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TEXTURE SPACES WITH IDEAL

MEMET KULE AND ŞENOL DOST

Abstract. In this paper, the authors define the notion of ideal on texture spaces. The concept of di-local function is also introduced here by utilizing the families of neighborhood structure for a ditopological texture space. These concepts are discussed with a view to finding new ditopological texture spaces from the original one. Finally, we introduce and give some properties of weakly bicontinuous difunction, a subclass of bicontinuous difunction.

1. Introduction

In general topological spaces, by introducing the notion of ideals was carried out in the classical text by Kuratowski [16] and also in [21]. There has been the generalization of some important properties in general topology via topological ideals in the paper of Jankovic and Hamlett [14], [11], Rancin [19] and Samuels [20]. The properties like decomposition of continuity [2], compactness [13], separation axioms [12] and connectedness [10] have been generalized using the concept of topological ideals.

The fundamental concept of a texture space was introduced by Brown and the primary motivation ditopological texture spaces are to offer a new extension of classical fuzzy sets. Since then various aspects of general topology were investigated and carried out in ditopological texture space sense by several authors of this field. In recent papers on textures show that they are also a useful model for rough set theory [8], [9] and semi-separation axioms in soft fuzzy topological spaces [15].

Our aim in this paper is the topic of the ditopological texture spaces with ideal which is discussed from the textural point of view. Also, in the literature there are many "weakened" forms of continuity, we characterized the weakened form of continuity on the ditopological texture space with ideal.

The paper is structured as follows: Section 2 introduces the notion of textures which will be used along this work. In section 3, we define di-ideal on a texture and

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introduce the notion of di-local function. We have deduced some characterization theorems for the ditopological texture space with ideal. In section 4, a class of difunction called weakly ideal bicontinuous are introduced and studied.

2. Preliminaries

In this section, we give some basic definitions and results of the theory of ditopological texture spaces which is needed in the sequel [3–7].

Texture Space: Let S be a set. Then $\mathcal{S} \subseteq \mathcal{P}(S)$ is said to be a texturing of S, and $(S, \mathcal{S})$ is said to be a *texture space,* or simply a *texture*, if $\mathcal{S}$ is a point-separating, complete, completely distributive lattice containing S and $\emptyset$, such that arbitrary meets coincide with intersection, and finite joins coincide with unions.

For a texture $(S, \mathcal{S})$, the sets $P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$ and, as a dually, $Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}$ are called the *p-sets* and the *q-sets*, respectively. Also, for $A \in \mathcal{S}$ the core A^b of $A = \bigvee\{A_i \mid i \in I\}$ is given by

$$A^b = \bigcap \left\{ \bigcup \{A_i \mid i \in I\} \mid \{A_i \mid i \in I\} \subseteq \mathcal{S} \right\}.$$

Complementation: Since a texturing of S need not be closed under the operation of taking the set-complement, but it may be that there exists a mapping $\sigma : \mathcal{S} \to \mathcal{S}$ verifying the condition $\sigma(\sigma(A)) = A$ for all $A \in \mathcal{S}$ and for all $A, B \in \mathcal{S}$, $A \subseteq B \Longrightarrow \sigma(B) \subseteq \sigma(A)$. Thus, a texture $(S, \mathcal{S})$ with a complementation σ is said to be a complemented texture $(S, \mathcal{S}, \sigma)$.

Examples 2.1. Now, we give well-known reference examples [3,5].

- (i) The pair $(X, \mathcal{P}(X))$ is called the *discrete texture* on X where $\mathcal{P}(X)$ is the power set of X. Obviously, for all $x \in X$ we have $P_x = \{x\}$, $Q_x = X \setminus \{x\}$.
- (ii) For $\mathbb{I} = [0, 1]$ define $\mathcal{I} = \{[0, t] \mid t \in [0, 1]\} \cup \{[0, t) \mid t \in [0, 1]\}$, $\iota([0, t)) = [0, 1 - t]$ and $\iota([0, t]) = [0, 1 - t)$, $t \in [0, 1]$. $(\mathbb{I}, \mathcal{I}, \iota)$ gives the *unit interval texture*, where $Q_t = [0, t)$ and $P_t = [0, t]$ for all $t \in \mathbb{I}$.
- (iii) For textures $(S, \mathcal{S})$, $(T, \mathcal{T})$, we will denote by $\mathcal{S} \otimes \mathcal{T}$ the *product texturing* of $S \times T$. Thus, $\mathcal{S} \otimes \mathcal{T}$ consists of arbitrary intersections of sets of the form $(A \times T) \cup (S \times B)$, $A \in \mathcal{S}, B \in \mathcal{T}$, and $(S \times T, \mathcal{S} \otimes \mathcal{T})$ is said to be the *product* of $(S, \mathcal{S})$ and $(T, \mathcal{T})$. For $s \in S, t \in T$ we obviously get $P_{(s,t)} = P_s \times P_t$ and $Q_{(s,t)} = (Q_s \times T) \cup (S \times Q_t)$.

We begin recalling that a texture $(S, \mathcal{S})$ is said to be *plain* if $\mathcal{S}$ is closed under arbitrary unions; equivalently if arbitrary joins coincide with unions or if $P_s \nsubseteq Q_s$ for all $s \in S$. For the above examples, $(X, \mathcal{P}(X))$ and $(\mathbb{I}, \mathcal{I})$ are plain.

Definition 2.2. A *ditopology* on a texture space $(S, \mathcal{S})$ is a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ and the set of *closed sets* κ verifies

(1)	$S, \emptyset \in \tau$,	$S, \emptyset \in \kappa$,
(2)	$G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$,	$K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$,
(3)	$G_i \in \tau, i \in I \implies \bigvee G_i \in \tau$.	$K_i \in \kappa, i \in I \implies \bigcap K_i \in \kappa$.

Therefore a ditopology is essentially a "topology" for which there is no *a priori* relation between the open and closed sets. We refer to τ as the *topology* and to κ as the *cotopology* of $(\tau,\ \kappa)$. Let $(\tau_1,\ \kappa_1)$ and $(\tau_2,\ \kappa_2)$ be ditopologies on a texture $(S, \mathcal{S})$. Then, we say that $(\tau_2,\ \kappa_2)$ is finer than $(\tau_1,\ \kappa_1)$ if $\tau_1 \subseteq \tau_2$ and $\kappa_1 \subseteq \kappa_2$.

The *closure* $cl\,(A)$ and the *interior* $int\,(A)$ of $A \in \mathcal{S}$ is defined, respectively:

$$cl\,(A) = \bigcap\{K \in \kappa \mid A \subseteq K\} \text{ and } int\,(A) = \bigvee\{G \in \tau \mid G \subseteq A\}.$$

If (τ, κ) is a ditopology on a complemented texture $(S, \mathcal{S}, \sigma)$ we say (τ, κ) is complemented if $\kappa = \sigma(\tau)$. In this case we get $\sigma(cl\,(A)) = int(\sigma(A))$ and $\sigma(int(A)) = cl(\sigma(A))$.

