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MATRIX FORMULATION OF REAL QUATERNIONS

REEL KUATERNİYONLARIN MATRİS FORMÜLASYONU

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ABSTRACT

Real quaternions have been expressed in terms of 4×4 matrices by means of Hamilton operators. These matrices are applied for rotations in Euclidean 4-space, and are determined also a Hamilton motions in E^4. We study these matrices and show that the set of these matrices with the group operation of matrix multiplication is Lie group of 6-dimension.

Keywords: De Moivre's formula, Homothetic motion, Lie group, Rotation, Real quaternion.

ÖZET

Reel kuaterniyonlar Hamilton operatörleri aracılığıyla 4×4 matrisler cinsinden ifade edilmiştir. Bu matrisler Öklid 4-uzayda rotasyonlar için uygulanır ve aynı zamanda E^4 bir Hamilton hareketleri tespit edilir. Biz bu matrisleri inceledik ve matris çarpımı grup ile bu matrislerin kümesi 6-boyutun Lie grubu olduğunu göstermektedir.

Anahtar Kelimeler: De Moivre formülü, Homotetik hareket, Lie grubu, Dönme, Reel Kuaterniyon

1. INTRODUCTION

The quaternion algebra is an associative and non-commutative 4-dimensional Clifford algebra. Arthur Cayley was first who discovered that quaternion could be used to represent the rotations in E^4. Some algebraic properties of Hamilton operators are considered in (Agrawal, 1987) where real quaternions have been expressed in terms of 4×4 matrices by means of these operators. These matrices have applications in mechanics, quantum physics and computer-aided geometric design (Adler, 1995). In addition, the homothetic motions has been considered with aid of the Hamilton operators in four-dimensional Euclidean space(Yayli, 1992). The eigenvalues, eigenvectors and the others algebraic properties of these matrices are studied by several authors (Farebrother *et al.*, 2003,

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Zhang, 1997). De-Moivre's and Euler's formulae for matrices associated with real quaternions is studied in (Jafari *et al.*, 2011) and every power of these matrices are immediately obtained. After a review of some fundamental properties of the real quaternions, we study the algebraic properties of matrices corresponding to real quaternions, completely. The set of these matrices with the group operation of matrix multiplication is Lie group of 6-dimension and its Lie algebra is found.

2. PRELIMINARIES

Definition 1. A real quaternion q is an expression of the form

$$q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

where a_0, a_1, a_2 and a_3 are real numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units satisfying the equalities

$$\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = \vec{i}\vec{j}\vec{k} = -1,$$
$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i}, \quad \vec{j}\vec{k} = \vec{i} = -\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i} = \vec{j} = -\vec{i}\vec{k}.$$

The set of all real quaternions is denoted by H. A real quaternion q is a sum of a scalar and a vector, called scalar part, $S_q = a_0$, and vector part $\vec{V}_q = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$. Therefore H is form a 4-dimensional real space which contains the real axis R and a 3-dimensional real linear space E^3, so that, $\mathrm{H} = \mathrm{R} \oplus \mathrm{E}^3$.

The conjugate of the quaternion $q = S_q + \vec{V}_q$ is denoted by $\bar{q}$ and defined as $\bar{q} = S_q - \vec{V}_q$. The norm of a quaternion $q = (a_0, a_1, a_2, a_3)$ is defined by $N_q = q\bar{q} = \bar{q}q = a_0^2 + a_1^2 + a_2^2 + a_3^2$ and we say that $q_0 = q/N_q$ is a unit quaternion where $q \neq 0$ (Ward, 1997).

Definition 2. A Lie group is a group G, equipped with a manifold structure such that the group operations

Mult : $G \times G \to G$, $(g_1, g_2) \to g_1 g_2$

Inv: $G \to G$, $g \to g^{-1}$ are smooth (Meinrenken 2010).

