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Research Article

Coefficient estimates for m-fold symmetric bi-subordinate functions

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Abstract

A function is said to be bi-univalent in the open unit disk $\mathbb{U}$ if both the function and its inverse map are univalent in $\mathbb{U}$. By the same token, a function is said to be bi-subordinate in $\mathbb{U}$ if both the function and its inverse map are subordinate to a given function in $\mathbb{U}$. In this paper, we consider the m-fold symmtric transform of such functions and use their Faber polynomial expansions to find upper bounds for their n-th ($n \geq 3$) coefficients subject to a given gap series condition. We also determine bounds for the first two coefficients of such functions with no restrictions imposed.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : \ |z| < 1\}$ and let $\mathcal{S}$ be the class of functions f that are analytic and univalent in $\mathbb{U}$ and are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

For $f(z)$ and $F(z)$ analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $F(z)$, written $f \prec F$, if there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{U}$ such that $f(z) = F(w(z))$. We note that $f(\mathbb{U}) \subset F(\mathbb{U})$ if both f and F are in S. Moreover, for the Schwarz function $w(z) = \sum_{n=1}^{\infty} w_n z^n$ we have $|w_n| \leq 1$ (e.g. see [3]).

For each function $f \in \mathcal{S}$, the m-fold symmetric function given by

$$f_m(z) = \sqrt[m]{f(z^m)} \ = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in \mathbb{U}, m \in \mathbb{N}),$$

is univalent in the unit disk $\mathbb{U}$ (e.g. see [3]). We denote the class of such functions by $\mathcal{S}_m$. The functions in the class $\mathcal{S}_1 = \mathcal{S}$ are univalent one-fold symmetric.

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Since the functions in $\mathcal{S}$ are one-to-one, they are invertible and their inverse maps need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem (e.g. see [3]) ensures that every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus every function $f \in \mathcal{S}$ has an inverse map f^{-1}, which is defined by $f^{-1}(f(z)) = z$ and $f(f^{-1}(w)) = w$ where $z \in \mathbb{U}$ and $|w| < r_0(f) \geq 1/4$.

It is easy to verify that for $f \in \mathcal{S}_1 = \mathcal{S}$ of the form (1.1), the inverse function $g = f^{-1}$ is given by

$$g(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

Similarly, for the m-fold symmetric function $f_m \in \mathcal{S}_m$, its inverse function $g_m = f_m^{-1}$ is of the form

$$\begin{aligned} & g_m(w) \qquad\qquad (1.3)\\ = \; & w - a_{m+1} w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}] w^{2m+1} \\ & - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots . \end{aligned}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both f and its inverse map $g = f^{-1}$ are univalent in $\mathbb{U}$. Similsarly, a function $f_m \in \mathcal{A}$ is said to be m-fold symmetric *bi-univalent* in $\mathbb{U}$ if both f_m and its inverse map $g_m = f_m^{-1}$ are univalent in $\mathbb{U}$. We let Σ_m be the class of all m-fold symmetric bi-univalent functions in $\mathbb{U}$. Obviously, for $m = 1$, the formula (1.3) coincides with the formula (1.2) of the class $\Sigma_1 = \Sigma$. For a brief history of functions in the class Σ, see the work of Srivastava et al. [9] and the references cited therein. The concept of m-fold symmetric bi-univalent functions has been introduced concurrently by Hamidi and Jahangiri [5] and Srivastava et al. [10]. Not much was known about the bounds of the general coefficients a_n $(n \geqq 4)$ of subclasses of bi-univalent functions up until the publication of the article [7] by Jahangiri and Hamidi who used the Faber polynomial series expansions to obtain bounds for the $n-$th coefficients a_n $(n \geqq 3)$ of certain subclasses of the normalized bi-univalent functions subject to a given gap series condition. Here we consider the m-fold symmetric transformation of a subordination version of a class of functions considered in [7] and obtain the upper bounds for the general coefficients $|a_{m(n-1)+1}|$ of such functions subject to a given gap series condition. We also determine the upper bounds for their first two coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ as well as bounds for their Feket-Szego coefficient body $\left|a_{2m+1} - \frac{m+1}{2}a_{m+1}^2\right|$. In general, our results are new on their own rights and in particular improve a few of the previously known results.