Examples 2.3. (i) Let $(X, \mathcal{T})$ be a topological space. Then the pair $(\mathcal{T}, \mathcal{T}')$ is a ditopology on the discrete texture $(X, \mathcal{P}(X))$ where $\mathcal{T}' = \{X \setminus G | G \in \mathcal{T}\}$.
(ii) For any texture $(S, \mathcal{S})$ a ditopology (τ, κ) with $\tau = \mathcal{S}$ is said to be *discrete,* and one with $\kappa = \mathcal{S}$ is said to be *codiscrete.*
(iii) For any texture $(S, \mathcal{S})$ a ditopology (τ, κ) with $\tau = \{\emptyset, S\}$ is said to be *indiscrete,* and one with $\kappa = \{\emptyset, S\}$ is said to be *co-indiscrete.*

The suitable morphisms between textures have two parts which are dual to each other. Namely, a direlation on a texture space is a pair (r, R) where r is a relation and R a corelation are the elements of a textural product verifying certain conditions [5]. One of the most useful notions of ditopological texture spaces is that of difunction. A difunction is derived from that of direlation as follows.

Difunction: A *difunction from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ is a direlation (f, F) from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ verifying the following two conditions.

DF1 For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \implies$ there exist $t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.

DF2 For $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t$.

Definition 2.4. Let $(f, F) : (S, \mathcal{S}) \to (T, \mathcal{T})$ be a difunction. For $A \in \mathcal{S}$ and $B \in \mathcal{T}$, the A-sections and the B-presections with respect to (f, F) are given as follows:

$$\begin{aligned} f^{\to}A &= \bigcap\{Q_t \mid \text{for all } s,\ f \not\subseteq \overline{Q}_{(s,t)} \Longrightarrow A \subseteq Q_s\}, \\ F^{\to}A &= \bigvee\{P_t \mid \text{for all } s,\ \overline{P}_{(s,t)} \not\subseteq F \Longrightarrow P_s \subseteq A\}, \\ &\text{and} \\ f^{\leftarrow}B &= \bigvee\{P_s \mid \text{for all } t,\ f \not\subseteq \overline{Q}_{(s,t)} \Longrightarrow P_t \subseteq B\}, \\ F^{\leftarrow}B &= \bigcap\{Q_s \mid \text{for all } t,\ \overline{P}_{(s,t)} \not\subseteq F \Longrightarrow B \subseteq Q_t\}, \end{aligned}$$

respectively.

For a given difunction, the inverse image and the inverse co-image are equal; and the image and co-image are usually not.

We note that $((f^{\leftarrow})^{\leftarrow})(A) = f^{\to}(A)$ and $((F^{\leftarrow})^{\leftarrow})(A) = F^{\to}(A)$ by [5, Lemma 2.9].

Examples 2.5. (1)

(2) The identity difunction (i, I) is given by

$$i = \bigvee \left\{ P_{(s,s)} \mid s \in S \right\} \text{ and } I = \bigcap \left\{ Q_{(s,s)} \mid s \in S^b \right\}.$$

(3) Let $f : X \to Y$ be a point function. Then (f, f') is a difunction from $(X, \mathcal{P}(X))$ to $(Y, \mathcal{P}(Y))$ where $f' = (X \times Y) \setminus f$.

Let (τ, κ) be a ditopology on $(S, \mathcal{S})$. We recall [7] that an element N of $\mathcal{S}$ is called a neighborhood of $s \in S^b$ if there exists $U \in \tau$ such that $P_s \subseteq U \subseteq N \nsubseteq Q_s$. The set of neighborhoods of s is denoted by $\eta(s)$. Dually, an element M of $\mathcal{S}$ is called a coneighborhood of $s \in S$ if there exists $K \in \kappa$ such that $P_s \nsubseteq M \subseteq K \subseteq Q_s$, and the set of coneighborhoods of s is denoted by $\mu(s)$. Furthermore, for all s, the sets G in τ are characterized by the condition that $G \in \eta(s)$ with $G \nsubseteq Q_s$ and the sets K in κ are characterized by the condition that $K \in \mu(s)$ with $P_s \nsubseteq K$.

Definition 2.6. A *difilter* on a texture $(S, \mathcal{S})$ is $\mathcal{F} \times \mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are nonempty and subsets of $\mathcal{S}$ satisfies

(1) $\emptyset \notin \mathcal{F}$, $\quad S \notin \mathcal{G}$,

(2) $F \in \mathcal{F}$, $F \subseteq F' \in \mathcal{S} \implies F' \in \mathcal{F}$, $\quad G \in \mathcal{G}$, $G' \subseteq G \implies G' \in \mathcal{G}$,

(3) $F_1, F_2 \in \mathcal{F} \implies F_1 \cap F_2 \in \mathcal{F}$. $\quad G_1, G_2 \in \mathcal{G} \implies G_1 \cup G_2 \in \mathcal{G}$.

3. The notion of ideal in ditopological texture spaces

Firstly we recall that a nonempty collection $\mathcal{I}$ of subsets on a nonempty set X is said to be an ideal if it verifies the following two conditions: (i) $B \in \mathcal{I}$ and $A \subseteq B \Longrightarrow A \in \mathcal{I}$ (heredity); (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I} \Longrightarrow A \cup B \in \mathcal{I}$ (finitely additive) [16]. Then $(X, \mathcal{I})$ is called a set with ideal. This leads to the following analogous concepts in a texture space.

Definition 3.1. Let $(S, \mathcal{S})$ be a texture space.

(a) A subset $\mathcal{L} \subseteq \mathcal{S}$ is called a $\mathcal{S}$-ideal, or an ideal on $(S, \mathcal{S})$ if $\mathcal{L} \neq \emptyset$ and verifies,
  (i) $L \in \mathcal{L}$ and $L \supseteq L' \Longrightarrow L' \in \mathcal{L}$,
  (ii) $L_1, L_2 \in \mathcal{L} \Longrightarrow L_1 \cup L_2 \in \mathcal{L}$.

(b) A subset $\mathcal{G} \subseteq \mathcal{S}$ is called a $\mathcal{S}$-co-ideal, or a co-ideal on $(S, \mathcal{S})$ if it verifies,
  (i) $G \in \mathcal{G}$ and $G \subseteq G' \Longrightarrow G' \in \mathcal{G}$,
  (ii) $G_1, G_2 \in \mathcal{G} \Longrightarrow G_1 \cap G_2 \in \mathcal{G}$.

(c) A pair $(\mathcal{L}, \mathcal{G})$, where $\mathcal{L}$ is an ideal and $\mathcal{G}$ is a co-ideal on $(S, \mathcal{S})$, is said to be a di-ideal on $(S, \mathcal{S})$.

Clear from the definition that the empty set $\emptyset$ and the set S always belongs to $\mathcal{L}$ and $\mathcal{G}$, respectively.

Now we investigate the effect of a complementation on a di-ideal. For this, we introduce the notions of an ideal $\mathcal{L}$ on the complemented texture $(S, \mathcal{S}, \sigma)$ as $\sigma(\mathcal{L}) = \{\sigma(L) \mid L \in \mathcal{L}\}$. Likewise, $\sigma(\mathcal{G})$ defined by $\mathcal{G}$.

