connection; Ricci curvature.*
AMS Subject Classification (2010): *53B05 ; 53B15; 53C40.*

1. Introduction

In [8], the idea of a Ricci quarter-symmetric metric connection on a Riemannian manifold was introduced and presented by Kamilya and De. They also found necessary and sufficient conditions for the symmetry of the Ricci tensor of a Ricci quarter-symmetric metric connection and showed that conformal curvature tensor of induced connection ∇ and linear connection $\widetilde{\nabla}$ are equal [8]. Before this work, a few papers had been written about the studies of various types of a quarter-symmetric metric connection and their properties in [11] and [12].

In 1993, Chen [4] introduced a new Riemannian invariant for a Riemannian manifold M as follows:

$$\delta_M = \tau(p) - \inf(K)(p), \tag{1.1}$$

where $\tau(p)$ is scalar curvature of M and

$$\inf(K)(p) = \inf\{K(\Pi) : K(\Pi) \text{ is a plane section of } T_pM\}.$$

Chen gave the following general optimal inequality involving the new intrinsic invariant δ_M, the squared mean curvature $\|H\|^2$ for an $n-$dimensional submanifold M in a real space form $R(c)$ of constant sectional curvature c :

$$\delta_M \leq \frac{n^2(n-2)}{2(n-2)} \|H\|^2 + \frac{1}{2}(n+1)(n-2)c. \tag{1.2}$$

[3].

Also, Chen established a sharp inequality between the main intrinsic curvatures (the sectional curvature and the scalar curvature) and the main extrinsic curvatures (the squared mean curvature) for a submanifold in real space form $R^m(\bar{c})$, well-known as *Chen inequalities*, in [2] as follows:

For each unit tangent vector $X \in T_pM^n$,

$$H^2(p) \geq \frac{4}{n^2}\{Ric(X) - (n-1)\bar{c}\}, \tag{1.3}$$

where H^2 is the squared mean curvature and $Ric(X)$ is Ricci curvature of M^n at X.

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*** Corresponding author***

In [7], Hong and Tripathi presented a general inequality for submanifolds of a Riemannian manifold by using (1.3). In [13], this inequality was named Chen-Ricci inequality by Tripathi. In fact, the general inequality obtained in [7] is a special case of Theorem 3.1 of [5]. Later, Mihai and Özgür in [10] proved Chen inequalities for submanifolds of real space forms endowed with a semi-symmetric metric connection. Moreover, several works in this direction is studied [1, 6, 9, 13].

The paper is organized as follows: Section 1 is concerned with introduction. In section 2, we give some basic concepts on submanifolds of Riemannian manifold endowed with Ricci quarter-symmetric metric conection which will be used throughout this paper. In section 3, we find some inequalities for submanifolds of real space forms endowed with a Ricci quarter-symmetric metric connection. Considering these inequalities, we obtain the relation between Ricci curvature, scalar curvature and the mean curvature endowed with the Ricci quarter-symmetric metric connection.

2. Preliminaries

Let $\widetilde{M}$ be an m-dimensional Riemannian manifold and $\widetilde{\nabla}$ a linear connection on $\widetilde{M}$. A linear connection $\widetilde{\nabla}$ is said to be Ricci *quarter-symmetric connection* if the torsion tensor $\widetilde{T}$ is of the form

$$\widetilde{T}(X,Y) = \pi(Y)LX - \pi(X)LY, \tag{2.1}$$

where $\widetilde{\pi}$ is a 1-form and L is the $(1,1)$ Ricci tensor defined by

$$\widetilde{g}(LX,Y) = S(X,Y) \tag{2.2}$$

S is the Ricci tensor of $\widetilde{M}$.

A linear connection $\widetilde{\nabla}$ is called a metric connection if

$$(\widetilde{\nabla}_X \widetilde{g})(Y,Z) = 0. \tag{2.3}$$

Following [8], a Ricci quarter-symmetric metric connection $\widetilde{\nabla}$ on $\widetilde{M}$ is given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} + \pi(\widetilde{Y})L\widetilde{X} - S(\widetilde{X},\widetilde{Y})P \tag{2.4}$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ of $\widetilde{M}$, where $\overset{\circ}{\widetilde{\nabla}}$ denotes the Levi-Civita connection with respect to the Riemannian metric $\widetilde{g}$, π is a 1−form and P is the vector field defined by

$$\widetilde{g}(P,\widetilde{X}) = \pi(\widetilde{X})$$

for an arbitrary vector field $\widetilde{X}$ of $\widetilde{M}$.

