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ON STRONG $N_{\theta}^{\alpha}(A,F)-$CONVERGENCE

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Abstract. In the papers [T. Bilgin, Studia Univ. Babeş-Bolyai Math. 46(4), (2001), 39–46] and [T. Bilgin, Appl. Math. Comput. 151(3), (2004), 595–600], author defined the spaces of strongly $N_{\theta}\left(A,f\right)-$convergent with respect to a modulus sequences and strongly $N_{\theta}\left(A,F\right)-$convergent with respect to a sequence of modulus functions sequences. In this paper, we introduce strong $N_{\theta}^{\alpha}\left(A,F\right)-$convergence with respect to a sequence of modulus functions and give some connections between sets of strongly $N_{\theta}^{\alpha}\left(A,F\right)-$convergent with respect to a sequence of modulus functions sequences and $S_{\theta}^{\alpha}(A)-$convergent sequences.

1. Introduction

In 1951, Steinhaus [33] and Fast [17] introduced the concept of statistical convergence and later in 1959, Schoenberg [32] reintroduced independently. Bhardwaj and Dhawan [4], Caserta et al. [5], Connor [6], Çakallı [9], Çınar et al. [10], Çolak [11], Et et al. ([13], [15]), Fridy [19], Işık [24], Salat [31], Di Maio and Kočinac [12], Demirci [7] and many authors investigated some arguments related to this notion.

A modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

i) $f(x)=0$ if and only if $x=0$,

ii) $f(x+y)\leq f(x)+f(y)$ for $x,y\geq 0$,

iii) f is increasing,

iv) f is continuous from the right at 0.

It follows that f must be continuous in everywhere on $[0,\infty)$. A modulus may be unbounded or bounded.

By a lacunary sequence we mean an increasing integer sequence $\theta=(k_r)$ of non-negative integers such that $k_0=0$ and $h_r=(k_r-k_{r-1})\rightarrow\infty$ as $r\rightarrow\infty$. The intervals determined by θ will be denoted by $I_r=(k_{r-1},k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1=k_1$ for convenience.

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In [20], Fridy and Orhan introduced the concept of lacunary statistically convergence in the sense that a sequence (x_k) of real numbers is called lacunary statistically convergent to a real number ℓ, if

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\{k\in I_r : |x_k-\ell|\geq\varepsilon\}\right|=0$$

for every positive real number ε.

Lacunary convergence and lacunary statistical convergence were studied in ([1], [8],[16],[18],[20],[22],[23],[25],[35],[29],[37],[38]).

The notion of a modulus was given by Nakano [27]. Maddox [26] used a modulus function to construct some sequence spaces. Afterwards different sequence spaces defined by modulus have been studied by Altın and Et [3], Et et al. [14], Işık [24], Gaur and Mursaleen [21], Nuray and Savaş [28], Pehlivan and Fisher [30], Şengül [34] and everybody else.

2. Main Results

In this section, we will give the definition of lacunary strong $N_\theta^\alpha(A,F)-$convergence where $A=(a_{ik})$ is an infinite matrix of complex numbers and $0<\alpha\leq 1$ and give some results related to this concept.

Definition 1. [2] *Let $A=(a_{ik})$ be an infinite matrix of complex numbers. If $A_i(x)=\sum_{k=1}^{\infty}a_{ik}x_k$ converges for each i then $Ax=(A_i(x))$ such that*

$$N_\theta(A,F)=\left\{x=(x_i):\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}f_i\left(|A_i(x)-\ell|\right)=0 \text{ for some } \ell\right\}$$

where $F=(f_i)$ is a sequence of modulus functions such that $\lim_{u\longrightarrow 0^+}\sup_i f_i(u)=0$.

Definition 2. *Let $A=(a_{ik})$ be an infinite matrix of complex numbers, $F=(f_i)$ be a sequence of modulus functions and $0<\alpha\leq 1$. We say that the sequence $x=(x_k)$ is lacunary strong $A-$convergent of order α to a number ℓ with respect to a sequence of modulus functions (or $N_\theta^\alpha(A,F)-$convergent to ℓ) if*

$$N_\theta^\alpha(A,F)=\left\{x=(x_i):\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{i\in I_r}f_i(|A_i(x)-\ell|)=0 \text{ for some } \ell\right\}.$$

In this case, we write $x_i\to\ell(N_\theta^\alpha(A,F))$ or $N_\theta^\alpha(A,F)-\lim x_i=\ell$. Note that, if we get $f_i=f$, then $N_\theta^\alpha(A,F)=N_\theta^\alpha(A,f)$. If $A=I$ unit matrix, we write $N_\theta^\alpha(F)$ for $N_\theta^\alpha(A,F)$.