Proposition 3.2. *Let $(S, \mathcal{S}, \sigma)$ be a complemented texture space and $\mathcal{L} \subseteq \mathcal{S}$. Then $\mathcal{L}$ is an ideal if and only if $\sigma(\mathcal{L})$ is a co-ideal.*

Proof. Suppose that $\mathcal{L}$ is an ideal on S. Then

(i) $G \in \sigma(\mathcal{L})$ and $G \subseteq G' \Longrightarrow \sigma(G) \in \mathcal{L}$ and $\sigma(G) \supseteq \sigma(G')$. By the ideal of $\mathcal{L}$, we have $\sigma(G') \in \mathcal{L}$, thus $G' \in \sigma(\mathcal{L})$.

(ii) $G_1, G_2 \in \sigma(\mathcal{L}) \Longrightarrow \sigma(G_1), \sigma(G_2) \in \mathcal{L}$ and by the ideal of $\mathcal{L}$, we have $\sigma(G_1) \cup \sigma(G_2) \in \mathcal{L}$. Thus $\sigma(\sigma(G_1) \cup \sigma(G_2)) \in \sigma(\mathcal{L})$ implies $G_1 \cap G_2 \in \sigma(\mathcal{L})$.

Hence, we see that $\sigma(\mathcal{L})$ is a co-ideal.

Using dual arguments, it can be obtained the other direction. □

Definition 3.3. Let $(S, \mathcal{S}, \sigma)$ be a complemented texture space and $(\mathcal{L}, \mathcal{G})$ be a di-ideal on $(S, \mathcal{S})$. In this case,

(i) The pair $(\sigma(\mathcal{G}), \sigma(\mathcal{L}))$ is called the conjugate of $(\mathcal{L}, \mathcal{G})$.

(ii) If $(\mathcal{L}, \mathcal{G}) = (\sigma(\mathcal{G}), \sigma(\mathcal{L}))$, then $(\mathcal{L}, \mathcal{G})$ is called a complemented di-ideal on $(S, \mathcal{S}, \sigma)$.

Remark 3.4. Let $\mathcal{I}$ be an ideal on a set X. Then it is easy to see that the pair $(\mathcal{I}, \mathcal{I}^c)$ is a di-ideal on the discrete texture $(X, \mathcal{P}(X))$ where $\mathcal{I}^c = \{X \setminus A | A \in \mathcal{I}\}$.

Let $(S, \mathcal{S}, \sigma)$ be a complemented texture space with a di-ideal $(\mathcal{L}, \mathcal{G})$. If $S \notin \mathcal{L}$ and $\emptyset \notin \mathcal{G}$, then $\mathcal{F} = \{\sigma(L) : L \in \mathcal{L}\}$ is a filter and $\mathcal{E} = \{\sigma(G) : G \in \mathcal{G}\}$ is a cofilter. Conditions (1) and (3) of Definition 2.6 are clear. Let us check condition (2). For $L \in \mathcal{L}$, consider $\sigma(L) \in \mathcal{F}$ and $\sigma(L) \subseteq \sigma(L') \in \mathcal{S} \Longrightarrow L' \subseteq L$ and $L' \in \mathcal{L}$, thus we have $\sigma(L') \in \mathcal{F}$. Also, we have to check the following cases:

$$\sigma(G) \in \mathcal{E} \text{ and } \sigma(G) \supseteq \sigma(G') \in \mathcal{S} \Longrightarrow G \subseteq G' \text{ and } G' \in \mathcal{G}$$

for $G \in \mathcal{G}$, thus we have $\sigma(G') \in \mathcal{E}$.

Furthermore, let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space. It can easy to satisfy that $\mu(s)$ is a $\mathcal{S}$-ideal, but in general $\eta(s)$ need not be a $\mathcal{S}$-co-ideal where $\eta(s)$ (resp. $\mu(s)$) is the set of neighborhoods (resp. coneighborhoods) of s. The reason is that if $N_2 \nsubseteq Q_s$ and $N_1 \nsubseteq Q_s$ it is not necessarily the case that $N_1 \cap N_2 \nsubseteq Q_s$ and thus $\eta(s)$ need not verify Definition 3.1 (b)(ii). On the other hand, if choose $(S, \mathcal{S})$ as a plain then $N_1 \cap N_2 \nsubseteq Q_s$ is equivalent to $P_s \subseteq N_1 \cap N_2$, which obviously holds whenever $N_2 \nsubseteq Q_s$ and $N_1 \nsubseteq Q_s$, because then $P_s \subseteq N_2$ and $P_s \subseteq N_1$. Hence for ditopology (τ, κ) on a plain texture $(S, \mathcal{S})$ the product $\mu(s) \times \eta(s)$ is a $\mathcal{S}$-di-ideal for all $s \in S^b = S$.

Definition 3.5. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $(\mathcal{L}, \mathcal{G})$ be a di-ideal on $(S, \mathcal{S})$ and $A \in \mathcal{S}$. Then

(i)

$$A^*(\mathcal{L}) = \bigvee \{P_s \mid \forall P_s \nsubseteq Q_r, \forall N \in \eta(r) \text{ with } A \cap N \notin \mathcal{L}\}$$

is called the local function of A.

(ii)

$$A_*(\mathcal{G}) = \bigcap \{Q_s \ \mid \forall P_r \nsubseteq Q_s, \forall M \in \mu(r) \text{ with } A \cup M \in \mathcal{G}\}$$

is called the co-local function of A.

(iii) A pair $(A^*(\mathcal{L}), A_*(\mathcal{G}))$ where $A^*(\mathcal{L})$ is local function of A and $A_*(\mathcal{G})$ is a co-local function of A, is said to be di-local function of A.

When no ambiguity is present we simply write A^* (resp. A_*) instead of $A^*(\mathcal{L})$ (resp. $A_*(\mathcal{G})$).

Example 3.6. Let (X, τ) be a topological space. Consider the discrete texture space $(X, \mathcal{P}(X))$. Obviously, the pairs $(\{\emptyset\}, \mathcal{P}(X) \setminus \{\emptyset\})$ and $(\mathcal{P}(X), \{\emptyset\})$ are di-ideals on $(X, \mathcal{P}(X))$. That is, $\mathcal{L} = \{\emptyset\}$ iff $A^* = cl(A)$ and $\mathcal{G} = \mathcal{P}(X) \setminus \{\emptyset\}$ iff $A_* = int(A)$, for any set $A \in \mathcal{P}(X)$. Likewise, $\mathcal{L} = \mathcal{P}(X)$ iff $A^* = \{\emptyset\}$ and $\mathcal{G} = \{\emptyset\}$ iff $A_* = S$, for any set $A \in \mathcal{P}(X)$.

The following theorem contains basic results and useful facts concerning the local function.

Theorem 3.7. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $A, B \in \mathcal{S}$. Then*

(1) *Let $\mathcal{L}$ and $\mathcal{J}$ be two ideals on $(S, \mathcal{S})$. Then the following are satisfied:*
 (i) $A \subseteq B \Longrightarrow A^* \subseteq B^*$.
 (ii) $\mathcal{L} \subseteq \mathcal{J} \Longrightarrow A^*(\mathcal{J}) \subseteq A^*(\mathcal{L})$.
 (iii) $A^* = cl(A^*) \subseteq cl(A)$ *(A^* is a closed subset of $cl(A)$).*
 (iv) $(A^*)^* \subseteq A^*$.
 (v) $(A \cup B)^* = A^* \cup B^*$.
 (vi) $I \in \mathcal{L} \Longrightarrow (A \cup I)^* = A^*$.