From now on, we will consider a Riemannian manifold $\widetilde{M}$ endowed with a Ricci quarter-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection denoted by $\overset{\circ}{\widetilde{\nabla}}$.

Let M^n be an n−dimensional submanifold of an m−dimensional Riemannian manifold $\widetilde{M}$. On the submanifold M^n we consider the induced Ricci quarter-symmetric metric connection denoted by ∇ and the induced Levi-Civita connection denoted by $\overset{\circ}{\nabla}$.

Let $\widetilde{R}$ be the curvature tensor of $\widetilde{M}$ with respect to $\widetilde{\nabla}$ and $\overset{\circ}{\widetilde{R}}$ the curvature tensor of $\widetilde{M}$ with respect to $\overset{\circ}{\widetilde{\nabla}}$. We also denote by R and $\overset{\circ}{R}$ the curvature tensors of ∇ and $\overset{\circ}{\nabla}$, respectively, on M.

The *Gauss formulas* with respect to ∇ and $\overset{\circ}{\nabla}$, respectively, can be written as:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.5}$$

$$\overset{\circ}{\widetilde{\nabla}}_X Y = \overset{\circ}{\nabla}_X Y + \overset{\circ}{h}(X,Y), \tag{2.6}$$

where $\overset{\circ}{h}$ is the second fundemental form of M in $\widetilde{M}$ and h is a $(0,2)$-tensor on M.

For any orthonormal basis $\{e_1, ..., e_n\}$ of the tangent space T_pM^n, the mean curvature vector $H(p)$ is given by

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

If $h = 0$(respectively $H = 0$), then the submanifold M^n is called totally geodesic (minimal) in $\widetilde{M}$. If $h(X, Y) = g(X, Y)H$ for all $X, Y \in TM$, then M^n is said to be totally umbilical.

The Gauss equation with respect to the Ricci quarter-symmetric metric connection is

$$R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)). \tag{2.7}$$

The curvature tensor $\overset{\circ}{\tilde{R}}$ with respect to the Levi-Civita connection $\overset{\circ}{\tilde{\nabla}}$ on $\widetilde{M}\ (c)$ is expressed by

$$\overset{\circ}{\tilde{R}}(X, Y, Z, W) = c\{g(X, W)g(Y, Z) - g(Y, W)g(X, Z)\}. \tag{2.8}$$

Let $\Pi = Span\{ei_i, e_j\}$ be $2-$dimensional non-degenerate plane of the tangent space TpM at $p \in M$. Then the number

$$K_{ij} = \frac{g(R(e_j, e_i)e_i, e_j)}{g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)^2} \tag{2.9}$$

is called the sectional curvature of the section Π at $p \in M$.

Let M^n be an n-dimensional Riemanian manifold. We denote by $K(\pi)$ the sectional curvature of M^n associated with a plane section $\pi \subset TpM^n$, $p \in M^n$. If $\{e_1, ..., e_n\}$ is an orthonormal basis of the tangent space TpM^n, then the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{1 \le i < j \le n} K_{ij}.$$

Let M^n be an n-dimensional Riemannian manifold, L be a k-plane section of TpM^n, $p \in M^n$, and X be a unit vector in L. We choose an orthonormal basis $\{e_1, ..., e_k\}$ of L such that $e_1 = X$.

One defines [2] the Ricci curvature (or k-Ricci curvature) of L at X by

$$Ric_L(X) = K_{12} + K_{13} + ... + K_{1k},$$

where K_{ij} denotes, as usual, the sectional curvature of the 2-plane section spanned by e_i, e_j. For each integer k, $2 \le k \le n$, the Riemannian invariant θ_k on M^n is defined by:

$$\theta_k(p) = \frac{1}{k-1}\inf_{L,X} Ric_L(X), \;\; p \in M, \tag{2.10}$$

where L runs over all k-plane sections in TpM^n and X runs over all unit vectors in L.