2. Main results

Let the function $\varphi \in \mathcal{A}$ have positive real part in $\mathbb{U}$ so that φ maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1, symmetric with respect to the real axis, $\varphi(0) = 1$ and $\varphi'(0) > 0$ (e.g. see [8]). Here we use the m-fold symmetric transformation of the function $\varphi \in \mathcal{A}$, denoted by $\varphi_m \in \mathcal{A}$. Obviously, by the properties of m-fold symmetric analytic functions (e.g. see [3]), φ_m is an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\varphi_m(0) = 1$, $\varphi_m^{(m)}(0) > 0$ and symmetric with respect to the real axis having the power series expansion

$$\varphi_m(z) = 1 + B_m z^m + B_{2m} z^{2m} + B_{3m} z^{3m} + \cdots \qquad (B_m > 0).$$

Using the above definition of functions $\varphi_m \in \mathcal{A}$ we introduce the following

Definition 2.1. A function $f_m \in \Sigma_m$ is said to be in the class $\Sigma_m(\lambda; \varphi_m)$ if

$$(1-\lambda)\frac{f_m(z)}{z} + \lambda f_m'(z) \prec \varphi_m(z) \qquad (z \in \mathbb{U}),$$

and

$$(1-\lambda)\frac{g_m(w)}{w}+\lambda g_m'(w)\prec\varphi_m(w)\qquad(w\in\mathbb{U}),$$

where $\lambda\geq 0,\ m\in\mathbb{N}$ and g_m is given by (1.3).

In order to prove our theorems in this section, we need to use the Faber polynomial expansions of inverse functions. For the function $f\in\mathcal{S}$ of the form (1.1), the coefficients of its inverse map $g=f^{-1}$ may be expressed (e.g. see [1] and [2]) by

$$g(w)=f^{-1}(w)=w+\sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\cdots)w^n,$$

where

$$\begin{aligned}K_{n-1}^{-n}&=\frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1}+\frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3\\&+\frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4+\frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}\\&\times[a_5+(-n+2)a_3^2]+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6+(-2n+5)a_3a_4]\\&+\sum_{j\geq7}a_2^{n-j}V_j,\end{aligned}$$

such that V_j with $7\leq j\leq n$ is a homogeneous polynomial in the variables $a_2,a_3,\cdots,a_n$. In particular, the first three terms of K_{n-1}^{-n} are

$$\frac{1}{2}K_1^{-2}=-a_2,\quad\frac{1}{3}K_2^{-3}=2a_2^2-a_3,\quad\frac{1}{4}K_3^{-4}=-(5a_2^3-5a_2a_3+a_4).$$

In general, for $n\geq1$ and real values of p, an expansion of K_{n-1}^p is (see [1,12] or [2, page 349])

$$K_{n-1}^p=pa_n+\frac{p(p-1)}{2}D_{n-1}^2+\frac{p!}{(p-3)!3!}D_{n-1}^3+\cdots+\frac{p!}{(p-n+1)!(n-1)!}D_{n-1}^{n-1},$$

where $D_{n-1}^p=D_{n-1}^p(a_2,a_3,\cdots,a_n)$ are homogeneous polynomials explicated in

$$D_{n-1}^p(a_2,a_3,\cdots,a_n)=\sum_{n=2}^{\infty}\frac{m!(a_2)^{\mu_1}\cdots(a_n)^{\mu_{n-1}}}{\mu_1!\cdots\mu_{n-1}!}\quad for\quad p\leq n-1,$$

and the sum is taken over all nonnegative integers $\mu_1,...,\mu_{n-1}$ satisfying

$$\begin{cases}\mu_1+\mu_2+\cdots+\mu_{n-1}=p,\\\mu_1+2\mu_2+\cdots+(n-1)\mu_{n-1}=n-1.\end{cases}$$

It is clear that $D_{n-1}^{n-1}(a_2,a_3,\cdots,a_n)=a_2^{n-1}$.

Now we are ready to state and prove our first theorem which provides an upper bound for the general coefficients of functions in $\Sigma_m(\lambda;\varphi_m)$ subject to a given gap series condition.