$N_\theta^\alpha(A,F)$ are linear spaces. Suppose that $x_i\to\ell(N_\theta^\alpha(A,F))$ and $y_i\to\ell'(N_\theta^\alpha(A,F))$ to show it. Then there exist integers T_1 and T_2 such that $|a|\leq T_1$ and $|b|\leq T_2$. We have

$$\frac{1}{h_r^\alpha}\sum_{i\in I_r}f_i(\left|A_i(ax+by)-(a\ell+b\ell^{'})\right|)$$

$$\begin{aligned} &= \frac{1}{h_r^\alpha}\sum_{i\in I_r} f_i(\left|a(A_i(x)-\ell)+b(A_i(y)-\ell^{'})\right|) \\ &\leq \frac{1}{h_r^\alpha}\sum_{i\in I_r}(f_i(|a(A_i(x)-\ell)|)+f_i(\left|b(A_i(y)-\ell^{'})\right|)) \\ &\leq T_1\,\frac{1}{h_r^\alpha}\sum_{i\in I_r} f_i(|A_i(x)-\ell|)+T_2\,\,\frac{1}{h_r^\alpha}\sum_{i\in I_r} f_i(\left|A_i(y)-\ell^{'}\right|). \end{aligned}$$

This implies that $ax+by\longrightarrow a\ell+b\ell^{'}(N_\theta^\alpha(A,F))$.

Definition 3. *Let* $A=(a_{ik})$ *be an infinite matrix of complex numbers,* $F=(f_i)$ *be a sequence of modulus functions and* $0<\alpha\leq 1$. *We say that the sequence* $x=(x_k)$ *is strong* $A-$*convergent of order* α *to a number* ℓ *with respect to a sequence of modulus functions (or* $w^\alpha(A,F)-$*convergent to* ℓ*) if*

$$w^\alpha(A,F)=\left\{x=(x_i):\lim_{n\to\infty}\frac{1}{n^\alpha}\sum_{i=1}^{n} f_i(|A_i\,(x)-\ell|)=0 \text{ for some } \ell\right\}.$$

In this case, we write $x_i\to\ell(w^\alpha(A,F))$. *Note that, if we get* $f_i=f$, *then* $w^\alpha(A,F)=w^\alpha(A,f)$. *If* $A=I$ *unit matrix, we write* $w^\alpha(F)$ *for* $w^\alpha(A,F)$.

Definition 4. [36] *Let* $A=(a_{ik})$ *be an infinite matrix of complex numbers. Then a sequence* $x=(x_k)$ *is said to be lacunary* $A-$*statistical convergent to a number* ℓ *(or* $S_\theta^\alpha(A)-$*convergent to* ℓ *) if for every* $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\left|\{i\in I_r:|A_i(x)-\ell|\geq\varepsilon\}\right|=0.$$

The set of all lacunary $A-$*statistical convergence sequences of order* α *will be denoted by* $S_\theta^\alpha(A)$. *If* $\theta=2^r$, *we write* $S^\alpha(A)$ *instead of* $S_\theta^\alpha(A)$.

Theorem 5. *If* $N_\theta^\alpha(A,F)-\lim x_i=\ell_1$ *and* $N_\theta^\alpha(A,F)-\lim x_i=\ell_2$, *then* $\ell_1=\ell_2$.

Proof. Since $N_\theta^\alpha(A,F)-\lim x_i=\ell_1$ and $N_\theta^\alpha(A,F)-\lim x_i=\ell_2$, we can write

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{i\in I_r} f_i(|A_i\,(x)-\ell_1|)=0$$

and

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{i\in I_r} f_i(|A_i\,(x)-\ell_2|)=0.$$

We have

$$\begin{aligned} |\ell_1-\ell_2| &= |\ell_1-\ell_2+A_i(x)-A_i(x)| \\ &\leq |A_i(x)-\ell_1|+|A_i(x)-\ell_2|\,. \end{aligned}$$

We get

$$\begin{aligned}\frac{1}{h_r^\alpha}\sum_{i\in I_r} f_i(|\ell_1-\ell_2|) &= \frac{1}{h_r^\alpha}\sum_{i\in I_r} f_i(|\ell_1-\ell_2+A_i(x)-A_i(x)|)\\ &\le \frac{1}{h_r^\alpha}\sum_{i\in I_r} f_i(|A_i(x)-\ell_1|)+\frac{1}{h_r^\alpha}\sum_{i\in I_r} f_i(|A_i(x)-\ell_2|).\end{aligned}$$

