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BAUES COFIBRATION FOR QUADRATIC MODULES OF LIE ALGEBRAS

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Abstract. In this paper; free quadratic modules and totally free objects in the category of quadratic modules are constructed over Lie algebras. We use the free quadratic modules of Lie algebras to show that the category of quadratic module of Lie algebras is a cofibration category by means of Baues.

1. Introduction

Whitehead defined the notion of crossed modules of groups in[12]. Simplicial groups introduced in [9]. Using simplicial methods Conduch defined 2-crossed modules [6]. Ellis in [8] defined the notion of free crossed modules and free 2-crossed modules in the category of Lie algebras and gave the relations among 2-crossed modules of Lie algebras and simplicial Lie algebras. He also proved some classical results for Lie algebraic versions. In the Moore complex of a simplicial Lie algebra using the image of the higher order Peiffer elements Akça and Arvasi in [2] explained the relations among 2-crossed modules of Lie algebras and simplicial Lie algebras .

Quadratic modules of groups are algebraic models for homotopy connected 3-types introduced by Baues [4]. Baues in [4] constructed a functor from the category of simplicial groups to the category of quadratic modules. In [11], Lie algebra versions of quadratic modules was defined, and the connections between 2-crossed modules ,quadratic modules and simplicial Lie algebras were explored by using simplicial properties in [2].

2. Quadratic Modules of Lie Algebras

We will denote the category of Lie algebras by $\mathfrak{LieAlg}$ and every object we will examine in $\mathfrak{LieAlg}$ will be over a fixed commutative ring. In [10] Kassel and Loday

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introduced crossed modules of Lie algebras. Let A and B be two objects in $\mathfrak{LieAlg}$. An action of B on A is a bilinear map $B \times A \to A$, $(b, a) \mapsto b \cdot a$ satisfying

$$\begin{array}{rcl} [b, b'] \cdot a & = & b \cdot (b' \cdot a) - b'(b \cdot a) \\ b \cdot [a, a'] & = & [b \cdot a, a'] + [a, b \cdot a'] \end{array}$$

for all $a, a' \in A$ and $b, b' \in B$.

Let $\partial : A \to B$ be a Lie algebra homomorphism. If the following condition is satisfied

$$CM1) \quad \partial(b \cdot a) = [b, \partial a]$$

for all $b \in B$ and $a \in A$, $\partial : A \to B$ is called a pre-crossed module of Lie algebras[8]. If $\partial : A \to B$ satisfy the extra condition

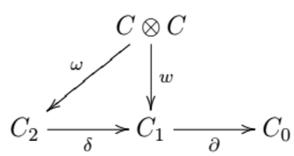
$$CM2) \quad (\partial a) \cdot a' = [a, a']$$

for all $a, a' \in A$. then $\partial : A \to B$ is called a crossed module. Let $\partial : A \to B$ be a pre-crossed module. The *Peiffer element* for $a_1, a_2 \in A$ is the Peiffer Lie ideal of A is $P_2(\partial)$ generated by elements of the form

$$\langle a_1, a_2 \rangle = (\partial a_1) \cdot a_2 - [a_1, a_2]$$

Here $[a_1, a_2]$ is the Lie bracket of elements a_1, a_2 in the Lie algebra A. We recall the following notations from [4].

Definition 1. *The diagram of homomorphisms of Lie algebras*



satisfying the following axioms is called quadratic module ([11])$(\omega, \delta, \partial)$ *of Lie algebras .*

$QM1)$ $\partial : C_1 \to C_0$ *is a nil(2)-module and the quotient map* $C_1 \twoheadrightarrow C = C_1^{cr}/[(C_1^{cr}), (C_1^{cr})]$ *is given by* $c_1 \mapsto [c_1]$, *where* $[c_1] \in C$ *denotes the class represented by* $c_1 \in C_1$. *The map* w *is defined by Peiffer multiplication, that is for* $c_1, c_1' \in C_1$

$$w([c_1] \otimes [c_1']) = \partial(c_1) \cdot c_1' - [c_1, c_1'].$$

$QM2)$ *The bottom row is a complex of Lie algebras and for* $c_1, c_1' \in C_1$

$$\delta\omega([c_1] \otimes [c_1']) = w([c_1] \otimes [c_1']) = \partial(c_1) \cdot c_1' - [c_1, c_1'].$$

