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Elastoplastik-Mikro Yapı Modellerinde Ortaya Çıkan Doğrusal Olmayan Evolüsyon Denklemi İçin Varlık Sonuçları

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Öz

Bu çalışmada, sınırlı bir alanda elastoplastik-mikroyapı modellerinde ortaya çıkan doğrusal olmayan bir evrim denklemi için global varlık sonuçları potential well metodu kullanılarak oluşturulmuştur. Potential well yöntemi için bir fonksiyonel tanımlanmış ve bu fonksiyonelin işaret değişmezliği kullanılarak yüksek başlangıç enerjili durumda global varlık kanıtlanmıştır.

***Anahtar Kelimeler**:* Evolüsyon denklemi, Yüksek başlangıç enerjisi, Global çözüm, Başlangıç-sınır değer problemi.

Existence Results for a Nonlinear Evolution Equation Arising in Elastoplastic-Microstructure Models

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Abstract

We establish global existence results for a nonlinear evolution equation which arises in elastoplastic-microstructure models on a bounded domain, employing potential well method. A functional is defined for the potential well method, and global existence is proved by use of sign invariance of this functional in the case of high initial energy.

Keywords: Evolution equation, High initial energy level, Global solution, Initial-boundary value problem.

1. INTRODUCTION

The present paper considers the nonlinear evolution equation of the form

$$u_{tt}+\Delta^2 u+\gamma u_t+\sum_{k=1}^{n}\left(\rho_k\left(u_{x_k}\right)\right)_{x_k}=0,\ \ x\in\Omega,\ \ t>0, \tag{1.1}$$

$$u(x,0)=\varphi(x),\ \ u_t(x,0)=\psi(x),\ \ x\in\Omega, \tag{1.2}$$

$$u=\frac{\partial u}{\partial\eta}=0,\ \ x\in\partial\Omega,\ \ t\geq 0, \tag{1.3}$$

where $\gamma\geq 0,\ \Omega\subset R^n$ is a bounded domain and η denotes unit outward normal of Ω.

For $n=1$, Eq. (1.1) without a dissipative term and with a source term becomes

$$u_{tt}+u_{xxxx}=a(u_x^2)_x+f(x,t) \tag{1.4}$$

which is a class of nonlinear evolution equations describing the motion of an elastoplastic bar [1]. Eq. (1.4) with several source terms and damping terms has been investigated in may papers [2-6, 7,8]Initial-boundary value problem of (1.4) is considered in [2,3]. The authors proved in [2] that under different conditions imposed on the term a and its first two derivatives, their problem has a unique generalized global solution and unique classical global solution. Furthermore, some blow-up results were given in [2]. Global existence of weak solutions, classical solutions and generalized solutions were proved in [3] via potential well method. Problem (1.1)-(1.3) was studied in [4] with a source and without a damping term. Qualitative behavior of solutions is analyzed, blow up, convergence to the ground state as $t\to\infty$ and boundedness are characterized by using this method. Problem (1.1)-(1.3) was also studied in [5,6]. In both [5,6] existence of global solutions were proved by using the same method. In [6], existence of global solution was proved for $0<E(0)<d_0$, and it was

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extended to $0 \le E(0) < d,\ d > d_0$ in [5]. But the case of high initial energy ,i.e. $E(0) > d$ for arbitrary d is still open. In this paper, we deal with this open problem. For this purpose, we use potential well method with a functional which include both of the initial data. The case of $E(0) > d$, by using potential well method, is considered in few papers [9-13].

Throughout this paper, $\|f\|_p$, $\|f\|$ and $\|f\|_\infty$ will be used instead of norms of $L^p(\mathbb{R}^n)$, $L^2(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$, respectively. We also use the following abbreviations:

$$W_0^{m,p} = W_0^{m,p}(\Omega),\ C^k = C^k(\Omega),\ C_0^\infty = C_0^\infty(\Omega)$$
$$H^m = W^{m,2},\ H_0^m = W_0^{m,2}.$$

$(\cdot,\cdot)$ denotes the L^2 inner product.

2. PRELIMINARIES

The present section refers to some preliminaries to achieve the main results of this paper.