This is possible with $\ell_1=\ell_2$. □

Theorem 6. *Let $0<\alpha\le 1$. If $\lim\limits_{u\to\infty}\inf_i \frac{f_i(u)}{u}>0$, then $N_\theta^\alpha(A,F)\subseteq N_\theta^\alpha(A)$.*

Proof. If $\lim\limits_{u\to\infty}\inf_i \frac{f_i(u)}{u}>0$, then there exist a number $\beta>0$ such that $f_i(u)\ge\beta u$ for all $u>0$ and $i\in\mathbb{N}$. Let $x\in N_\theta^\alpha(A,F)$. It is clear that

$$\frac{1}{h_r^\alpha}\sum_{i\in I_r} f_i(|A_i(x)-\ell|)\ge\frac{1}{h_r^\alpha}\sum_{i\in I_r}\beta\,|A_i(x)-\ell|=\beta\frac{1}{h_r^\alpha}\sum_{i\in I_r}|A_i(x)-\ell|\,.$$

Therefore $x_i\to\ell(N_\theta^\alpha(A))$.

If $\beta=0$, then $N_\theta^\alpha(A,F)\subseteq N_\theta^\alpha(A)$ may not be provided. Consider $A=I$ and $f_i(x)=x^{\frac{2}{i}}(i\ge 1,x>0)$. Define $x=(x_i)$ by for $r=1,2,3,...$

$$x_i=\begin{cases}\sqrt{h_r}, & \text{if}\quad i=k_r\\ 0, & \quad\text{otherwise.}\end{cases}$$

We can write

$$\frac{1}{h_r^\alpha}\sum_{i\in I_r} f_i(|A_i(x)|)=\frac{1}{h_r^\alpha}f_{k_r}(\sqrt{h_r})=\frac{1}{h_r^\alpha}h_r^{\frac{1}{k_r}}\to 0,\quad (\text{as } r\to\infty)$$

for $\alpha>\frac{1}{k_r}$ and so $x\in N_\theta^{0,\alpha}(A,F)\subseteq N_\theta^\alpha(A,F)$. But

$$\frac{1}{h_r^\alpha}\sum_{i\in I_r}|A_i(x)|=\frac{1}{h_r^\alpha}\sum_{i\in I_r}|x_i|=\frac{1}{h_r^\alpha}\sqrt{h_r}\to 1,\quad(\text{as } r\to\infty)$$

for $\alpha=\frac{1}{2}$ and

$$\frac{1}{h_r^\alpha}\sum_{i\in I_r}|A_i(x)|=\frac{1}{h_r^\alpha}\sum_{i\in I_r}|x_i|=\frac{1}{h_r^\alpha}\sqrt{h_r}\to\infty,\ (\text{as } r\to\infty)$$

for $\alpha<\frac{1}{2}$. $x\notin N_\theta^{0,\alpha}(A)\subseteq N_\theta^\alpha(A)$ is obtained. As a result $\beta>0$ must be. □

Theorem 7. *Let (f_i) be pointwise convergent. If $\lim\limits_i f_i(u)>0$ for $u>0$, then $N_\theta^\alpha(A,F)\subseteq S_\theta^\alpha(A)$ for $0<\alpha\le 1$.*

Proof. Let $\varepsilon > 0$ and $x_i \to \ell(N_\theta^\alpha(A,F))$. If $\lim_i f_i(u) > 0$, then there exist a number $\rho > 0$ such that $f_i(\varepsilon) > \rho$ for $u > \varepsilon$ and $i \in \mathbb{N}$. We have

$$\begin{aligned}\frac{1}{h_r^\alpha}\sum_{i\in I_r} f_i(|A_i(x)-\ell|) &\geq \frac{1}{h_r^\alpha}\sum_{\substack{i\in I_r \\ |A_i(x)-\ell|\geq\varepsilon}} f_i(|A_i(x)-\ell|) \\ &\geq \frac{1}{h_r^\alpha}\left|\{i\in I_r : |A_i(x)-\ell|\geq\varepsilon\}\right| f_i(\varepsilon) \\ &\geq \rho\frac{1}{h_r^\alpha}\left|\{i\in I_r : |A_i(x)-\ell|\geq\varepsilon\}\right|\end{aligned}$$

for $0 < \alpha \leq 1$. It follows that $x_i \to \ell(S_\theta^\alpha(A))$. □

Theorem 8. *Let* $0 < \alpha \leq 1$. *If* $\lim_i f_i(u) > 0$ *for* $u > 0$, *then* $w^\alpha(A,F) \subseteq S^\alpha(A)$.