3. PROPERTIES OF HAMILTON OPERATORS

Left multiplication by a real quaternion q is a linear map

$$\overset{+}{h}_q(x) = qx, \qquad x \in \mathrm{H},$$

from the quaternions into the quaternions, as is right multiplication,

$$\overset{-}{h}_q(x) = xq \qquad x \in \mathrm{H}.$$

Since these multiplications are linear maps from four dimensional vector space into itself, we can find a matrix representation of each. The Hamilton operators $\overset{+}{H}$ and $\overset{-}{H}$, could be represented as the matrices

$$\overset{+}{H}(q) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}$$

and

$$\overset{-}{H}(q) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}.$$

Theorem 1. If q and p are two real quaternions, λ is a real number and $\overset{+}{H}$ and $\overset{-}{H}$ are operators as defined in equations (1) and (2), respectively, then the following identities hold:

i. $q = p \Leftrightarrow \overset{+}{H}(q) = \overset{+}{H}(p) \Leftrightarrow \overset{-}{H}(q) = \overset{-}{H}(p).$

ii. $\overset{+}{H}(q+p) = \overset{+}{H}(q) + \overset{+}{H}(p), \quad \overset{-}{H}(q+p) = \overset{-}{H}(q) + \overset{-}{H}(p).$

iii. $\overset{+}{H}(\lambda q) = \lambda \overset{+}{H}(q), \ \overset{-}{H}(\lambda q) = \lambda \overset{-}{H}(q).$

iv. $\overset{+}{H}(qp) = \overset{+}{H}(q)\overset{+}{H}(p), \quad \overset{-}{H}(qp) = \overset{-}{H}(p)\overset{-}{H}(q).$

v. $\overset{+}{H}(q^{-1}) = \left[\overset{+}{H}(q)\right]^{-1}, \quad \overset{-}{H}(q^{-1}) = \left[\overset{-}{H}(q)\right]^{-1}, \quad (N_q)^2 \neq 0.$

vi. $\overset{+}{H}(\bar{q}) = \left[\overset{+}{H}(q)\right]^T, \quad \overset{-}{H}(\bar{q}) = \left[\overset{-}{H}(q)\right]^T.$

vii. $\det\left[\overset{+}{H}(q)\right] = (N_q)^2, \quad \det\left[\overset{-}{H}(q)\right] = (N_q)^2.$

viii. $tr\left[\overset{+}{H}(q)\right] = 4a_0, \quad tr\left[\overset{-}{H}(q)\right] = 4a_0.$

Proof: The proof can be found in (Agrawal, 1987) and (GroB *et al.* 2001).

Theorem 2. The map

$$\psi : (\mathrm{H}, +, .) \to (\mathrm{M}_{(4,\mathrm{R})}, \oplus, \otimes)$$

defined as

$$\psi(a_0 + a_1\vec{\mathrm{i}} + a_2\vec{\mathrm{j}} + a_3\vec{\mathrm{k}}) \mapsto \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}$$

is an isomorphism of algebras.

Proof: The proof can be found in (Ward, 1997).

Theorem 3. Let Ω be the set of all of 4×4 real matrices A *i.e.*

$$\Omega = \left\{ A_{4\times 4} \;\middle|\; A = \begin{bmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{bmatrix}, \; x_i \in \mathrm{R},\; 1 \le i \le 4 \right\}.$$

Ω is a differentiable manifold (Yayli 1988).

Proof: Let us consider the following function:

$$f : \Omega \to \mathrm{R}^4$$
$$A \to f(A) = (x_0, x_1, x_2, x_3),$$

f is one-to-one and on to function, and since $f(\Omega) = \mathrm{R}^4$ then $f(\Omega)$ is open set. Furthermore, since x_i, $i = 1,2,3,4$ are continuously, then

f, f^{-1} are continuously functions. $\{(f\ ,\Omega)\}$ is a differentiable atlas with one chart, Thus Ω has the structure of a differentiable manifold.

Theorem 4. The set $\Omega^* = \Omega - \{0\}$ is a Lie group of dimension 6.

Proof: Ω^* under matrix multiplication is a non-commutative matrix group. Also, Ω^* is a submanifold of Ω. The following maps

$$\text{Mult} : \Omega^* \times \Omega^* \to \Omega^*,$$

sending (A_1, A_2) to A_1A_2 and

$$\text{Inv} : \Omega^* \to \Omega^*,$$

sending A to A^{-1} are differentiable(smooth).