Theorem 2.2. *For $\lambda\geq0,\ m\in\mathbb{N}$, let the function $f_m\in\Sigma_m(\lambda;\varphi_m)$ be given by (1.3). If $a_k=0$ for $m+1\leq k\leq(n-2)m+1$, then*

$$|a_{(n-1)m+1}|\leq\frac{B_m}{[1+(n-1)m\lambda]}\qquad n\geq3.$$

Proof. By definition, for function $f_m\in\Sigma_m(\lambda;\varphi_m)$, we have

$$(1-\lambda)\frac{f_m(z)}{z}+\lambda f_m'(z)=1+\sum_{n=2}^{\infty}[1+(n-1)m\lambda]a_{(n-1)m+1}z^{(n-1)m},\tag{2.1}$$

and for its inverse map, $g_m = f_m^{-1}$, we obtain

$$\begin{aligned} & (1-\lambda)\frac{g_m(w)}{w} + \lambda g'_m(w) \qquad (2.2) \\ = \; & 1 + \sum_{n=2}^{\infty}[1+(n-1)m\lambda] b_{(n-1)m+1} w^{(n-1)m} \\ = \; & 1 + \sum_{n=2}^{\infty}[1+(n-1)m\lambda] \frac{1}{n} K_{n-1}^{-n}(a_{m+1}, a_{2m+1}, \cdots, a_{(n-1)m+1}) w^{(n-1)m}. \end{aligned}$$

On the other hand, since $f_m \in \Sigma_m(\lambda; \varphi_m)$, by the definition of subordination, there exist two Schwarz functions $P_m, Q_m : \mathbb{U} \to \mathbb{U}$ with

$$P_m(z) = \sum_{n=1}^{\infty} p_{nm} z^{nm} = p_m z^m + \cdots \qquad \textit{and}$$

$$Q_m(w) = \sum_{n=1}^{\infty} q_{nm} w^{nm} = q_m w^m + \cdots,$$

so that

$$\begin{aligned} & (1-\lambda)\frac{f_m(z)}{z} + \lambda f'_m(z) \qquad (2.3) \\ = \; & \varphi_m(P_m(z)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} B_{km} D_n^k(p_m, p_{2m}, \cdots, p_{nm}) z^{nm}, \end{aligned}$$

and

$$\begin{aligned} & (1-\lambda)\frac{g_m(w)}{w} + \lambda g'_m(w) \qquad (2.4) \\ = \; & \varphi_m(Q_m(w)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} B_{km} D_n^k(q_m, q_{2m}, \cdots, q_{nm}) w^{nm}. \end{aligned}$$

Comparing the corresponding coefficients of (2.1) and (2.3), we obtain

$$[1+(n-1)m\lambda] a_{(n-1)m+1} = \sum_{k=1}^{n-1} B_{km} D_{n-1}^k(p_m, p_{2m}, \cdots, p_{(n-1)m}). \qquad (2.5)$$

Similarly, by comparing the corresponding coefficients of (2.2) and (2.4), we obtain

$$\begin{aligned} & [1+(n-1)m\lambda] \frac{1}{n} K_{n-1}^{-n}(a_{m+1}, a_{2m+1}, \cdots, a_{(n-1)m+1}) \\ = \; & \sum_{k=1}^{n-1} B_{km} D_{n-1}^k(q_m, q_{2m}, \cdots, q_{(n-1)m}). \qquad (2.6) \end{aligned}$$

Letting $a_k = 0$ for $m+1 \leq k \leq (n-2)m+1$ yields $b_{(n-1)m+1} = -a_{(n-1)m+1}$ and hence

$$[1+(n-1)m\lambda] a_{(n-1)m+1} = B_m p_{(n-1)m},$$

and

$$-[1+(n-1)m\lambda] a_{(n-1)m+1} = B_m q_{(n-1)m}.$$

Now taking the absolute values of either of the above two equations and using the facts that $|p_{(n-1)m}| \leq 1$ and $|q_{(n-1)m}| \leq 1$, we obtain

$$|a_{(n-1)m+1}| \leq \frac{B_m |p_{(n-1)m}|}{[1+(n-1)m\lambda]} = \frac{B_m |q_{(n-1)m}|}{[1+(n-1)m\lambda]} \leq \frac{B_m}{[1+(n-1)m\lambda]}.$$

□

Our next two theorems provide bounds for the first two coefficients of certain subclasses of $\Sigma_m(\lambda;\varphi_m)$ with no gap series restrictions imposed.