Then the curvature tensor $\widetilde{R}$ with respect to the Ricci quarter-symmetric metric connection $\widetilde{\nabla}$ on $\widetilde{M}$ can be shown that [8]

$$\begin{aligned}\widetilde{R}(X, Y)Z &= \overset{\circ}{\tilde{R}}(X, Y)Z - M(Y, Z)LX + M(X, Z)LY \\ &\quad -S(Y, Z)QX + S(X, Z)QY + \pi(Z)[(\overset{\circ}{\tilde{\nabla}}_X L)Y - (\overset{\circ}{\tilde{\nabla}}_Y L)X] \\ &\quad -[(\overset{\circ}{\tilde{\nabla}}_X S)(Y, Z) - (\overset{\circ}{\tilde{\nabla}}_Y S)(X, Z)]P,\end{aligned} \tag{2.11}$$

where M is tensor field of type $(0, 2)$ defined by

$$M(X, Y) = g(QX, Y) = (\overset{\circ}{\tilde{\nabla}}_X \pi)Y - \pi(Y)\pi(LX) + \frac{1}{2}\pi(P)S(X, Y) \tag{2.12}$$

and Q is a tensor field of type $(2, 1)$ defined by

$$QX = \overset{\circ}{\tilde{\nabla}}_X P - \pi(LX)P + \frac{1}{2}\pi(P)LX. \tag{2.13}$$

Here we shall consider M^n to be an Einstein manifold, that is,

$$S(X,Y) = \frac{\overset{\circ}{\widetilde{\tau}}}{n} g(X,Y), \tag{2.14}$$

where $\overset{\circ}{\widetilde{\tau}}$ is the scalar curvature.

Throughout this paper, we assume that M^n is an Einstein manifold.

Considering (2.11) and (2.14) we get

$$\widetilde{R}(X,Y)Z = \overset{\circ}{\widetilde{R}}(X,Y)Z - \frac{\overset{\circ}{\widetilde{\tau}}}{n}[M(Y,Z)X - M(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{2.15}$$

Contracting (2.15) with respect to X, we get

$$\widetilde{S}(Y,Z) = \frac{\overset{\circ}{\widetilde{\tau}}}{n}[g(Y,Z) - \{(n-2)M(Y,Z) + mg(Y,Z)\}], \tag{2.16}$$

where $\widetilde{S}$ is the Ricci tensor of $\widetilde{\nabla}$ and m is the trace of $M(Y,Z)$. Now putting $Y = Z = e_i$, where $\{e_1, ..., e_n\}$ is an orthonormal basis of the tangent space at any point, we get by taking the sum for $1 \le i \le n$ in the relation (2.16)

$$\widetilde{\tau} = \frac{\overset{\circ}{\widetilde{\tau}}}{n}[n - 2(n-1)m], \tag{2.17}$$

where $\widetilde{\tau}$ is the scalar curvature of $\widetilde{\nabla}$.

3. k-Ricci Curvature and k-Scalar Curvature

In this section, a sharp relation between the Ricci curvature in the direction of unit tangent vector X and the mean curvature H with respect to Ricci quarter-symmetric metric connection $\widetilde{\nabla}$ is established. Using this inequality, a relationship between the $k-$Ricci curvature of M^n and the squared mean curvature $\|H\|^2$ is showed. From now on, we assume that the vector field P is tangent to M^n.

Denote by

$$N(p) = \{X \in T_pM^n \mid h(X,Y) = 0, \forall Y \in T_pM^n\}.$$

Theorem 3.1. *Let M^n be an $n-$dimensional submanifold of an $m-$dimensional real space form $\widetilde{M}(c)$ of constant sectional curvature c endowed with Ricci quarter-symmetric metric connection $\widetilde{\nabla}$. Then, the following statements are true.*

(i) For each unit vector $X \in T_pM^n$ we have

$$Ric(X) \le \frac{1}{4}n^2 \|H\|^2 - (n-1)c\,[1 - m + (n-2)M(X,X)]\,, \tag{3.1}$$

where m is the trace of M.

(ii) The equality case of (3.1) is satisfied by unit vector $X \in T_pM^n$ if and only if

$$\begin{aligned} h(X,Y) &= 0, \text{ for all } Y \in TpM^n \text{ orthogonal to } X, \\ h(X,X) &= \frac{n}{2}H(p). \end{aligned} \tag{3.2}$$

(iii) The equality case of (3.1) holds for all unit vector $X \in T_pM^n$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.