Proof. Let $x_i \to \ell(w^\alpha(A,F))$ be. From Theorem 7, we can write

$$\begin{aligned}\frac{1}{n^\alpha}\sum_{i=1}^{n} f_i(|A_i(x)-\ell|) &\geq \frac{1}{n^\alpha}\sum_{\substack{i=1 \\ |A_i(x)-\ell|\geq\varepsilon}}^{n} f_i(|A_i(x)-\ell|) \\ &\geq \frac{1}{n^\alpha}\left|\{i\leq n : |A_i(x)-\ell|\geq\varepsilon\}\right| f_i(\varepsilon) \\ &\geq \rho\frac{1}{n^\alpha}\left|\{i\leq n : |A_i(x)-\ell|\geq\varepsilon\}\right|\end{aligned}$$

and so $x_i \to \ell(S^\alpha(A))$. □

Theorem 9. *i) If* $\liminf q_r > 1$, *then* $w^\alpha(A,F) \subseteq N_\theta^\alpha(A,F)$, *for* $0 < \alpha \leq 1$.
ii) If $\limsup \frac{k_r}{k_{r-1}^\alpha} < \infty$, *then* $N_\theta(A,F) \subseteq w^\alpha(A,F)$, *for* $0 < \alpha \leq 1$.

Proof. *i)* Let $x_i \to \ell(w^\alpha(A,F))$ and $\liminf q_r > 1$. There exist a $\delta > 0$ such that $q_r = \frac{k_r}{k_{r-1}} \geq 1 + \delta$. We have

$$\left(\frac{h_r}{k_r}\right) \geq \frac{\delta}{\delta+1} \Rightarrow \left(\frac{h_r}{k_r}\right)^\alpha \geq \left(\frac{\delta}{\delta+1}\right)^\alpha$$

for $0 < \alpha \leq 1$. We can write

$$\begin{aligned}\frac{1}{k_r^\alpha}\sum_{i=1}^{k_r} f_i(|A_i(x)-\ell|) &\geq \frac{1}{k_r^\alpha}\sum_{i\in I_r} f_i(|A_i(x)-\ell|) \\ &= \left(\frac{h_r^\alpha}{k_r^\alpha}\right)\frac{1}{h_r^\alpha}\sum_{i\in I_r} f_i(|A_i(x)-\ell|) \\ &\geq \left(\frac{\delta}{\delta+1}\right)^\alpha\frac{1}{h_r^\alpha}\sum_{i\in I_r} f_i(|A_i(x)-\ell|).\end{aligned}$$

$x_i \to \ell(N_\theta^\alpha(A,F))$ is obtained.

ii) If $\limsup \frac{k_r}{k_{r-1}^{\alpha}} < \infty$, then there is $M > 0$ such that $\frac{k_r}{k_{r-1}^{\alpha}} < M$ for $r \geq 1$. Now suppose that $x \in N_{\theta}^{0}(A, F)$ and $\varepsilon > 0$. We can find $R > 0$ and $K > 0$ numbers such that $\sup_{i \geq R} \tau_i < \varepsilon$ and $\tau_i < K$ for every $i = 1, 2, 3, ...$ Let t be any integer with $k_{r-1} < t \leq k_r$. For $r > R$ and $0 < \alpha \leq 1$

$$\frac{1}{t^{\alpha}} \sum_{i=1}^{t} f_i(|A_i(x)|) \leq \frac{1}{k_{r-1}^{\alpha}} \sum_{i=1}^{k_r} f_i(|A_i(x)|)$$

$$= \frac{1}{k_{r-1}^{\alpha}} \left(\sum_{I_1} f_i(|A_i(x)|) + \sum_{I_2} f_i(|A_i(x)|) + ... + \sum_{I_r} f_i(|A_i(x)|) \right)$$

$$= \frac{k_1}{k_{r-1}^{\alpha}} \tau_1 + \frac{k_2 - k_1}{k_{r-1}^{\alpha}} \tau_2 + ... + \frac{k_R - k_{R-1}}{k_{r-1}^{\alpha}} \tau_R + \frac{k_{R+1} - k_R}{k_{r-1}^{\alpha}} \tau_{R+1} + ... + \frac{k_r - k_{r-1}}{k_{r-1}^{\alpha}} \tau_r$$

$$\leq \left(\sup_{i \geq 1} \tau_i \right) \frac{k_R}{k_{r-1}^{\alpha}} + \left(\sup_{i \geq R} \tau_i \right) \frac{k_r - k_R}{k_{r-1}^{\alpha}} < K \frac{k_R}{k_{r-1}^{\alpha}} + \varepsilon M.$$