$QM3)$ *For* $c_2 \in C_2, c_0 \in C_0$

$$\partial(c_0) \cdot c_2 = \omega([\delta c_2] \otimes [c_0] + [c_0] \otimes [\delta c_2]).$$

QM4) For $c_2, c_2' \in C_2$,

$$\omega([\delta c_2] \otimes [\delta c_2']) = [c_2, c_2'].$$

A morphism $\varphi = (f_2, f_1, f_0) : (\omega, \delta, \partial) \to (\omega', \delta', \partial')$ in the category of quadratic modules of Lie algebras is a commutative diagram,

$$\begin{array}{ccccccc} C \otimes C & \xrightarrow{\omega} & C_2 & \xrightarrow{\delta} & C_1 & \xrightarrow{\partial} & C_0 \\ \downarrow{\scriptstyle \varphi_* \otimes \varphi_*} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ C' \otimes C' & \xrightarrow[\omega']{} & C_2' & \xrightarrow[\delta']{} & C_1' & \xrightarrow[\partial']{} & C_0' \end{array}$$

We will denote this category by $\mathfrak{LieQM}$.

3. Free Quadratic Module Construction in $\mathfrak{LieQM}$.

We recall the free simplicial Lie algebra construction by use of the 'step-by-step' construction[1]. For more details regarding the simplicial analogue of André's construction, we refer to the paper [3]. Now, we will give briefly from [3] ' skeleton of a free simplicial Lie algebra up to dimension 2 and a free simplicial Lie algebra's step-by-step' construction in order to construct a (totally) free object in $\mathfrak{LieQM}$.

From [3], we recall the 2-skeleton of the free simplicial Lie algebra. We should point out that there is an additional structure such as augmentation $L(X) \to L$ given by sending all X to zero, where $L(X)$ is the free Lie algebra over X. Thus we get the augmentation ideal $L^+(X)$.

Let P be a Lie algebra and $I = (x_1, x_2, ..., x_n)$ be an ideal of P generated by the elements $x_1, x_2, ..., x_n \in P$. Let $\mathbf{K(P, 0)}$ denote the simplicial Lie algebra and for all i, j $d_i = s_j = id$. There is an obvious epimorphism $f : P \to P/I$. Then we have an isomorphism $P/\ker f \cong P/I$. Let $\Omega^0 = \{x_1, x_2, ..., x_n\} \subset \ker f$. This 1-skeleton $\mathbf{L^{(1)}}$ of P/I can be built by adding new determinates $X = \{X_1, X_2, ..., X_n\}$ into $L_1^{(0)} = P$ to form $L_1^{(1)} = L_1^{(0)}(X) = P[X_1, X_2, ..., X_n]$, the free Lie algebra on X with

$$P[X_1, ..., X_n] \overset{d_0, d_1}{\underset{s_0}{\rightleftarrows}} P$$

where $d_1(X_i) = x_i \in \ker f$, $d_0(X_i) = 0$ and $s_0(p) = p \in P$. Thus the 1-skeleton looks like

$$\mathbf{L^{(1)}} : \cdots P[s_0X, s_1X] \rightleftarrows P[X] \rightleftarrows P \xrightarrow{f} P/I = B.$$

Define $\Omega^1 = \{y_1, y_2, ...y_m\} \subset \pi_1(\mathbf{L^{(1)}})$ set of generators and killed off the elements in $\pi_1(\mathbf{L^{(1)}})$. We add a new generator $Y = \{Y_1, Y_2, ..., Y_m\}$ into $L_2^{(1)}$ to get

$$L_2^{(2)} = L_2^{(1)}[Y] = P[s_0X, s_1X][Y]$$

with the maps $d_0^2(Y_i) = 0, d_1^2(Y_i) = 0$ and $d_2^2(Y_i) = y_i$. Thus the augmented 2-skeleton looks like:

$$\mathbf{L^{(2)}} : \cdots P[s_0X, s_1X][Y] \underset{s_0,s_1}{\overset{d_0^2,d_1^2,d_2^2}{\rightrightarrows}} P[X] \underset{s_0}{\overset{d_0^1,d_1^1}{\rightrightarrows}} P \xrightarrow{f} P/I.$$