Proof. From (2.7) and (2.15) we get

$$2\tau(p) = n(n-1)c - 2(n-1)^2cm + n^2\|H\|^2 - \|h\|^2\,, \tag{3.3}$$

where m is the trace of M and denote by

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

From (3.3), we get

$$\begin{aligned} \frac{1}{4}n^2 \|H\|^2 &= \tau(p) - \frac{n(n-1)c}{2} + (n-1)^2 cm \\ &+\frac{1}{4}\sum_{r=n+1}^{m}(h_{11}^r - h_{22}^r - ... - h_{nn}^r)^2 + \sum_{r=n+1}^{m}\sum_{j=2}^{n}(h_{1j}^r)^2 \\ &- \sum_{r=n+1}^{m}\sum_{2\leq i<j\leq n}^{m}(h_{ii}^r h_{jj}^r - (h_{ij}^r)^2), \end{aligned} \tag{3.4}$$

where

$$h_{ij}^r = g(h(e_i, e_j), e_r).$$

Using (2.7) and (2.15) we also have

$$\begin{aligned} \sum_{r=n+1}^{m}\sum_{2\leq i<j\leq n}^{m}(h_{ii}^r h_{jj}^r - (h_{ij}^r)^2) &= \sum_{2\leq i<j\leq n} K_{ij} - \sum_{2\leq i<j\leq n} \tilde{K}_{ij} \\ &= \sum_{2\leq i<j\leq n} K_{ij} - (n-1)(n-2)c\left(\frac{1}{2} - m + M(e_1, e_1)\right). \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$\begin{aligned} Ric(e_1) &= \frac{1}{4}n^2 \|H\|^2 - (n-1)c\,(1 - m + (n-2)M(e_1, e_1)) \\ &- \sum_{r=n+1}^{m}\sum_{j=2}^{n}(h_{1j}^r)^2 - \frac{1}{4}\sum_{r=n+1}^{m}(h_{11}^r - h_{22}^r - ... - h_{nn}^r)^2. \end{aligned} \tag{3.6}$$

If we choose $e_1 = X$ as any unit vector of T_pM^n in the above equation, one obtains (3.1).

Taking into consideration equation (3.6) and $X = e_1$, the equality case of (3.1) holds if and only if

$$h_{12}^r = h_{13}^r = ... = h_{1n}^r = 0 \text{ and } h_{11}^r = h_{22}^r + ... + h_{nn}^r,\ r \in \{n+1, ..., m\} \tag{3.7}$$

which shows that (3.2) holds.

We now suppose that the equality case of (3.1) holds for all unit vector $X \in T_pM^n$. Then , in view of (3.7), for each $r \in \{n+1, ..., m\}$ we have $i \in \{1, ..., n\}$,

$$h_{ij}^r = 0, \quad i \neq j \tag{3.8}$$

$$2h_{ii}^r = h_{11}^r + h_{22}^r + ... + h_{nn}^r, \quad i \in \{1, ..., n\}. \tag{3.9}$$

From (3.9), we have $2h_{11}^r = 2h_{22}^r = ... = 2h_{nn}^r = h_{11}^r + h_{22}^r + ... + h_{nn}^r$, which implies that

$$(n-2)(h_{11}^r + h_{22}^r + ... + h_{nn}^r) = 0. \tag{3.10}$$

Thus, either $h_{11}^r + h_{22}^r + ... + h_{nn}^r = 0$ or $n = 2$. If $h_{11}^r + h_{22}^r + ... + h_{nn}^r = 0$, then in view of (3.9), we get $h_{ii}^r = 0$ for all $i \in \{1, ..., n\}$. This together with (3.8) gives $h_{ij}^r = 0$ for all $i, j \in \{1, ..., n\}$ and $r \in \{n+1, ..., m\}$, that is, p is a totally geodesic point. If $n = 2$, then from (3.9) $2h_{11}^r = 2h_{22}^r = h_{11}^r + h_{22}^r$, which shows that p is a totally umbilical point. The proof of the converse part is straightforward. □

Corollary 3.1. *If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case of (3.1) if and only if $X \in N(p)$.*

Theorem 3.2. *Let M^n be an $n-$dimensional submanifold of an $m-$dimensional real space form $\widetilde{M}(c)$ of constant sectional curvature c endowed with Ricci quarter-symmetric metric connection $\widetilde{\nabla}$*

$$\tau(p) \leq \frac{(n-1)}{2}\left(n\|H\|^2 + nc - 2c(n-1)m\right). \tag{3.11}$$

Equality case of (3.11) holds at $p \in M^n$ if and only if p is a totally umbilical point.