Assume that $\rho_k(s),\ 1 \le k \le n$ satisfy

i. $\rho_k(s) \in C(R) \cap C^1(R),\ \rho_k(s) > 0$ for $s \ne 0,$
$s\rho_k'(s) - \rho_k(s) > 0$ for $s > 0,$

There exists ϑ satisfying $1 < \vartheta < \infty$ for $n = 1, 2;$ $\vartheta < \left(\frac{n+2}{n-2}\right)$ for $n \ge 3$ such that

ii. $|\rho_k(s)| \le \beta |s|^\vartheta$ for $s \in \mathbb{R}$ and some $\beta > 0$.

Let us define

$$E(t) = \frac{1}{2}\left[\|u_t\|^2 + \|\Delta u\|^2\right] - \frac{\beta}{\vartheta+1}\|\nabla u\|_{\vartheta+1}^{\vartheta+1} \quad (2.1)$$

$$E(t) + \lambda\int_0^t \|u_\tau\| d\tau = E(0),$$

$$J(u) = \frac{1}{2}\|\Delta u\|^2 - \frac{\beta}{\vartheta+1}\|\nabla u\|_{\vartheta+1}^{\vartheta+1},$$

$$I(u) = \|\Delta u\|^2 - \beta\|\nabla u\|_{\vartheta+1}^{\vartheta+1}, \quad (2.2)$$

$$d = \inf_{u \in N} J(u),$$

where $N = \left\{u \in H_0^2(\mathbb{R}^n) \,\middle|\, I(u) = 0, \nabla u \ne 0\right\}$, d and $J(u)$ describes the depth of potential well and the potential energy, respectively.

Lemma([14]): For any $u \in H_0^2$, $\|\Delta u\|$ is equivalent to $\|u\|_{H^2}$.

Lemma([14]): Let $\vartheta \le \frac{n+2}{n-2}$ if $n > 2$. Then the imbedding $H_0^2 \hookrightarrow W^{1,\vartheta+1}$ is compact, and we have $\|\nabla u\|_{\vartheta+1} \le C\|\Delta u\|$.

For the case of $0 < E(0) < d_0 = \left(\frac{(\vartheta-1)}{(4(\vartheta+1)}\right)\left(\frac{1}{bC_*}\right)^{\frac{2}{\vartheta-1}}$ and $E(0) < d$, the existence of global solutions of (1.1)-(1.3) are proved respectively in [6] and [5] by the sign invariance of (2.2) and some tresholds were given by aid of an augmented functional I_δ

$$I_\sigma(u) = \sigma\|\Delta u\|^2 - \sum_{i=1}^n \int_\Omega u_{x_k}\rho_k(u_{x_k})dx,\ \sigma > 0.$$

In case of $0 < E(0) < d$, the existence of global weak solution and blow up of solution may be proved with the functional $I(u)$. But for $E(0) > d$, the existence of global solution does not depend only on u_0 . Some additional conditions should be imposed on initial data and a functional will be constructed according to these conditions. This will be achieved in the next section.

We complete this section by a corollary that can be proved as in [10]. For this purpose, we firstly introduce some notations: For $d > -\frac{J-1}{2}$

$$I_\delta(u) = (1-\delta)\|\Delta u\|^2 - \sum_{i=1}^n \int_\Omega u_{x_k}\rho_k(u_{x_k})dx,\ \delta > 0.$$

$$d_\delta = \inf_{u \in N_\delta} J(u),$$

$$N_\delta = \left\{u \in H_0{}^2(R^n) : I_\delta(u) = 0,\ \|u\|_{H_0{}^2} \ne 0\right\}.$$

Corollary 2.1. Suppose $u_0 \in H^1(\mathbb{R}^n),\ u_1 \in L^2(\mathbb{R}^n)$. Let $I_0(u_0) > 0$ and $0 < E(0) < d$. Then, for every $t > 0$

$$0 < I_0(u(t)) < \delta_m \|u\|_{H_0^1}^2 ,$$

where δ_m is maximum of positive root of $d_\delta = E(0)$.