Construction of this 2-dimensional data consists of a function $\vartheta : X \to P$, which is 1-dimensional data, used to induce $d_1 : P[X] \to P$ and a function $\psi : Y \to P^+[X]$ which is 2-dimensional construction data used to induce $d_2 : P[s_0X, s_1X][Y] \to P[X]$. The constructed 2-dimensional construction data will be denoted by (ϑ, ψ, P).

Definition 2. *Let $\vartheta : Y \to C_2$ be a function and the following diagram*

$$\begin{array}{ccccc} & & C \otimes C & & \\ & \swarrow^{\omega} & \downarrow w & & \\ C_2 & \xrightarrow[\partial_2]{} & C_1 & \xrightarrow[\partial_1]{} & C_0. \end{array}$$

be an object in $\mathfrak{LieQM}$. If for any object in $\mathfrak{LieQM}$ say $(\omega', \partial_2', \partial_1)$ and $\vartheta' : Y' \to C_2'$ such that $\partial_2'\vartheta' = \partial_2\vartheta$, then there exists a unique morphism $\Phi : C_2 \to C_2'$ such that $\partial_2'\Phi = \partial_2$. We say that $(\omega, \partial_2, \partial_1)$ is the free object on the function $\partial_2\vartheta : Y \to C_1$ in $\mathfrak{LieQM}$. If ∂_1 is a free nil(2)-module then $(\omega, \partial_2, \partial_1)$ is a totally free object in $\mathfrak{LieQM}$.

Theorem 3. *A totally free object in $\mathfrak{LieQM}$ exists on (ϑ, ψ, P).*

Proof. Let $\vartheta : X \to P$ and $\psi : Y \to P^+[X]$ be two functions to form (ϑ, ψ, P). Since there is an action of P on $P^+[X]$, $\partial : P^+[X] \to P$ is a free pre-crossed module in $\mathfrak{LieAlg}$ with basis $\vartheta : X \to P$. Here $P^+[X] = NL_1^{(1)} = \ker d_0$ is the positively degree part of $P[X]$. Define

$$q_1 : P^+[X] \to P^+[X]/P_3 = M.$$

Then we get the following commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{i} & P^+[X] \\ {\scriptstyle\vartheta}\downarrow & & \downarrow{\scriptstyle q_1} \\ P & \xleftarrow[\partial_1]{} & P^+[X]/P_3 \end{array}$$

where the morphism

$$\begin{array}{rcl}\partial_1 : P^+[X]/P_3 & \to & P \\ x + P_3 & \mapsto & \partial(x)\end{array}$$

is in $\mathfrak{LieAlg}$ and i is the inclusion. Since $\partial : P^+[X] \to P$ is a pre-crossed module in $\mathfrak{LieAlg}$ we have $\partial(P_3) = 0$. Thus $\partial_1 : M = P^+[X]/P_3 \to P$ is a free nil(2) -module on the function $\partial_1 q_1 i$. Now, from the 2-skeleton $\mathbf{L^{(2)}}$, take

$$D = NL_2^{(2)} = P[s_0X]^+[s_1X, Y] \cap ((s_0 - s_1)(X)).$$

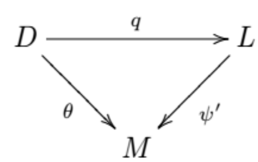
Then, by using the function

$$q_1\psi : Y \to M = P^+[X]/P_3$$

we get a morphism of Lie algebras

$$\theta : P[s_0X]^+[s_1X, Y] \cap ((s_0 - s_1)(X)) \to P^+[X]/P_3$$

such that $\theta(y) = q_1\psi(y)$. Let the second order Peiffer Lie ideal be $P' = \partial_3(NL_3^{(2)} \cap D_3^{(2)}) \subset D$. The generators of this ideal were obtained by the images of the functions $M_{\alpha,\beta}$ given in [2]. Then, $q_1\psi(P') = 0$. Taking $L = D/P'$ we get a morphism $\psi' : L \to M$ making the following diagram commutative.