Proof. Let $p \in M^n$ and $\{e_1, ..., e_n\}$ be orthonormal basis of T_pM^n. The relation (3.3) is equivalent to

$$n^2 \|H\|^2 = 2\tau(p) + \|h\|^2 + (n-1)c\,(2(n-1)m - n)\,. \tag{3.12}$$

We choose an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_m\}$ at p such that e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1, ..., e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} = \begin{bmatrix} a_1 & 0 & . & . & . & 0 \\ 0 & a_2 & . & . & . & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ 0 & 0 & . & . & . & a_n \end{bmatrix} \tag{3.13}$$

$$A_{e_r} = \left(h_{ij}^r\right), \quad i,j = 1, ..., n; \quad r = n+2, ..., m, \quad traceA_{e_r} = 0. \tag{3.14}$$

From (3.12), we get

$$n^2 \|H\|^2 = 2\tau(p) + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + (n-1)c\,(2(n-1)m - n)\,. \tag{3.15}$$

On the other hand, since

$$0 \leq \sum_{i<j} (a_i - a_j)^2 = (n-1)\sum_i a_i^2 - 2\sum_{i<j} a_i a_j \tag{3.16}$$

we obtain

$$n^2 \|H\|^2 = \left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \leq n\sum_{i=1}^{n} a_i^2 \tag{3.17}$$

which implies

$$\sum_{i=1}^{n} a_i^2 \geq n\|H\|^2\,. \tag{3.18}$$

So from (3.15) and (3.18), we have

$$n^2 \|H\|^2 \geq 2\tau(p) + n\|H\|^2 + \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + (n-1)c\,(2(n-1)m - n)\,. \tag{3.19}$$

If the equality case of (3.11) holds, then from (3.16) and (3.19) it follows that

$$a_1 = a_2 = ... = a_n \text{ and } A_{e_r} = 0, \; r = n+2, ..., m. \tag{3.20}$$

Therefore, p is a totally umbilical point. The converse is straightforward. □

Theorem 3.3. *Let M^n be an $n-$dimensional submanifold of an $m-$dimensional real space form $\widetilde{M}(c)$ of constant sectional curvature c endowed with Ricci quarter-symmetric metric connection $\widetilde{\nabla}$. Then we have*

$$\theta_k(p) \leq \|H\|^2 + c\left(2 - \frac{4(n-1)m}{n}\right). \tag{3.21}$$

Proof. Let $\{e_1, ..., e_n\}$ be an orthonormal basis of T_pM^n. Denote by $L_{i_1...i_k}$ the k-plane section spanned by $\{e_{i_1}, ..., e_{i_k}\}$. Using the definitions of the ricci and scalar curvatures, we have

$$\tau\left(L_{i_1...i_k}\right) = \frac{1}{2} \sum_{i \in \{i_1,...,i_k\}} Ric_{L_{i_1...i_k}}(e_i), \tag{3.22}$$

$$\tau(p) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \leq i_1 < ... \leq i_k \leq n} \tau\left(L_{i_1...i_k}\right). \tag{3.23}$$

From (2.10), (3.22) and (3.23), we get

$$\tau(p) \geq \frac{n(n-1)}{2}\theta_k(p). \tag{3.24}$$

Using (3.11) and (3.24) we obtain (3.21). □

Lemma 3.1. *If $n > k \geq 2$ and $a_1, ..., a_n, a$ are real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-k+1)\left(\sum_{i=1}^{n} a_i^2 + a\right) \tag{3.25}$$

then

$$2\sum_{1\leq i<j\leq k} a_i a_j \geq a \tag{3.26}$$

with equality holding if and only if

$$a_1 + a_2 + ... + a_k = a_{k+1} = ... = a_n. \tag{3.27}$$

Theorem 3.4. *Let M^n be an $n-$dimensional submanifold of an $m-$dimensional real space form $\widetilde{M}(c)$ of constant sectional curvature c endowed with Ricci quarter-symmetric metric connection $\widetilde{\nabla}$. Then, for each point $p \in M^n$ and each k-plane section $\Pi_k \subset TpM^n$ $(n > k \geq 2)$, we have*