3. MAIN RESULTS

This section is devoted to the global existence of weak solutions of problem (1.1)-(1.3). We prove firstly the sign invariance of a functional, which includes both u_0 and u_1 . The sign invariance of this functional plays a key role in the proof of main theorem about existence of global solution.. Let us define this functional

$$K(u,t)=\|\Delta u\|^2-\beta\|\nabla u\|_{\vartheta+1}^{\vartheta+1}-\|u_t\|^2$$
$$=I(u)-\|u_t\|^2. \quad (3.1)$$

Theorem 3.1: Let $\phi\in H_0^2$, $\psi\in H^1$, and $E(0)>0$. For $\delta>\delta_k$, assume that

$$(\psi,\phi)+\frac{1}{2}\|\phi\|^2\le-\frac{(\vartheta+1)\delta}{(\vartheta+3)\delta+\vartheta-1}E(0) \quad (3.2)$$

Then the sign of $K(u,t)$ is invariant for every $t\in[0,\infty)$.

Proof: We try to get a contradiciton for the proof. We define

$$\theta(t)=\|u\|^2+\gamma\int_0^t\|u\|^2\,d\tau.$$

Then

$$\theta'(t)=2(u_t,u)+\gamma\|u\|,$$
$$\theta''(t)=2\|u_t\|^2+2(u_{tt},u)+\gamma 2(u_t,u)$$
$$=-2K(u,t).$$

Let there exists some $t'>0$ such that $K(u,t')=0$ and t' is the first time with this property. $\theta''(t)<0$ results in $\theta'(t)$ is strictly decreasing on $[0,t')$. It follows from (3.2) that $\theta'(0)<0$ and therefore $\theta'(t)<0$ in $[0,t']$. On account of this, we conclude that $\theta(t)$ is strictly decreasing on $[0,t']$. From the energy identity and $K(u,t')=0$, we get

$$E(0)=\frac{1}{2}\|u_t(t')\|^2+\frac{\vartheta-1}{2(\vartheta+1)}\|\Delta u(t')\|^2+\frac{1}{\vartheta+1}I(u(t'))$$
$$=\left(\frac{1}{2}+\frac{1}{\vartheta+1}\right)\|u_t(t')\|^2+\frac{\vartheta-1}{2(\vartheta+1)}\|\Delta u(t')\|^2 \quad (3.3)$$

The corollary in the previous section and $K(u,t')=0$ provides the following inequality

$$\|\Delta u\|^2\ge\delta_m^{-1}I_0(u(t'))\ge\delta^{-1}\|u_t(t')\|^2.$$

The above inequality in (3.3) yields

$$E(0)\ge\left(\frac{1}{2}+\frac{1}{\vartheta+1}+\frac{\vartheta-1}{2(\vartheta+1)\delta}\right)\|u_t(t')\|^2$$
$$=\frac{(\vartheta+3)\delta+\vartheta-1}{2(\vartheta+1)\delta}\Big[\|(u_t(t')+u(t'))\|^2$$
$$-2(u_t(t'),u(t'))-\|u(t')\|^2\Big]$$

From the monotonocity of $\theta(t)$ and $\theta'(t)$, we get

$$E(0)\ge\frac{(\vartheta+3)\delta+\vartheta-1}{(\vartheta+1)\delta}\left[-(\psi,\varphi)-\frac{1}{2}\|\varphi\|^2\right]$$

Then we have a contradiction with (3.2). The proof is completed.

Theorem: Let $\phi\in H_0^2(\mathbb{R}^n)$, $\psi\in L^2(\mathbb{R}^n)$, $1<\vartheta<\infty$ for $n=1,2$; $1<\vartheta<\frac{n+2}{n-2}$ for $n\ge3$ and. Suppose that condition (3.2) holds, $K(u,0)>0, E(0)>0$ and for some $\delta>\delta_k$. Then, the solution of problem (1.1), (1.3) is global for every $t\in[0,\infty)$.

Proof: Since $K(u,t)$ is invariant under the flow of (1.1)-(1.3) , we have $I(u)>0$ for every $t>0$. By $E(t)$, we have

$$E(0)\ge\frac{1}{2}\|u_t\|^2+\frac{\vartheta-1}{2(\vartheta+1)}\|\Delta u\|^2+\frac{1}{\vartheta+1}I(u)$$
$$\ge\frac{1}{2}\|u_t\|^2+\frac{\vartheta-1}{2(\vartheta+1)}\|\Delta u\|^2.$$

This yields the boundedness of $\|u\|_{H_0^2}$ and $\|u_t\|_{L^2}$ for every $t>0$. The combination of previously mentioned local existence theory [15] and the above estimate yield existence of global solution. Thus we complete the proof.

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