For $X_i, X_j \in P^+[X]$ the Peiffer elements are given by

$$\langle X_i, X_j\rangle = [X_i, X_j] - \partial(X_i)X_j$$

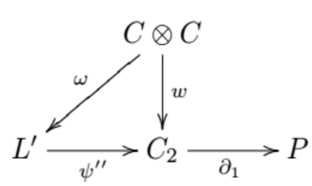
and for all $X_i, X_j, X_k \in P^+[X]$ we have the elements of P_3 as

$$[s_1(\langle X_i, X_j\rangle), s_1(X_k) - s_0(X_k)] + \partial_3(NL_3^{(2)} \cap D_3^{(2)})$$

and

$$[s_1(X_i), s_1(\langle X_j, X_k\rangle) - s_0\langle X_j, X_k\rangle] + \partial_3(NL_3^{(2)} \cap D_3^{(2)}).$$

Let $q_2 : L \to L' = L/P_3'$ where $L' = L/P_3'$. Since $\psi(P_3') = P_3$, we define $\psi'':L' \to C_2$ such that $\psi'' q_2 = q_1\psi$. Then, the diagram



is a totally free object in $\mathfrak{LieQM}$. Here quadratic map ω is defined as follows:

$$\omega(\{q_1X_i\} \otimes \{q_1X_j\}) \quad = \quad q_2([s_1X_i,(s_1X_j - s_0X_j)] + \partial_3(NL_3^{(2)} \cap D_3^{(2)}))$$

for $X_i \in P^+[X]$, $q_1(X_i) \in M = P^+[X]/P_3$ and $\{q_1X_i\} \in C = M^{cr}/[(M^{cr}),(M^{cr})]$.

For another object in $\mathfrak{LieQM}$ say

$$C \otimes C \xrightarrow{\omega'} D \xrightarrow{\partial_2'} C_2 \xrightarrow{\partial_1} P$$

and a function $\vartheta' : Y \to D$ there is a unique morphism

$$\begin{array}{ccc} \Phi : L' & \to & D \\ (q_2(y+P')) & \mapsto & \vartheta'(y) \end{array}$$

satisfying $\partial_2'\Phi = \psi''$. Thus $(\omega, \psi'', \partial_1)$ is the required totally free object in $\mathfrak{LieQM}$ with basis $q_1\psi : Y \to C_2$. $\square$

4. Quadratic Chain Complexes of Lie Algebras and Simplicial Lie Algebras

A functor from the category of simplicial Lie algebras to quadratic chain complexes of Lie algebras is defined in [11]. By using this functor from a free simplicial Lie algebra we can get a totally free quadratic chain complex of Lie algebras. Now we will give the definition of a quadratic chain complex of Lie algebras. The group case is given by Baues [4].

Definition 4. *A diagram of Lie homomorphisms between Lie algebras*

$$\begin{array}{ccccccccc} & & & & & & C \otimes C & & \\ & & & & & \overset{\omega}{\swarrow} & \downarrow w & & \\ \mathbf{C} : \cdots & \longrightarrow & C_3 & \xrightarrow[\partial_3]{} & C_2 & \xrightarrow[\partial_2]{} & C_1 & \xrightarrow[\partial_1]{} & C_0 \end{array}$$

is called quadratic chain complex of Lie algebras if
(i) for $n \geqslant 1$ C_0 acts on C_n ,and for $n \geqslant 3$ $\partial_1(C_1)$ acts on trivial on C_n ;
(ii) for all $i \geqslant 1$ $\partial_i\partial_{i+1} = 0$ and each ∂_n is a Lie C_0-module homomorphism;
(iii)

$$\begin{array}{ccccc} & & C \otimes C & & \\ & \overset{\omega}{\swarrow} & \downarrow w & & \\ C_2 & \xrightarrow[\partial_2]{} & C_1 & \xrightarrow[\partial_1]{} & C_0 \end{array}$$

is in $\mathfrak{LieQM}$.