(2) *Let $\mathcal{G}$ and $\mathcal{F}$ be two co-ideals on $(S, \mathcal{S})$. Then the following are satisfied:*
 (i) $A \subseteq B \Longrightarrow A_* \subseteq B_*$.
 (ii) $\mathcal{F} \subseteq \mathcal{G} \Longrightarrow A_*(\mathcal{G}) \subseteq A_*(\mathcal{F})$.
 (iii) $int(A) \subseteq int(A_*) = A_*$ *($int(A)$ is an open subset of A_*).*
 (iv) $A_* \subseteq (A_*)_*$.
 (v) $(A \cap B)_* = A_* \cap B_*$.
 (vi) $G \in \mathcal{G} \Longrightarrow (A \cap G)_* = A_*$.

Proof. The proof of (2) is dual to the proof of (1) and is left to the interested reader.

(i) Let $A \subseteq B$. Suppose that $A^* \nsubseteq B^*$. Then, there exists some $s \in S$ such that $A^* \nsubseteq Q_s$ and $P_s \nsubseteq B^*$. Thus $A^* \nsubseteq Q_s$ implies $P_{s'} \nsubseteq Q_s$ for $\exists s' \in S$ and by definition of A^*, we have:

$$\forall P_{s'} \nsubseteq Q_m, \forall N \in \eta(m) \text{ with } A \cap N \notin \mathcal{L}. \tag{3.1}$$

Now, let $P_s \nsubseteq Q_r$ and $N \in \eta(r)$. Then $P_{s'} \nsubseteq Q_r$. By (3.1) we have $A \cap N \notin \mathcal{L}$. Since $A \cap N \subseteq B \cap N$, we have $B \cap N \notin \mathcal{L}$. So we have $P_s \subseteq B^*$. This is a contradiction.

(ii) Assume that $\mathcal{L} \subseteq \mathcal{J}$ and $A^*(\mathcal{J}) \nsubseteq A^*(\mathcal{L})$. Then, there exist $\exists s \in S$ such that $A^*(\mathcal{J}) \nsubseteq Q_s$ and $P_s \nsubseteq A^*(\mathcal{L})$. Considering $A^*(\mathcal{J}) \nsubseteq Q_s$, we have $P_{s'} \nsubseteq Q_s$ for any point $s' \in S$ and $\forall P_{s'} \nsubseteq Q_r$ such that $A \cap N \notin \mathcal{J}$ for $\forall N \in \eta(r)$. Suppose that $P_s \nsubseteq Q_m$ with $N \in \eta(m)$. Thus $P_{s'} \nsubseteq Q_m$ and $A \cap N \notin \mathcal{J}$ and $\mathcal{L} \subseteq \mathcal{J}$ so $A \cap N \notin \mathcal{L}$. Thus we get $P_s \subseteq A^*(\mathcal{L})$. But, this is a contradiction.

(iii) Since $\{\emptyset\} \subseteq \mathcal{L}$ for any ideal $\mathcal{L}$ on $(S, \mathcal{S})$, therefore by (ii) and Example 3.6, $A^*(\mathcal{L}) \subseteq A^*(\{\emptyset\}) = cl(A)$, for $A \in \mathcal{S}$.

Clearly, $A^* \subseteq cl(A^*)$ and hence we must show that $cl(A^*) \subseteq A^*$. Let us suppose the contrary, that is $cl(A^*) \nsubseteq A^*$. There exist $s \in S$ such that $cl(A^*) \nsubseteq Q_s$ and $P_s \nsubseteq A^*$. Then, $A^* \nsubseteq Q_s$, and so $P_{s'} \nsubseteq Q_s$ for $\exists s' \in S$ and we have:

$$\forall P_{s'} \nsubseteq Q_m, \forall N \in \eta(m) \text{ with } A \cap N \notin \mathcal{L}.$$

Now let $P_s \nsubseteq Q_r$ and $N \in \eta(r)$. Then we have $P_{s'} \nsubseteq Q_r$, and so $A \cap N \notin \mathcal{L}$ for all $N \in \eta(r)$. Hence we have $P_s \subseteq A^*$. But, this is a contradiction.

(iv) By (iii), we have $(A^*)^* = cl\left((A^*)^*\right) \subseteq cl(A^*) = A^*$.

(v) Assume that $A^* \cup B^* \nsubseteq (A \cup B)^*$, so there exist $\exists s \in S$ such that $A^* \cup B^* \nsubseteq Q_s$ and $P_s \nsubseteq (A \cup B)^*$. Considering $A^* \cup B^* \nsubseteq Q_s$, there exist $r, r' \in S$ such that $P_r \nsubseteq Q_s$ or $P_{r'} \nsubseteq Q_s$ and $\exists N \in \eta(s)$ with $A \cap N \notin \mathcal{L}$ or $B \cap N \notin \mathcal{L}$ implies $(A \cup B) \cap N \notin \mathcal{L}$. Then $(A \cup B)^* \nsubseteq Q_s$ which is a contradiction.

Contrary assume that $(A \cup B)^* \nsubseteq A^* \cup B^*$. Take $s \in S$ where $(A \cup B)^* \nsubseteq Q_s$ and $P_s \nsubseteq A^* \cup B^*$. For $(A \cup B)^* \nsubseteq Q_s$, there exists $\exists s' \in S$ and $\forall r$ such that $P_{s'} \nsubseteq Q_s$ and $P_s \nsubseteq Q_r$. Thus we have $\exists N \in \eta(r)$ with $(A \cup B) \cap N \notin \mathcal{L}$ which leads to $(A \cap N) \cup (B \cap N) \notin \mathcal{L}$ implies $(A \cap N) \notin \mathcal{L}$ and $(B \cap N) \notin \mathcal{L}$. This gives the contradiction $P_s \subseteq A^* \cup B^*$.

(vi) It is clear that $I \in \mathcal{L}$ satisfies $I^* = \emptyset$ so that $(A \cup I)^* = A^* \cup I^* = A^*$.

□

Theorem 3.8. *Let $(S, \mathcal{S})$ be a texture space and $(\mathcal{L}, \mathcal{G})$ be a di-ideal on $(S, \mathcal{S})$.*

(1) *An operator $\mathbf{cl}^* : \mathcal{S} \to \mathcal{S}$, $\mathbf{cl}^*(A) = A \cup A^*$ has the following properties:*
(i) $\mathbf{cl}^*(\emptyset) = \emptyset$,
(ii) $A \in \mathcal{S} \Rightarrow A \subseteq \mathbf{cl}^*(A)$,
(iii) $A, B \in \mathcal{S} \Rightarrow \mathbf{cl}^*(A \cup B) = \mathbf{cl}^*(A) \cup \mathbf{cl}^*(B)$,
(iv) $A \in \mathcal{S} \Rightarrow \mathbf{cl}^*(\mathbf{cl}^*(A)) = \mathbf{cl}^*(A)$.