$$\begin{aligned}\tau(p) - \tau(\pi_k) \quad &\leq \quad (n-k)\left[\frac{n^2}{2(n-k+1)}\|H\|^2 - \frac{(n+k-1)}{2}c - (n-1)cm\right] \\ &\quad -(n-1)(k-1)c\, trace(m_{|_{\pi_k^\perp}}).\end{aligned} \tag{3.28}$$

The equality case of (3.28) *holds at $p \in M^n$ if and only if there exist an orthonormal basis $\{e_1, ..., e_n\}$ of TpM^n and an orthonormal basis $\{e_{n+1}, ..., e_m\}$ of $T_p^\perp M^n$ such that (a) Π_k=Span$\{e_1, ..., e_k\}$ and (b) the forms of shape operators A_{e_r}, $r = n+1, ..., m$, take the forms*

$$A_{e_{n+1}} = \begin{bmatrix} h_{11}^{n+1} & 0 & . & . & . & 0 & \\ 0 & h_{22}^{n+1} & . & . & . & 0 & \\ . & . & . & & & & 0 \\ . & . & & . & & & \\ . & . & & & . & & \\ 0 & 0 & . & . & . & h_{kk}^{n+1} & \\ & & 0 & & & & \left(\sum_{i=1}^{k} h_{ii}^{n+1}\right) I_{n-k} \end{bmatrix}, \tag{3.29}$$

$$A_{e_r} = \begin{bmatrix} h_{11}^r & h_{12}^r & . & . & . & h_{1k}^r & \\ h_{12}^r & h_{22}^r & . & . & . & h_{2k}^r & \\ . & . & . & & & & 0 \\ . & . & & . & & & \\ . & . & & & . & & \\ h_{1k}^r & h_{2k}^r & . & . & . & -\sum_{i=1}^{k-1} h_{ii}^r & \\ & & 0 & & & & 0_{n-k} \end{bmatrix}, \quad r \in \{n+2, ..., m\}. \tag{3.30}$$

Proof. Let $\Pi_k \subset TpM^n$ be a $k-$plane section. We choose an orthonormal basis $\{e_1, ..., e_n\}$ for TpM^n and $\{e_{n+1}, ..., e_m\}$ for the normal space $T_p^\perp M^n$ at p such that Π_k=Span$\{e_1, ..., e_k\}$, the mean curvature vector H is in the direction of the normal vector to e_{n+1} and $e_1, ..., e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms (3.13) and (3.14). We rewrite (3.3) as

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-k+1)\left(\sum_{i=1}^{n}\left(h_{ii}^{n+1}\right)^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + \epsilon\right), \tag{3.31}$$

where

$$\epsilon = 2\tau(p) - n(n-1)c + 2(n-1)^2cm - \frac{n^2(n-k)}{(n-k+1)}\|H\|^2. \tag{3.32}$$

Applying Lemma 3.1 in (3.31), we get

$$2\sum_{1\leq i<j\leq k} h_{ii}^{n+1}h_{jj}^{n+1} \geq \epsilon + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^r)^2. \tag{3.33}$$

From equation (2.7) and (2.15) it also follows that

$$\tau(\pi_k) = \frac{k(k-1)c}{2} - (n-1)(k-1)c\sum_{i=1}^{k} M(e_i,e_i) + \sum_{1\le i<j\le k} h_{ii}^{n+1}h_{jj}^{n+1} + \sum_{r=n+2}^{m}\sum_{1\le i<j\le k}^{m}\left(h_{ii}^{r}h_{jj}^{r} - (h_{ij}^{r})^2\right). \tag{3.34}$$