Theorem 2.3. *For $\lambda \geq 0,\ m \in \mathbb{N}$ and $0 \leq \beta < 1$ let $f_m \in \Sigma_m\left(\lambda; \frac{1+(1-2\beta)z^m}{1-z^m}\right)$. Then*

$$|a_{m+1}| \leq \min\left\{\frac{2(1-\beta)}{1+m\lambda}, \sqrt{\frac{4(1-\beta)}{(1+2m\lambda)(m+1)}}\right\}$$

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{1+2m\lambda},$$

and

$$\left|a_{2m+1} - \frac{m+1}{2}a_{m+1}^2\right| \leq \frac{2(1-\beta)}{1+2m\lambda}.$$

Proof. The equations (2.5) and (2.6) for $n = 2$ and $n = 3$, respectively, imply

$$(1+m\lambda)a_{m+1} = 2(1-\beta)p_m, \tag{2.7}$$

$$(1+2m\lambda)a_{2m+1} = 2(1-\beta)p_{2m} + 2(1-\beta)p_m^2, \tag{2.8}$$

$$-(1+m\lambda)a_{m+1} = 2(1-\beta)q_m, \tag{2.9}$$

$$(1+2m\lambda)[(m+1)a_{m+1}^2 - a_{2m+1}] = 2(1-\beta)q_{2m} + 2(1-\beta)q_m^2. \tag{2.10}$$

Taking absolute values of (2.7) or (2.9), we get

$$|a_{m+1}| \leq \frac{2(1-\beta)}{1+m\lambda}.$$

Also by adding (2.8) and (2.10), we have

$$(1+2m\lambda)(m+1)a_{m+1}^2 = 2(1-\beta)\left[(p_{2m}+p_m^2) + (q_{2m}+q_m^2)\right].$$

Taking the absolute values of the above equation yields

$$(1+2m\lambda)(m+1)|a_{m+1}|^2 \leq 2(1-\beta)\left[|p_{2m}+p_m^2| + |q_{2m}+q_m^2|\right].$$

Now by using [6, Corollary 2.3], we have

$$(1+2m\lambda)(m+1)|a_{m+1}|^2 \leq 2(1-\beta)\left[1+(1-1)|p_m|^2 + 1 + (1-1)|q_m|^2\right].$$

Therefore,

$$|a_{m+1}| \leq \sqrt{\frac{4(1-\beta)}{(1+2m\lambda)(m+1)}}.$$

Next, by solving (2.8) for a_{2m+1}, taking the absolute values and using [6, Corollary 2.3] we get

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{1+2m\lambda}\left[1+(1-1)|p_m|^2\right] = \frac{2(1-\beta)}{1+2m\lambda}.$$

Finally, subtracting (2.10) from (2.8) and considering the fact that $p_m^2 = q_m^2$ we obtain

$$2(1+2m\lambda)\left(a_{2m+1} - \frac{m+1}{2}a_{m+1}^2\right) = 2(1-\beta)\left(p_{2m}-q_{2m}\right).$$

Taking the absolute values of both sides and using the fact that $|p_{2m}| \leq 1$ and $|q_{2m}| \leq 1$ we obtain

$$\left|a_{2m+1} - \frac{m+1}{2}a_{m+1}^2\right| \leq \frac{2(1-\beta)}{1+2m\lambda}.$$

This completes the proof. □

Theorem 2.4. *For $\lambda \geq 0$, $m \in \mathbb{N}$ and $0 < \alpha \leq 1$ let $f_m \in \Sigma_m\left(\lambda; \left(\frac{1+z^m}{1-z^m}\right)^{\alpha}\right)$. Then*

$$|a_{m+1}| \leq \min\left\{\frac{2\alpha}{1+m\lambda}, \frac{2\alpha}{\sqrt{(1+m\lambda)^2+\alpha m(1+2m\lambda-m\lambda^2)}}\right\} \tag{2.11}$$

$$|a_{2m+1}| \leq \frac{2\alpha}{1+2m\lambda},$$

and

$$\left|a_{2m+1}-\frac{m+1}{2}a_{m+1}^2\right| \leq \frac{2\alpha}{1+2m\lambda}.$$