2. *An operator $\mathbf{int}^* : \mathcal{S} \to \mathcal{S}$, $\mathbf{int}^*(A) = A \cap A_*$ has the following properties:*
(i) $\mathbf{int}^*(S) = S$,
(ii) $A \in \mathcal{S} \Rightarrow \mathbf{int}^*(A) \subseteq A$,
(iii) $A, B \in \mathcal{S} \Rightarrow \mathbf{int}^*(A \cap B) = \mathbf{int}^*(A) \cap \mathbf{int}^*(B)$,
(iv) $A \in \mathcal{S} \Rightarrow \mathbf{int}^*(\mathbf{int}^*(A)) = \mathbf{int}^*(A)$.

Proof. The proof of (2) is dual to the proof of (1) and is left to the interested reader.

Since $\emptyset^* = \emptyset$ and $(A \cup B)^* = A^* \cup B^*$, (i) and (iii) are trivial, so we concentrate on (ii) and (iv).

(ii) For $A \in \mathcal{S}$ we have $\mathbf{cl}^*(A) = A \cup A^* \supseteq A$.

(iv) For $A \in \mathcal{S}$ we have $\mathbf{cl}^*(\mathbf{cl}^*(A)) = (\mathbf{cl}^*(A)) \cup (\mathbf{cl}^*(A))^* \supseteq \mathbf{cl}^*(A)$. Conversely, by hypothesis and (iii) we have

$$\begin{aligned}
\mathbf{cl}^*(\mathbf{cl}^*(A)) &= \mathbf{cl}^*(A \cup A^*) \\
&= \mathbf{cl}^*(A) \cup \mathbf{cl}^*(A^*) \\
&= (A \cup A^*) \cup \left(A^* \cup (A^*)^*\right) \\
&= A \cup A^* \cup (A^*)^* \\
&\subseteq A \cup A^* \cup A^* = A \cup A^* \\
&= \mathbf{cl}^*(A).
\end{aligned}$$

□

Theorem 3.9. *Let $(\mathcal{L}, \mathcal{G})$ be a di-ideal on $(S, \mathcal{S})$. Then the pair $(\tau^*(\mathcal{G}), \kappa^*(\mathcal{L}))$ is a ditopology on $(S, \mathcal{S})$ where,*

$$\tau^*(\mathcal{G}) = \{U \in \mathcal{S} : \mathbf{int}^*(U) = U\} \text{ and } \kappa^*(\mathcal{L}) = \{U \in \mathcal{S} : \mathbf{cl}^*(U) = U\}.$$

Proof. We can show that the family $\kappa^*(\mathcal{L})$ verifies cotopology conditions of Definition 2.2.

(1) Both S and $\emptyset$ are in $\kappa^*(\mathcal{L})$, since, by Theorem 3.8, we have that $\mathbf{cl}^*(S) = S \cup S^* = S$ and $\mathbf{cl}^*(\emptyset) = \emptyset$.

(2) If K_1, K_2 are nonempty elements of $\kappa^*(\mathcal{L})$, to show that $K_1 \cup K_2$ is in $\kappa^*(\mathcal{L})$, we compute

$$\mathbf{cl}^*(K_1 \cup K_2) = \mathbf{cl}^*(K_1) \cup \mathbf{cl}^*(K_2) = K_1 \cup K_2$$

because of $\mathbf{cl}^*(K_1) = K_1$ and $\mathbf{cl}^*(K_2) = K_2$.

(3) If $\{K_\alpha\}$ is an indexed family of nonempty elements of $\kappa^*(\mathcal{L})$, to show that $\bigcap K_\alpha$ is in $\kappa^*(\mathcal{L})$, we compute

$$\mathbf{cl}^*\left(\bigcap K_\alpha\right) \subseteq \bigcap \mathbf{cl}^*(K_\alpha) = \bigcap K_\alpha$$

because of $\bigcap K_\alpha \subseteq K_\alpha$ for all α.

Now, we can show that the family $\tau^*(\mathcal{G})$ verifies topology conditions of Definition 2.2.

(1) Both S and $\emptyset$ are in $\tau^*(\mathcal{G})$, since, by Theorem 3.8, we have that $\mathbf{int}^*(S) = S$ and $\mathbf{int}^*(\emptyset) = \emptyset \cap \emptyset_* = \emptyset$.

(2) If G_1, G_2 are nonempty elements of $\tau^*(\mathcal{G})$, to show that $G_1 \cap G_2$ is in $\tau^*(\mathcal{G})$, we compute

$$\mathbf{int}^*(G_1 \cap G_2) = \mathbf{int}^*(G_1) \cap \mathbf{int}^*(G_2) = G_1 \cap G_2$$

because of $\mathbf{int}^*(G_1) = G_1$ and $\mathbf{int}^*(G_2) = G_2$.

(3) If $\{G_\alpha\}$ is an indexed family of nonempty elements of $\tau^*(\mathcal{G})$ and $G = \bigcup G_\alpha$, then we need to show that G is in $\tau^*(\mathcal{G})$. By Theorem 3.8 (2)(ii), $\mathbf{int}^*(G) \subseteq G$. On the other hand, for each α, we compute

$$G_\alpha = \mathbf{int}^*(G_\alpha) = \mathbf{int}^*(G_\alpha \cap G) = \mathbf{int}^*(G_\alpha) \cap \mathbf{int}^*(G) \subseteq \mathbf{int}^*(G),$$

by Theorem 3.8 (2)(iii), hence $G = \bigcup G_\alpha \subseteq \mathbf{int}^*(G)$ and consequently $\mathbf{int}^*(G) = G$, $G \in \tau^*(\mathcal{G})$.

□

Remark 3.10. Let a ditopology (τ, κ) be on the discrete texture space $(X, \mathcal{P}(X))$. We have already observed that if $\mathcal{L} = \{\emptyset\}$ and $\mathcal{G} = \mathcal{P}(X) \setminus \{\emptyset\}$, then $\mathbf{cl}^*(A) = A \cup A^* = A \cup \mathbf{cl}(A) = \mathbf{cl}(A)$ and $\mathbf{int}^*(A) = A \cap A_* = A \cap \mathbf{int}(A) = \mathbf{int}(A)$, for all $A \in \mathcal{P}(X)$. So, $(\tau^*(\mathcal{G}), \kappa^*(\mathcal{L})) = (\tau, \kappa)$. Again, if $\mathcal{L} = \mathcal{P}(X)$ and $\mathcal{G} = \{\emptyset\}$, then $A^* = \{\emptyset\}$ and $A_* = S$, for all $A \in \mathcal{P}(X)$ and hence $\tau^*(\mathcal{G}) = \mathcal{P}(X) = \kappa^*(\mathcal{L})$ is the discrete and codiscrete ditopological texture space.