Let C, C' be two quadratic chain complexes of Lie algebras. A quadratic chain map $f : \mathbf{C} \to \mathbf{C}'$ is a family of Lie homomorphisms between Lie algebras ($n \geqslant 0$), $f_n : C_n \to C'_n$ with $f_n d_{n+1} = d_{n+1} f_{n+1}$ such that (f_2, f_1, f_0) is a morphism in $\mathfrak{LieQM}$. We denote the category of quadratic chain complexes of Lie algebras by $\mathfrak{QuadchainLie}$.

If the quadratic module at the base is a free object in $\mathfrak{LieQM}$ and for $n \geqslant 3$, C_n are free Lie algebras then quadratic chain complex of Lie algebras $\mathbf{C}$ will be called *free*. In addition if the base quadratic module is totally free $\mathfrak{LieQM}$ then it will be *totally free*.

A chain complex of Lie algebras

$$\rho : \qquad \cdots \longrightarrow \rho_2 \xrightarrow{\partial_2} \rho_1 \xrightarrow{\partial_1} \rho_0$$

is called crossed complex of Lie algebras if

(*i*) $\partial_1 : \rho_1 \to \rho_0$ is a crossed module in $\mathfrak{LieAlg}$,

(*ii*) Each ∂_n is a Lie ρ_0-module homomorphism and ρ_n is a Lie ρ_0-module for $n > 1$, and $\partial_1(\rho_1)$ acts trivially,

(*iii*) $\partial_n \partial_{n+1} = 0$ for $n \geqslant 1$.

Carrasco and Cegarra [5] constructed a functor from the category of simplicial groups to category of crossed complexes of groups. Now from [3], we give the Lie algebra version of this functor. For a simplicial Lie algebra $\mathbf{L}$, Arvasi defined in [3]

$$\mathbf{C}^{(1)}(\mathbf{L})_n = \rho_n = \frac{NL_n}{(NL_n \cap D_n) + d_{n+1}(NL_{n+1} \cap D_{n+1})}.$$

Thus we get a crossed complex $\mathbf{C}^{(1)}(\mathbf{L})$ in $\mathfrak{LieAlg}$. d^n_n induces a map $\partial_n : \mathbf{C}^{(1)}(\mathbf{L})_n \to \mathbf{C}^{(1)}(\mathbf{L})_{n-1}$. Arvasi also showed in [3] that $\mathbf{C}^{(1)}(\mathbf{L})$ is a free crossed complex in $\mathfrak{LieAlg}$ if $\mathbf{L}$ is a free simplicial Lie algebra.

Using the Moore complex of a simplicial Lie algebra we get a quadratic chain complex in $\mathfrak{LieAlg}$. Let $\mathbf{NL}$ be the Moore complex of a simplicial Lie algebra $\mathbf{L}$, define $C_n = \mathbf{C}^{(2)}(\mathbf{L})_n$ by $C_0 = NL_0$, $C_1 = NL_1/P_3$, $C_2 = (NL_2/\partial_3(NG_3 \cap D_3))/P'_3$, and for $n \geqslant 3$, $C_n = NL_n/(NL_n \cap D_n) + d_{n+1}(NL_{n+1} \cap D_{n+1})$ with ∂_n induced by the differential of $\mathbf{NL}$ and $C = (C_1^{cr})/[(C_1^{cr}), (C_1^{cr})]$ and where P_3 is the ideal of NL_1 generated by $\langle x, \langle y, z\rangle\rangle$ and $\langle\langle x, y\rangle, z\rangle$ and P'_3 is the ideal of $NL_2/\partial_3(NL_3 \cap D_3)$ generated by elements $x, y, z \in NL_1$;

$$[s_1\langle x, y\rangle, s_1 z - s_0 z] + \partial_3(NG_3 \cap D_3)$$

and

$$[s_1 x, s_1\langle y, z\rangle - s_0\langle y, z\rangle] + \partial_3(NG_3 \cap D_3)$$

as given in [11].