The Lie algebra of Ω^* is specified by left invariant vector field $\chi(\Omega^*)$. Since $\chi(\Omega^*) \cong T_e(\Omega^*)$, we determine tangent space $T_e(\Omega^*)$.

Let $\Phi : \mathbb{R}^4 - \{0\} \to \Omega^*$, defined by

$$\Phi(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{bmatrix},$$

is a one-to-one and on to map. The differential map $d\Phi = \Phi_*$ at point p is

$$\Phi_*\big|_p : T_p \mathbb{R}^4 \to T_{\Phi(p)} \Omega^*,$$

where $p = (1,0,0,0)$ and $\Phi(p) = e = I_4$ is identity element of Ω^*.

For every $V_p \in T_p \mathbb{R}^4$, we have

$$V_p = a_1 \partial/\partial x_1 + a_2 \partial/\partial x_2 + a_3 \partial/\partial x_3 + a_4 \partial/\partial x_4 ,$$

so

$$\Phi_*|_P(\mathrm{V_p})=\begin{bmatrix}\mathrm{V_p}[\mathrm{x_1}] & \mathrm{V_p}[\text{-}\mathrm{x_2}] & \mathrm{V_p}[\text{-}\mathrm{x_3}] & \mathrm{V_p}[\text{-}\mathrm{x_4}]\\ \mathrm{V_p}[\mathrm{x_2}] & \mathrm{V_p}[\mathrm{x_1}] & \mathrm{V_p}[\text{-}\mathrm{x_4}] & \mathrm{V_p}[\mathrm{x_3}]\\ \mathrm{V_p}[\mathrm{x_3}] & \mathrm{V_p}[\mathrm{x_4}] & \mathrm{V_p}[\mathrm{x_1}] & \mathrm{V_p}[\text{-}\mathrm{x_2}]\\ \mathrm{V_p}[\mathrm{x_4}] & \mathrm{V_p}[\text{-}\mathrm{x_3}] & \mathrm{V_p}[\mathrm{x_2}] & \mathrm{V_p}[\mathrm{x_1}]\end{bmatrix}$$

$$=\begin{bmatrix}a_1 & -a_2 & -a_3 & -a_4\\ a_2 & a_1 & -a_4 & a_3\\ a_3 & a_4 & a_1 & -a_2\\ a_4 & -a_3 & a_2 & a_1\end{bmatrix}=[0],$$

Then $a_1=a_2=a_3=a_4=0$. So, Φ_* an injective map.

On the other hand, $\dim \mathrm{T_p}\,\mathrm{R}^4=\dim \mathrm{T}_e\,\Omega^*=4,$ Φ_* is a linear isomorphism. Since every linear isomorphism maps any basis of space to another one. So we determine the basis of space $\mathrm{T}_{\Phi(p)}\,\Omega^*$.

It is obviously that $\mathrm{T_p}\,\mathrm{R}^4=\mathrm{Sp}\{\partial/\partial x_1,\partial/\partial x_2,\partial/\partial x_3,\partial/\partial x_4\}$. We find the images of this basis under the map Φ_*.

$$\Phi_*|_P\left(\frac{\partial}{\partial x_1}\right)=\frac{\partial}{\partial x_1}\begin{bmatrix}x_1 & -x_2 & -x_3 & -x_4\\ x_2 & x_1 & -x_4 & x_3\\ x_3 & x_4 & x_1 & -x_2\\ x_4 & -x_3 & x_2 & x_1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\Phi_*|_P\left(\frac{\partial}{\partial x_2}\right)=\frac{\partial}{\partial x_2}\begin{bmatrix}x_1 & -x_2 & -x_3 & -x_4\\ x_2 & x_1 & -x_4 & x_3\\ x_3 & x_4 & x_1 & -x_2\\ x_4 & -x_3 & x_2 & x_1\end{bmatrix}=\begin{bmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{bmatrix}$$

$$\Phi_*|_P\left(\frac{\partial}{\partial x_3}\right)=\frac{\partial}{\partial x_3}\begin{bmatrix}x_1 & -x_2 & -x_3 & -x_4\\ x_2 & x_1 & -x_4 & x_3\\ x_3 & x_4 & x_1 & -x_2\\ x_4 & -x_3 & x_2 & x_1\end{bmatrix}=\begin{bmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{bmatrix}$$

$$\Phi_*\big|_P\left(\frac{\partial}{\partial x_4}\right)=\frac{\partial}{\partial x_4}\begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