Hence, for any di-ideal $(\mathcal{L}, \mathcal{G})$ on $(X, \mathcal{P}(X))$ we get $\{\emptyset\} \subseteq \mathcal{L} \subseteq \mathcal{P}(X)$ and $\{\emptyset\} \subseteq \mathcal{G} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. So we can conclude by Theorem 3.7 (1) and (2), (ii), $\kappa^*(\{\emptyset\}) \subseteq \kappa^*(\mathcal{L}) \subseteq \kappa^*(\mathcal{P}(X))$ and $\tau^*(\mathcal{P}(X) \setminus \{\emptyset\}) \subseteq \tau^*(\mathcal{G}) \subseteq \tau^*(\{\emptyset\})$, for any di-ideal $(\mathcal{L}, \mathcal{G})$ on $(X, \mathcal{P}(X))$. In particular, we have, for any two di-ideals $(\mathcal{L}, \mathcal{G})$ and $(\mathcal{J}, \mathcal{F})$ on $(X, \mathcal{P}(X))$, $\mathcal{L} \subseteq \mathcal{J} \to \kappa^*(\mathcal{L}) \subseteq \kappa^*(\mathcal{J})$ and $\mathcal{G} \subseteq \mathcal{F} \to \tau^*(\mathcal{F}) \subseteq \tau^*(\mathcal{G})$.

Consequently, given a ditopology (τ, κ) on the discrete texture space $(X, \mathcal{P}(X))$ with di-ideal $(\mathcal{L}, \mathcal{G})$, we say that $(\tau^*(\mathcal{G}), \kappa^*(\mathcal{L}))$ is finer than (τ, κ).

Lemma 3.11. *Let $(\mathcal{L}, \mathcal{G})$ and $(\mathcal{J}, \mathcal{F})$ be di-ideals on $(S, \mathcal{S})$. Then*

(1) *$\mathcal{L} \bigvee \mathcal{J} = \{L \cup J : L \in \mathcal{L},\ J \in \mathcal{J}\}$ and $\mathcal{L} \cap \mathcal{J}$ are ideals on $(S, \mathcal{S})$.*
(2) *$\mathcal{G} \bigvee \mathcal{F} = \{G \cup F : G \in \mathcal{G},\ F \in \mathcal{F}\}$ and $\mathcal{G} \cap \mathcal{F}$ are co-ideals on $(S, \mathcal{S})$.*

Proof. (1) Let $\mathcal{L}$ and $\mathcal{J}$ be ideals over $(S, \mathcal{S})$. First we show that $\mathcal{L} \cap \mathcal{J}$ is an ideal. We prove the conditions (i) and (ii) of Definition 3.1 (a) and live the proof of the other result to the reader.

(i) Let $L \in \mathcal{L} \cap \mathcal{J}$ and $L' \subseteq L$, then $L \in \mathcal{L}$ and $L \in \mathcal{J}$. Therefore, we have $L' \in \mathcal{L}$ and $L' \in \mathcal{J}$. Consequently, $L' \in \mathcal{L} \cap \mathcal{J}$.
(ii) Let $L_1 \in \mathcal{L} \cap \mathcal{J}$ and $L_2 \in \mathcal{L} \cap \mathcal{J}$, then $L_1 \in \mathcal{L}$, $L_2 \in \mathcal{L}$, $L_1 \in \mathcal{J}$ and $L_2 \in \mathcal{J}$. Consequently, $L_1 \cup L_2 \in \mathcal{L}$ and $L_1 \cup L_2 \in \mathcal{J}$ and so $L_1 \cup L_2 \in \mathcal{L} \cap \mathcal{J}$.

(2) Let $\mathcal{G}$ and $\mathcal{F}$ be co-ideals over $(S, \mathcal{S})$. Now we show that $\mathcal{G} \bigvee \mathcal{F}$ is a co-ideal. We prove the conditions (i) and (ii) of Definition 3.1 (b) and live the proof of the other result to the reader.

(i) Let $G \in \mathcal{G} \bigvee \mathcal{F}$. Then $G \in \mathcal{G}$ or $G \in \mathcal{F}$. Now $G \in \mathcal{G}$ implies there is at least one G' such that $G' \in \mathcal{G}$. Again, $G \in \mathcal{F}$ implies there is at least one G' such that $G' \in \mathcal{F}$. Therefore, we have $G' \in \mathcal{G} \bigvee \mathcal{F}$.
(ii) Let $G_1 \in \mathcal{G} \bigvee \mathcal{F}$ and $G_2 \in \mathcal{G} \bigvee \mathcal{F}$. Since $G_1 \cap G_2$ is included in both G_1 and G_2, $G_1 \cap G_2$ is included in $\mathcal{G} \bigvee \mathcal{F}$.

□

Theorem 3.12. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space, and $A \in \mathcal{S}$. Hence*

(1) *Let* $\mathcal{L}$ *and* $\mathcal{J}$ *ideals on* $(S,\mathcal{S})$.
(i) $A^*(\mathcal{L}\cap\mathcal{J}) = A^*(\mathcal{L})\cup A^*(\mathcal{J})$,
(ii) $A^*(\mathcal{L}\bigvee\mathcal{J}) = A^*(\mathcal{L},\kappa^*(\mathcal{L}))\cap A^*(\mathcal{J},\kappa^*(\mathcal{J}))$.
(2) *Let* $\mathcal{G}$ *and* $\mathcal{F}$ *co-ideals on* $(S,\mathcal{S})$.
(i) $A_*(\mathcal{G}\bigvee\mathcal{F}) = A_*(\mathcal{G})\cap A_*(\mathcal{F})$,
(ii) $A_*(\mathcal{G}\cap\mathcal{F}) = A_*(\mathcal{G},\tau^*(\mathcal{G}))\cup A_*(\mathcal{F},\tau^*(\mathcal{F}))$.

Proof. We prove (1), leaving the dual proof of (2) to the interested reader.

(i) For by Theorem 3.7 (1)(ii) $A^*(\mathcal{L}\cap\mathcal{J}) \supseteq A^*(\mathcal{L})\cup A^*(\mathcal{J})$. To prove the reverse inclusion, suppose that $A^*(\mathcal{L}\cap\mathcal{J}) \not\subseteq A^*(\mathcal{L})\cup A^*(\mathcal{J})$ and take $s\in S$ with $A^*(\mathcal{L}\cap\mathcal{J})\not\subseteq Q_s$ and $P_s\not\subseteq A^*(\mathcal{L})\cup A^*(\mathcal{J})$. If $A^*(\mathcal{L}\cap\mathcal{J})\not\subseteq Q_s$ then $P_{s'}\not\subseteq Q_s$ for any $s'\in S$ and $\forall P_{s'}\not\subseteq Q_m$, $N\in\eta(m)$ with $A\cap N\notin(\mathcal{L}\cap\mathcal{J})$.

Let $P_s\not\subseteq Q_r$ with $N\in\eta(r)$. Then $P_{s'}\not\subseteq Q_r$, so we have $A\cap N\notin\mathcal{L}$ or $A\cap N\notin\mathcal{J}$. Hence $P_s\subseteq A^*(\mathcal{L})$ or $P_s\subseteq A^*(\mathcal{J})$. This is a contradiction.