Using (3.33) and (3.34) we get

$$\begin{aligned}\tau(\pi_k) &\ge \frac{k(k-1)c}{2} - (n-1)(k-1)c\sum_{i=1}^{k} M(e_i,e_i) + \frac{1}{2}\epsilon \\ &+\frac{1}{2}\sum_{r=n+2}^{m}(h_{11}^{r}+h_{22}^{r}+...+h_{kk}^{r})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j>k}^{n}(h_{ij}^{r})^2 \\ &+\sum_{r=n+2}^{m}\sum_{j>k}^{n}\left((h_{1j}^{r})^2+(h_{2j}^{r})^2+...+(h_{kj}^{r})^2\right)\end{aligned} \tag{3.35}$$

or

$$\tau(\pi_k) \ge \frac{k(k-1)c}{2} - (n-1)(k-1)c\sum_{i=1}^{k} M(e_i,e_i) + \frac{1}{2}\epsilon. \tag{3.36}$$

We remark that

$$M(e_1,e_1)+M(e_2,e_2)+...+M(e_k,e_k) = m - trace(m_{|_{\pi_k^{\perp}}}). \tag{3.37}$$

From (3.32), (3.36) and (3.37) we obtain

$$\begin{aligned}\tau(\pi_k) &\ge (n-k)\left(\frac{(n+k-1)}{2}c + (n-1)cm\right) + \tau(p) \\ &-\frac{n^2(n-k)}{2(n-k+1)}\|H\|^2 + (n-1)(k-1)ctrace(m_{|_{\pi_k^{\perp}}})\end{aligned} \tag{3.38}$$

which proves the inequality case of (3.28). □

If the equality case of (3.28) holds, then the inequalities given by (3.33) and (3.36) become equalities. In this case, for $r = n+2, ..., m$ we have

$$h_{1j}^{n+1} = h_{2j}^{n+1} = ... = h_{kj}^{n+1} = 0, \quad j = k+1, ..., n, \tag{3.39}$$

$$h_{ij}^{r} = 0, \quad i,j = k+1, ..., n, \tag{3.40}$$

$$h_{11}^{r} + h_{22}^{r} + ... + h_{kk}^{r} = 0. \tag{3.41}$$

Applying Lemma 3.1 we also have

$$h_{11}^{n+1} + h_{22}^{n+1} + ... + h_{kk}^{n+1} = h_{ll}^{n+1}, \quad l = k+1, ..., n. \tag{3.42}$$

Thus, after choosing a suitable orthonormal basis $\{e_1, ..., e_m\}$, the shape operator of M^n takes the form given by (3.29) and (3.30). The converse is easy to follow.

By Theorem 3.4 we get the following corollary.

Corollary 3.2. *Let M^n $n \ge 3$, be an n-dimensional submanifold of an $m-$dimensional real space form $\widetilde{M}(c)$ of constant sectional curvature c endowed with Ricci quarter-symmetric metric connection $\widetilde{\nabla}$. Then, for each point $p \in M^n$ and each 2-plane section $\Pi_2 \subset TpM^n$, we have*

$$\begin{aligned}\delta_M &\le (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 - \frac{(n+1)}{2}c - (n-1)cm\right] \\ &-(n-1)c\, trace(m_{|_{\pi^{\perp}}}).\end{aligned} \tag{3.43}$$

The equality case of (3.43) *holds at* $p \in M^n$ *if and only if there exist an orthonormal basis* $\{e_1, ..., e_2\}$ *of* TpM^n *and an orthonormal basis* $\{e_{n+1}, ..., e_m\}$ *of* $T_p^{\perp} M^n$ *such that* (a) Π_2=*Span*$\{e_1, e_2\}$ *and* (b) *the forms of shape operators* A_{e_r}*,* $r = n+1, ..., m$,*become*

$$A_{e_{n+1}} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (a+b)I_{n-2} \end{bmatrix}, A_{e_r} = \begin{bmatrix} c_r & d_r & 0 \\ d_r & -c_r & 0 \\ 0 & 0 & 0_{n-2} \end{bmatrix}, \ r = n+2, ..., m. \tag{3.44}$$

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Affiliations

Nergiz (Önen) Poyraz
Address: Cukurova University, Department of Mathematics,
01330, Adana-Turkey.
e-mail: nonen@cu.edu.tr
ORCID ID: orcid.org/0000-0002-8110-712X

Halil İbrahim Yoldaş
Address: Mersin University, Departmet of Mathematics,
33343, Mersin-Turkey.
e-mail: hibrahimyoldas@mersin.edu.tr
ORCID ID: orcid.org/0000-0002-3238-6484