Proof. The equations (2.5) and (2.6) for $n=2$ and $n=3$, respectively, imply

$$(1+m\lambda)a_{m+1} = 2\alpha p_m, \tag{2.12}$$

$$(1+2m\lambda)a_{2m+1} = 2\alpha p_{2m} + 2\alpha^2 p_m^2, \tag{2.13}$$

$$-(1+m\lambda)a_{m+1} = 2\alpha q_m, \tag{2.14}$$

$$(1+2m\lambda)[(m+1)a_{m+1}^2 - a_{2m+1}] = 2\alpha q_{2m} + 2\alpha^2 q_m^2. \tag{2.15}$$

Taking the absolute values of (2.12) or (2.14), we get

$$|a_{m+1}| \leq \frac{2\alpha}{1+m\lambda}. \tag{2.16}$$

Also by adding (2.13) and (2.15), we have

$$(1+2m\lambda)(m+1)a_{m+1}^2 = 2\alpha\left[(p_{2m}+\alpha p_m^2)+(q_{2m}+\alpha q_m^2)\right].$$

Taking the absolute values of the above equation yields

$$(1+2m\lambda)(m+1)|a_{m+1}|^2 \leq 2\alpha\left[|p_{2m}+\alpha p_m^2|+|q_{2m}+\alpha q_m^2|\right].$$

Now, for $0<\alpha\leq 1$ we use [6, Corollary 2.3], to obtain

$$(1+2m\lambda)(m+1)|a_{m+1}|^2 \leq 2\alpha\left[1+(\alpha-1)|p_m|^2+1+(\alpha-1)|q_m|^2\right].$$

Solve the above equation for $|a_{m+1}|$ and apply the fact that $|p_m|^2 = |q_m|^2 = \frac{(1+m\lambda)^2|a_{m+1}|^2}{4\alpha^2}$ to obtain

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(1+m\lambda)^2+\alpha m(1+2m\lambda-m\lambda^2)}}. \tag{2.17}$$

So, (2.16) in conjunction with (2.17) yield (2.11).

Next, we solve (2.13) for a_{2m+1}, take the absolute values and apply [6, Corollary 2.3] to obtain

$$|a_{2m+1}| \leq \frac{2\alpha}{1+2m\lambda}\left[1+(\alpha-1)|p_m|^2\right] \leq \frac{2\alpha}{1+2m\lambda}.$$

Finally, subtracting (2.15) from (2.13) and considering the fact that $p_m^2 = q_m^2$ we obtain

$$2(1+2m\lambda)\left(a_{2m+1}-\frac{m+1}{2}a_{m+1}^2\right) = 2\alpha\left(p_{2m}-q_{2m}\right).$$

Taking the absolute values of both sides and using the fact that $|p_{2m}| \leq 1$ and $|q_{2m}| \leq 1$ we obtain

$$\left|a_{2m+1}-\frac{m+1}{2}a_{m+1}^2\right| \leq \frac{2\alpha}{1+2m\lambda}.$$

This completes the proof. □

Remark 2.5. Theorem 2.2 for $m=1$ and $\varphi_1(z) = \frac{1+(1-2\beta)z}{1-z}$ yields the estimates obtained by Jahangiri and Hamidi [7, Theorem 1].

Remark 2.6. Theorems 2.3 and 2.4 are improvements of the estimates obtained by Sümer Eker [11, Theorems 2 and 1], respectively.

Remark 2.7. Theorems 2.3 and 2.4 for $m = 1$ are improvements of the estimates obtained by Frasin and Aouf [4, Theorems 3.2 and 2.2], respectively.

Remark 2.8. Theorems 2.3 and 2.4 for $\lambda = 1$ are improvements of the estimates obtained by Srivastava et al. [10, Theorems 3 and 2], respectively.

Remark 2.9. Letting $\lambda = 1$ in Theorem 2.3 yields the following bounds for $|a_2|$ and $|a_3|$ which are improvements of the estimates obtained by Srivastava et al. [9, Theorem 2]

$$|a_2| \leq \begin{cases} 1-\beta & \dfrac{1}{3} \leq \beta < 1, \\ \sqrt{\dfrac{2(1-\beta)}{3}} & 0 \leq \beta < \dfrac{1}{3}. \end{cases}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{3}.$$

Remark 2.10. Letting $\lambda = 1$ in Theorem 2.4 we obtain $|a_3| \leq (2\alpha/3)$ which is an improvement of the estimate obtained by Srivastava et al. [9, Theorem 1].

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