(ii) Suppose that $A^*(\mathcal{L},\kappa^*(\mathcal{L}))\cap A^*(\mathcal{J},\kappa^*(\mathcal{J}))\not\subseteq A^*(\mathcal{L}\bigvee\mathcal{J})$. Then, there exist $\exists s\in S$ such that $A^*(\mathcal{L},\kappa^*(\mathcal{L}))\cap A^*(\mathcal{J},\kappa^*(\mathcal{J}))\not\subseteq Q_s$ and $P_s\not\subseteq A^*(\mathcal{L}\bigvee\mathcal{J})$. Considering $A^*(\mathcal{L},\kappa^*(\mathcal{L}))\cap A^*(\mathcal{J},\kappa^*(\mathcal{J}))\not\subseteq Q_s$ implies there is at least one $s'\in S$, $P_{s'}\not\subseteq Q_s$ and $\forall P_{s'}\not\subseteq Q_r$, $\forall N$ in $\eta(r)$ with $A\cap N\notin\mathcal{L}$ and there is at least one $s''\in S$, $P_{s''}\not\subseteq Q_s$ and $\forall P_{s''}\not\subseteq Q_r$, $\forall N$ in $\eta(r)$ with $A\cap N\notin\mathcal{J}$ for any $r\in S$. Let $P_s\not\subseteq Q_m$ with $N\in\eta(m)$. Then $P_{s'}\not\subseteq Q_m$, so we have $A\cap N\notin(\mathcal{L}\bigvee\mathcal{J})$. This gives the contradiction $P_s\subseteq A^*(\mathcal{L}\bigvee\mathcal{J})$.

Conversely, let $A^*(\mathcal{L}\bigvee\mathcal{J})\not\subseteq A^*(\mathcal{L},\kappa^*(\mathcal{L}))\cap A^*(\mathcal{J},\kappa^*(\mathcal{J}))$. Take $s\in S$ where $A^*(\mathcal{L}\bigvee\mathcal{J})\not\subseteq Q_s$ and $P_s\not\subseteq A^*(\mathcal{L},\kappa^*(\mathcal{L}))\cap A^*(\mathcal{J},\kappa^*(\mathcal{J}))$. For $A^*(\mathcal{L}\bigvee\mathcal{J})\not\subseteq Q_s$, there exists $\exists s'\in S$ and $\forall r$ such that $P_{s'}\not\subseteq Q_s$ implies $\forall P_{s'}\not\subseteq Q_r$ and $\exists N\in\eta(r)$ with $A\cap N\notin\mathcal{L}\bigvee\mathcal{J}$. Then $A\cap N\notin\mathcal{L}$ or $A\cap N\notin\mathcal{J}$. Thus $P_s\not\subseteq Q_r$, $\forall N$ in $\eta(r)$ with $A\cap N\notin\mathcal{L}$ satisfies $P_s\subseteq A^*(\mathcal{L},\kappa^*(\mathcal{L}))$ or $P_s\not\subseteq Q_r$, $\forall N$ in $\eta(r)$ with $A\cap N\notin\mathcal{J}$ satisfies $P_s\subseteq A^*(\mathcal{J},\kappa^*(\mathcal{J}))$ which leads to $P_s\subseteq A^*(\mathcal{L},\kappa^*(\mathcal{L}))\cap A^*(\mathcal{J},\kappa^*(\mathcal{J}))$. This is a contradiction.

□

4. Weakly-bicontinuous difunction

We recall that a function $f: X\to Y$ between topological spaces X and Y is called weakly-continuous [17] if $f^{-1}(V)\subseteq int\left(f^{-1}(cl(V))\right)$ for all open set $V\subseteq Y$. This leads to the following concepts for a difunction between ditopological texture spaces.

Definition 4.1. Let $(S_k,\mathcal{S}_k,\tau_k,\kappa_k)$, $k=1,2$ be ditopological texture spaces and $(f,F):(S_1,\mathcal{S}_1)\to(S_2,\mathcal{S}_2)$ be a difunction. Then (f,F) is called:

(1) *weakly-continuous* $F^{\leftarrow}(V)\subseteq int\,(F^{\leftarrow}(cl(V)))$ for all $V\in\tau_2$,

(2) *weakly-cocontinuous* if $cl\,(f^{\leftarrow}(int(H)))\subseteq f^{\leftarrow}(H)$ for all $H\in\kappa_2$,

(3) *weakly-bicontinuous* if it is both weakly-continuous and weakly-cocontinuous.

Lemma 4.2. *Let $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$, $k = 1,2$ be ditopological texture spaces and $(f,F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ be a difunction. Then if (f,F) is continuous (cocontinuous, bicontinuous) then it is weakly- continuous (respectively, -cocontinuous, -bicontinuous)*

Proof. Suppose that (f,F) is a continuous difunction. Let $V \in \tau_2$. Then $F^{\leftarrow}(V) \in \tau_1$, and so $int(F^{\leftarrow}(V)) = F^{\leftarrow}(V)$. Hence we have, $V \subseteq cl(V) \Longrightarrow F^{\leftarrow}(V) \subseteq F^{\leftarrow}(cl(V)) \Longrightarrow int(F^{\leftarrow}(V)) = F^{\leftarrow}(V) \subseteq int\,(F^{\leftarrow}(cl(V))) \Longrightarrow (f,F)$ is weakly-continuous.

Now let (f,F) be a cocontinuous difunction and $H \in \kappa_2$. Then $f^{\leftarrow}(H) \in \kappa_1$, and so $cl(f^{\leftarrow}(H)) = f^{\leftarrow}(H)$. Then $int(H) \subseteq H \Longrightarrow f^{\leftarrow}(int(H)) \subseteq f^{\leftarrow}(H) \Longrightarrow cl(f^{\leftarrow}(int(H))) \subseteq cl(f^{\leftarrow}(H)) = f^{\leftarrow}(H) \Longrightarrow (f,F)$ is weakly-cocontinuous. □

Clearly, for any ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$, the identity difunction (i, I) is weakly-bicontinuous difunction, since $I^{\leftarrow}(B) = B = i^{\leftarrow}(B)$ for all $B \in \mathcal{S}_2$ by [5].

Recall that [5] $f : X \to Y$ is a point function if and only if (f, f') is a difunction from $(X, \mathcal{P}(X))$ to $(Y, \mathcal{P}(Y))$ such that

$$f^{\leftarrow} = (f^{-1})' = (f')^{-1}$$

where $f' = (X \times Y) \setminus f$. Note that $f^{\leftarrow}(B) = (f')^{\leftarrow}(B) = f^{-1}(B)$ for all $B \in \mathcal{P}(Y)$.