So we have

$$\mathrm{T}_e\,\Omega^*=\mathrm{Sp}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right\}.$$

Corollary 1. The tangent space of manifold Ω^* is at any point is isomorphic to Ω space, *i.e.* $\mathrm{T}_{\Phi(p)}(\Omega^*)\cong\Omega$ (Yayli, 1988).

4. DE MOIVRE'S FORMULA FOR REAL QAUTERNIONS

Theorem 7. (De Moivre's formula) For an integer *n* and matrix

$$A=\begin{bmatrix} \cos\theta & -u_1\sin\theta & -u_2\sin\theta & -u_3\sin\theta \\ u_1\sin\theta & \cos\theta & -u_3\sin\theta & u_2\sin\theta \\ u_2\sin\theta & u_3\sin\theta & \cos\theta & -u_1\sin\theta \\ u_3\sin\theta & -u_2\sin\theta & u_1\sin\theta & \cos\theta \end{bmatrix}, \qquad (1)$$

the *n*-th power of the matrix *A* reads as

$$A^n=\begin{bmatrix} \cos n\theta & -u_1\sin n\theta & -u_2\sin n\theta & -u_3\sin n\theta \\ u_1\sin n\theta & \cos n\theta & -u_3\sin n\theta & u_2\sin n\theta \\ u_2\sin n\theta & u_3\sin n\theta & \cos n\theta & -u_1\sin n\theta \\ u_3\sin n\theta & -u_2\sin n\theta & u_1\sin n\theta & \cos n\theta \end{bmatrix}.$$

Proof: The proof is easily followed by induction on *n*.

Teorem 8. Let q be a unit quaternion with the polar form $q=\cos\theta+\vec{u}\sin\theta$. And let $m=\dfrac{2\pi}{\theta}\in\mathrm{Z}^+-\{1\}$ and the matrix *A* correspond to q. Then $n \equiv p\,(\mathrm{mod}\,m)$ is true if and only if $A^n = A^p$.

Example 1. Let $q = 1 + (1,1,1)$ be a rael quaternion. Then the matrix corresponding to this quaternion is

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix},$$

Every power of this matrix is found to be with aid of Theorem 7, for example *28*-th power is

$$A^{28} = 2^{27} \begin{bmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \end{bmatrix}.$$

5. ROTATION IN FOUR DIMENSIONS VIA HAMILTON OPERATORS

In this section, we show how unit quaternions can be used to described the rotation in 4-space E^4.

Theorem 9. Let q be a unit real quaternion. Matrices generated by operators $\overset{+}{H}$ and $\overset{-}{H}$ are orthogonal matrices, *i.e.*

i) $\left[\overset{+}{H}(q)\right]^T \overset{+}{H}(q) = I_4,$

ii) $\left[\overset{-}{H}(q)\right]^T \overset{-}{H}(q) = I_4.$

Corollary 1: Let $q = \cos\theta + \vec{u}\sin\theta$ be a unit real quaternion. Then Hamilton operators $\overset{+}{h}_q$ and $\overset{-}{h}_q$ represent rotations of x in E^4.

The angle of rotation (using $\overset{+}{h}_q$) is easily determined. This is the angle ω between x and qx:

$$\cos\omega = \frac{S(x(\overline{qx}))}{\sqrt{N_x}\sqrt{N_{qx}}}$$

$$= \frac{S(x(\overline{x}\overline{q}))}{N_x\sqrt{N_q}} = \frac{S(q)}{\sqrt{N_q}} = S(q) = \cos\theta.$$

Therefore that the angle of rotation ω is the angle of q.

6. Hamilton Motions in E^4

In this section, we show how matrices corresponding to real quaternion can be used to described the homothetic motion 4-space E^4.