Proposition 5. *Let* $\mathbf{L}$ *be a simplicial Lie algebra (cf.*[7],[8]*), then* $\mathbf{C} = \mathbf{C}^{(2)}(\mathbf{L})$ *is a quadratic chain complex in* $\mathfrak{LieAlg}$.

Proof. We know from [11] $(C_2, C_1, C_0, \omega, w)$ is a quadratic module. Since $\partial_2\partial_3$ is a complex of Lie algebras the proof is straightforward. □

Then, we get a functor

$$\mathbf{C}^{(2)} : \mathfrak{SimpLieAlg} \to \mathfrak{LieQuadchain}.$$

Baues [4] constructed a functor,

$$\lambda : \mathfrak{Quadchain} \to \mathfrak{Xchain}.$$

Next we will give Lie algebraic version of this functor.

For a quadratic chain complex $\mathbf{C}$ in $\mathfrak{LieAlg}$, $\lambda(\mathbf{C}) = (\rho_n, d_n)_{n\geqslant 0}$ is a crossed complex in $\mathfrak{LieAlg}$. The extra structures are: $\rho_0 = \lambda(\mathbf{C})_0 = C_0$, $\rho_1 = \lambda(\mathbf{C})_1 = C_1/w(C \otimes C) = C_1^{cr}$, $\rho_2 = \lambda(\mathbf{C})_2 = C_2/(\omega(C \otimes C))$ and for $n \geqslant 3$ $\rho_n = C_n$. Then we have

$$\lambda(\mathbf{C}) : \cdots \longrightarrow C_4 \longrightarrow C_3 \xrightarrow{d_3} \rho_2 \xrightarrow{d_2} \rho_1 \xrightarrow{d_1} C_0$$

in which $d_1 : \rho_1 \to C_0$ is a crossed module in $\mathfrak{LieAlg}$.

Corollary 6. *For a simplicial Lie algebra* $\mathbf{L}$*,*

$$\lambda\mathbf{C}^{(2)}(\mathbf{L}) \cong \mathbf{C}^{(1)}(\mathbf{L}).$$

Proposition 7. $\mathbf{C}^{(2)}(\mathbf{L})$ *is a totally free quadratic chain complex if* $\mathbf{L}$ *is a free simplicial Lie algebra.*

Proof. From Theorem 3 $\mathbf{C}^{(2)}(\mathbf{L})$ is totally free on 2-dimensional construction data. And for $n \geqslant 3$ C_n are free proved in [3]. □

For a free simplicial Lie algebra $\mathbf{L}$, since $\lambda\mathbf{C}^{(2)}(\mathbf{L}) \cong \mathbf{C}^{(1)}(\mathbf{L})$, $\lambda\mathbf{C}^{(2)}(\mathbf{L})$ is a totally free crossed complex in $\mathfrak{LieAlg}$ as given in [3].

5. Cofibrations in the category $\mathfrak{QuadChainLie}$

In this section, we give the definition of Baues cofibration for quadratic modules over Lie algebras. For a quadratic chain map $f{:}\sigma \to \sigma'$ in $\mathfrak{LieAlg}$ we define homotopy modules

$$\pi_1(\sigma) = coker\,(d_2)$$

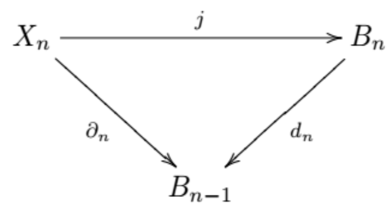
$$\pi_n(\sigma) = \frac{ker\,(d_n)}{Im\,(d_{n+1})} \quad ; \quad n \geq 2.$$

If f induces a morphism of $\pi_n(f_n)$ for $n \geqslant 1$ then f is a weak equivalence. It is clear that

$$\mathfrak{FreeQuadLie} \subset \mathfrak{QuadChainLie}$$

Definition 8. *For a map $f : A \to B$ in $\mathfrak{QuadLie}$ if f is a free extension in each degree n then f is a cofibration.*