Further, if $(X, \mathcal{T})$ be a topological space then $(\mathcal{T}, \mathcal{T}')$ is a ditopology on the discrete texture $(X, \mathcal{P}(X))$ where $\mathcal{T}' = \{X \setminus G \mid G \in \mathcal{T}\}$. Then we have

Proposition 4.3. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{V})$ be a point function between topological spaces and $(f, f') : (X, \mathcal{P}(X), \mathcal{T}, \mathcal{T}') \to (Y, \mathcal{P}(Y), \mathcal{V}, \mathcal{V}')$ be the corresponding difunction. Then f is weakly continuous $\Longleftrightarrow$ (f, f') is weakly bicontinuous difunction.*

Proof. Let $V \in \mathcal{V}$. Since $(f'^{\leftarrow} = f^{-1}$, we have

$$\begin{aligned} f \text{ is weakly continuous} &\Longleftrightarrow \quad f^{-1}(V) \subseteq int(f^{-1}(cl(V)) \\ &\Longleftrightarrow \quad (f'^{\leftarrow}(V) \subseteq int((f'^{\leftarrow}(cl(V)) \\ &\Longleftrightarrow \quad (f, f') \text{ is weakly-continuous} \end{aligned}$$

Now let $Y \setminus H = H' \in \mathcal{V}'$. Then $H \in \mathcal{V}$, and

$$\begin{aligned} f \text{ is weakly continuous} &\iff f^{-1}(H) \subseteq int(f^{-1}(cl(H))) \\ &\iff X \setminus (int(f^{-1}(cl(H)))) \subseteq X \setminus f^{-1}(H) \\ &\iff cl(X \setminus (f^{-1}(cl(H)))) \subseteq f^{-1}(Y \setminus H) = f^{-1}(H') \\ &\iff cl(f^{-1}(Y \setminus cl(H))) \subseteq f^{\leftarrow}(H') \\ &\iff cl(f^{\leftarrow}(int(H'))) \subseteq f^{\leftarrow}(H') \\ &\iff (f, f') \text{ is weakly-cocontinuous} \end{aligned}$$

Now let $(X, \mathcal{T})$ be a topological space and $\mathcal{I}$ be an ideal on X. Then $(X, \mathcal{T}, \mathcal{I})$ is said to be ideal topological space. For $A \subseteq X$, the set

$$A^* = \{x \in X \mid \forall N \in \mathcal{N}(x), A \cap N \notin \mathcal{I}\}$$

is called local function of the set A where $\mathcal{N}(x)$ is the neighborhood system of x [14]. □

Now suppose that $(X, \mathcal{T})$ is a topological space and $(Y, \mathcal{V}, \mathcal{I})$ is an ideal topological space. Then we recall that a function $f : X \to Y$ is called weakly ideal-continuous [1] if $f^{-1}(V) \subseteq int(f^{-1}(cl^*(V)))$ for all $V \in \mathcal{V}$. This leads to the following concepts for a difunction between ditopological texture spaces.

Definition 4.4. Let $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)$, $j = 1, 2$ be ditopological texture spaces and $(\mathcal{L}, \mathcal{G})$ be a di-ideal on $(S_2, \mathcal{S}_2)$. Then a difunction $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is called:

(1) *weakly ideal-continuous* $F^{\leftarrow}(V) \subseteq int(F^{\leftarrow}(cl^*(V)))$ for all $V \in \tau_2$,

(2) *weakly ideal-cocontinuous* if $cl(f^{\leftarrow}(int^*(H))) \subseteq f^{\leftarrow}(H)$ for all $H \in \kappa_2$,

(3) *weakly ideal-bicontinuous* if it is both weakly-ideal continuous and weakly ideal-cocontinuous.

Lemma 4.5. *Let $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$, $k = 1, 2$ are ditopological texture spaces and $(\mathcal{L}, \mathcal{G})$ be a di-ideal on $(S_2, \mathcal{S}_2)$ and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ be a difunction. Then*

(i) *if (f, F) is continuous (cocontinuous, -bicontinuous) then it is weakly ideal-continuous (respectively, -cocontinuous, -bicontinuous).*

(ii) *if (f, F) is weakly ideal-continuous (respectively, -cocontinuous, -bicontinuous) then it is weakly-continuous (respectively, -cocontinuous, -bicontinuous)*

Proof. (i) Since $V \subseteq cl^*(V) = V \cup V^*$ and $int^*(H) = H \cap H^* \subseteq H$, the proofs follows the same lines as that of Lemma 4.2, and is omitted.

(ii) Let $V \in \tau_2$ and $H \in \kappa_2$. Then $cl^*(V) \subseteq cl_{(\tau_2, \kappa_2)}(V)$ and $int_{(\tau_2, \kappa_2)}(H) \subseteq int^*(H)$, since $(\tau, \kappa) \subseteq (\tau^*, \kappa^*)$. Hence, the proof is completed. □

Proposition 4.6. *Let* $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$, $k = 1, 2$ *are ditopological texture spaces and* $(\mathcal{L}, \mathcal{G})$ *be a di-ideal on* $(S_2, \mathcal{S}_2)$ *where* $\mathcal{L} = \{\emptyset\}, \mathcal{G} = \mathcal{S}_2$. *Now let* $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ *be a difunction. Then* (f, F) *is a weakly ideal bicontinuous difunction if and only if it is a weakly bicontinuous difunction.*

Proof. The proof of necessity is clear from Lemma 4.5(ii). Now let $V \in \tau_2$ and $H \in \kappa_2$. By Example 3.6, we have $cl(V) = cl^*(V)$ and $int(H) = int^*(H)$. Consequently, if (f, F) is weakly continuous difunction then it is weakly-ideal bicontinuous difunction. □

We observe that if $\mathcal{I}$ is an ideal on a set Y then $(\mathcal{I}, \mathcal{I}')$ is a di-ideal on the discrete texture $(Y, \mathcal{P}(Y))$. Then we have:

Proposition 4.7. *Let* $(Y, \mathcal{V}, \mathcal{I})$ *be an ideal topological space and* $(Y, \mathcal{V}, \mathcal{V}', (\mathcal{I}, \mathcal{I}'))$ *be the corresponding di-ideal ditopological texture space. Now let* $f : (X, \mathcal{T}) \to (Y, \mathcal{V})$ *be a point function between topological spaces and* $(f, f') : (X, \mathcal{P}(X), \mathcal{T}, \mathcal{T}') \to (Y, \mathcal{P}(Y), \mathcal{V}, \mathcal{V}')$ *be the corresponding difunction. Then* f *is weakly ideal-continuous function* $\Longleftrightarrow$ (f, f') *is weakly ideal-bicontinuous difunction.*

Proof. The proof follows the same lines as that of Proposition 4.3, and is omitted. □

5. Conclusion

The main purpose of this paper is to introduce the notion of ditopological texture spaces with ideal, which is finer than the given ditopological texture space on the discrete texture space $(X, \mathcal{P}(X))$. We study the notions of di-local function and weakly di-ideal bicontinuous on ditopological texture space with di-ideal. Also, in the framework of this paper, there are still many other aspects of ditopological texture space with di-ideal, namely, compactness, uniformity and separation axiom, etc. which can be investigated further.

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Current address: Memet Kule: Kilis 7 Aralik University, Department of Mathematics, Faculty of Arts and Sciences, 79000 Kilis, Turkey.
E-mail address: memetkule@kilis.edu.tr
ORCID Address: http://orcid.org/0000-0002-2869-2358
Current address: Şenol Dost: Hacettepe University, Department of Secondary Science and Mathematics Education, 06800 Beytepe, Ankara, Turkey.
E-mail address: dost@hacettepe.edu.tr
ORCID Address: http://orcid.org/0000-0002-5762-8056