Let us consider the following curve:

$$a : I \subset \mathbb{R} \to E^4$$

defined by $a(t) = (a_0(t), a_1(t), a_2(t), a_3(t))$ for every $t \in I$.

We suppose that the unit velocity curve $a(t)$ is differentiable regular curve of order r. The operator B called the Hamiltonian operator, corresponding to $a(t)$ is defined by the following matrix;

$$B = \overset{+}{H}\left[a(t)\right] = \begin{bmatrix} a_0(t) & -a_1(t) & -a_2(t) & -a_3(t) \\ a_1(t) & a_0(t) & -a_3(t) & a_2(t) \\ a_2(t) & a_3(t) & a_0(t) & -a_1(t) \\ a_3(t) & -a_2(t) & a_1(t) & a_0(t) \end{bmatrix}. \quad (2)$$

Definition 3. The 1-parameter Hamilton motions of a body in E^4 are generated by transformation

$$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

or equivalently

$$Y = BX + C. \quad (3)$$

Here $B = \overset{+}{H}\left[a(t)\right]$ and Y, X and C are $n \times 1$ real matrices. Y and X correspond to the position vectors of the same point P.

Theorem 10. The Hamilton motion determined by equation (3) is a homothetic motion in E^4.

Proof: We suppose that length of $a(t)$ is not zero, so the matrix B can be represented as

$$B = h\begin{bmatrix} \frac{a_0(t)}{h} & -\frac{a_1(t)}{h} & -\frac{a_2(t)}{h} & -\frac{a_3(t)}{h} \\ \frac{a_1(t)}{h} & \frac{a_0(t)}{h} & -\frac{a_3(t)}{h} & \frac{a_2(t)}{h} \\ \frac{a_2(t)}{h} & \frac{a_3(t)}{h} & \frac{a_0(t)}{h} & -\frac{a_1(t)}{h} \\ \frac{a_3(t)}{h} & \frac{-a_2(t)}{h} & \frac{a_1(t)}{h} & \frac{a_0(t)}{h} \end{bmatrix}$$

where $h: I \subset \mathbb{R} \to \mathbb{R}$,

$$t \to h(t) = \|\alpha(t)\| = \sqrt{a_0^2(t) + a_1^2(t) + a_2^2(t) + a_3^2(t)}..$$

So, we find $A^T A = I_4$ and $\det A = 1$, thus B is a homothetic matrix and equation (3) determines a homothetic motion.

For detailed information about the homothetic motions, we refer the reader to (Yayli, 1992).

REFERENCES

Adler, S. L. 1995. Quaternionic quantum mechanics and quantum fields. Oxford University Press Inc., New York. Pp. 65.

Agrawal, O. P. 1987. Hamilton operators and dual-number-quaternions in spatial kinematics. Mechanism and Machine Theory 22 (6): 569-575.

Farebrother, R. W., GroB, J. & Troschke, S. 2003. Matrix representation of quaternions. Linear Algebra and its Applications 362: 251-255.

Groβ, J., Trenkler, G. & Troschke, S. 2001. Quaternions: futher contributions to a matrix oriented approach, Linear Algebra and its Applications 326: 205-213.

Jafari, M., Mortazaasl, H. & Yayli, Y. 2011. De-Moivre's formula for matrices of quaternions. JP Journal of Algebra, Number Theory and Applications 21(1): 57-67.

Meinrenken E., Lie groups and Lie algebras, Lecture Notes, University of Toronto, 2010.

Ward, J. P. 1997. Quaternions and Cayley numbers algebra and applications, Kluwer Academic Publishers, London. Pp.78.

Weiner, J. L. & Wilkens, G. R. 2005. Quaternions and rotations in . Mathematical Association of America 12: 69-76.

Yayli, Y. 1992. Homothetic motions at E^4, Mechanism and Machine Theory 27(3): 303-305.

Yayli, Y. 1988. , Hamilton Motions and Lie Grups, Ph.D. Thesis, Gazi University, Ankara, Turkey.

Zhang, F. 1997. Quaternions and matrices of quaternions. Linear Algebra and its Applications 251: 21-57.