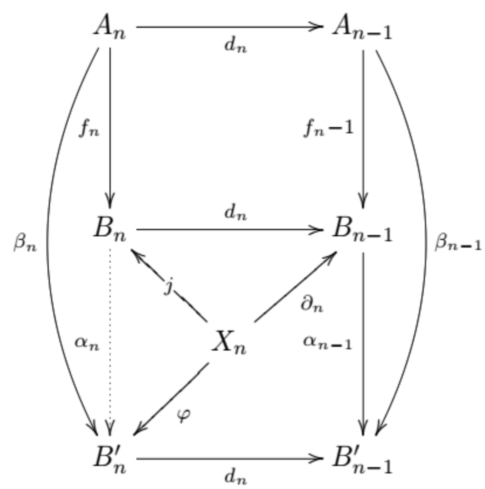
We define a free extension in degree n with basis $\partial_n : X_n \to B_{n-1}$ where the diagram



commutes. Define B' be any quadratic chain complex of Lie algebras and A^n, B^n, B'^n be the n-skeleton of A, B and B' respectively. Let $\beta : A \to B'$, $\alpha^{n-1} : B^{n-1} \to B'^{n-1}$ be quadratic chain maps of lie algebras such that

$$\alpha^{n-1} f^{n-1} = \beta^{n-1} : A^{n-1} \to B^{n-1}$$

and assume a function $\varphi : X_n \to B'_n$ is chosen for the diagram of unbroken arrows



commutes. Then the quadratic complex map $\alpha^n : B^n \to B'^n$ of Lie algebras is unique. Totally free quadratic chain complexes of Lie algebras are cofibrant objects in $\mathfrak{QuadChainLie}$. That is

$$\mathfrak{FreeQuadLie} = \mathfrak{QuadChainLie}_C$$

where $\mathfrak{QuadChainLie}_C$ denotes the full subcategory of quadratic complexes of Lie algebras consisting of cofibrant objects.

Lemma 9. *Let A^n be an n-skeleton. Assume that a function $\partial_n : X_n \to B_{n-1}$ and $f^{n-1} : A^{n-1} \to B^{n-1}$ are given. Then there exists a free extension in each degree $f^n : A^n \to B^n$ with basis ∂_n such that $\partial_n^2 = 1$.*

Proof. In degree 1;

$$B_1 = A_1 * F\ [X_1]^+$$

where $F\ [X_1]^+$ denotes the free module on X.

In degree 2;

$$\bar{d}_2 : \bar{B}_2 * F\,(A_2 \cup X_2) \times B_1 \to B$$

free nil(2) −module of Lie algebras with basis (f_1, d_2, ∂_2). However the map $i : A_2 \to \bar{B}_2$ is not a morphism of nil(2) −modules of Lie algebras. Let J be the ideal of $\bar{B}_2$ with relations

$$i\,(j_1)\, i\,(j_2)\, i\left(j_1 j_2^{-1}\right) \simeq 1$$

$$i\,(j_1^{\alpha}) \left({}^{f_1(\alpha)} i\,(j_1)\right)^{-1} \simeq 1$$

for $j_2, j_2 \in A_2, \alpha \in A_1$. Then

$$d_2 : B_2 = B_2/J \to B_1$$

becomes the required nil(2) −module of Lie algebras.

Finally for degree $n \geqslant 3$; B_n is the direct sum

$$B_n = (A_n \otimes \pi_1\,(A)) \oplus M$$

where M is the free R-module in $\mathfrak{LieAlg}$. □

Proposition 10. *The category* $\mathfrak{QuadChainLie}$ *with weak equivalences and cofibrations is a cofibration structure.*

Pushout object in $\mathfrak{QuadChainLie}$

$$\begin{array}{ccc} B & \xrightarrow{\ \overline{f}\ } & B' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ A & \longrightarrow & A' \end{array}$$

can be defined as follows. Let B_n be the free extension of A. Defining B' as a free extension of A, the basis of B' can be given as

$$f_{n-1} d_{/x_n} : X_n \to A_{n-1} \to B_{n-1}; n \geqslant 2.$$

For a cofibration C, C_C with weak equivalences and cofibrations is a cofibration structure.

Corollary 11. $\mathfrak{QuadChainLie}$ *is a cofibration structure.*

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