We deduce $x \in w^{0,\alpha}(A, F)$. □

Theorem 10. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let $\alpha_1, \alpha_2, \beta_1$ and β_2 be such that $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$,*

(i) If

$$\lim_{r \to \infty} \inf \frac{h_r^{\alpha_1}}{\ell_r^{\alpha_2}} > 0 \tag{1}$$

then $N_{\theta',\alpha_2}^{\beta_2}(A, F) \subset N_{\theta,\alpha_1}^{\beta_1}(A, F)$,

(ii) If the modulus $F = (f_i)$ is bounded and

$$\lim_{r \to \infty} \frac{\ell_r}{h_r^{\alpha_2}} = 1 \tag{2}$$

then $N_{\theta,\alpha_1}^{\beta_2}(A, F) \subset N_{\theta',\alpha_2}^{\beta_1}(A, F)$.

Proof. (i) Let $x \in N_{\theta',\alpha_2}^{\beta_2}(A, F)$. We can write

$$\frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k \in J_r} f_i(|A_i(x) - \ell|) \right)^{\beta_2} \geq \frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k \in I_r} f_i(|A_i(x) - \ell|) \right)^{\beta_2}$$

$$\geq \frac{h_r^{\alpha_1}}{\ell_r^{\alpha_2}} \frac{1}{h_r^{\alpha_1}} \left(\sum_{k \in I_r} f_i(|A_i(x) - \ell|) \right)^{\beta_1}.$$

Thus if $x \in N_{\theta',\alpha_2}^{\beta_2}(A, F)$, then $x \in N_{\theta,\alpha_1}^{\beta_1}(A, F)$.

(ii) Let $x = (x_k) \in N_{\theta,\alpha_1}^{\beta_2}(A, F)$ and (2) holds. Assume that $F = (f_i)$ is bounded. Therefore $f_i(x) \leq K$, for a positive integer K and all $x \geq 0$. Now, since $I_r \subseteq J_r$

and $h_r \leq \ell_r$ for all $r \in N$, we can write

$$\begin{aligned}
\frac{1}{\ell_r^{\alpha_2}}\left(\sum_{k\in J_r} f_i(|A_i(x)-\ell|)\right)^{\beta_1} &= \frac{1}{\ell_r^{\alpha_2}}\left(\sum_{k\in J_r-I_r} f_i(|A_i(x)-\ell|)\right)^{\beta_1} \\
&\quad +\frac{1}{\ell_r^{\alpha_2}}\left(\sum_{k\in I_r} f_i(|A_i(x)-\ell|)\right)^{\beta_1} \\
&\leq \left(\frac{\ell_r-h_r}{\ell_r^{\alpha_2}}\right)^{\beta_1} K^{\beta_1}+\frac{1}{\ell_r^{\alpha_2}}\left(\sum_{k\in I_r} f_i(|A_i(x)-\ell|)\right)^{\beta_1} \\
&\leq \left(\frac{\ell_r-h_r^{\alpha_2}}{h_r^{\alpha_2}}\right) K^{\beta_1}+\frac{1}{h_r^{\alpha_2}}\left(\sum_{k\in I_r} f_i(|A_i(x)-\ell|)\right)^{\beta_2} \\
&\leq \left(\frac{\ell_r}{h_r^{\alpha_2}}-1\right) K^{\beta_1}+\frac{1}{h_r^{\alpha_1}}\left(\sum_{k\in I_r} f_i(|A_i(x)-\ell|)\right)^{\beta_2}
\end{aligned}$$

for every $r \in \mathbb{N}$. Therefore $N_{\theta,\alpha_1}^{\beta_2}(A,F) \subset N_{\theta',\alpha_2}^{\beta_1}(A,F)$. □

Now as a result of Theorem 10 we have the following Corollary 11.

Corollary 11. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$.*

(i) If (1) holds then, $N_{\theta'}(A,F) \subset N_{\theta}(A,F)$ for $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 1$.

(ii) If (2) holds then, $N_{\theta}(A,F) \subset N_{\theta'}(A,F)$ for $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 1$.

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