Some New Inequalities for Lipschitz Functions via a Functional

Bir Fonksiyonel yardımı ile Lipschitz Fonksiyonları için Bazı Yeni Eşitsizlikler

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Abstract

This study is about getting some new integral inequalities for Lipschitz functions by using a functional defined via a Lipschitz function. Here, some new Hermite-Hadamard (H-H) type inequalities are first found out as a corollary of main theorems. Afterwards, some new H-H type inequalities for Lipschitz functions by means of inequalities which are used for p-convex functions are obtained.

Keywords: Convex function, Lipschitz function,p-Convex function

Öz

Bu çalışma, bir Lipschitz fonksiyonu yardımı ile tanımlanmış bir fonksiyonel kullanarak Lipschitz fonksiyonları için bazı yeni integral eşitsizliklerin elde edilmesi ile ilgilidir. Burada ilk önce, bazı yeni Hermite-Hadamard tipi eşitsizlikler, ana teoremlerin bir sonucu olarak ortaya çıkarılacaktır. Daha sonra ise, p-konveks fonksiyonlar için kullanılan eşitsizlikler aracılığıyla Lipschitz fonksiyonları için yeni Hermite Hadamard tipi eşitsizlikler elde edilecektir.

Anahtar kelimeler: Konveks fonksiyon, Lipschitz fonksiyonu,p-konveks fonksiyon

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1. Preliminaries and Fundamentals

Definition 1.1. (Pečarić, 1992) A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the following inequality

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v)$$

holds for all $u, v \in I$ and $\lambda \in [0,1]$. If the above inequality reverses, then the f is said to be concave on the interval $I \neq \emptyset$.

The most important integral inequalities for convex functions are the Hermite-Hadamard inequalities. The following double integral inequalities are well known as the Hermite-Hadamard inequalities in the literature.

Theorem 1.2. Given $f: [u, v] \to \mathbb{R}$ be a convex function. Then the following inequalities

$$f\left(\frac{u + v}{2}\right) \leq \frac{1}{v - u}\int_u^v f(x)dx \leq \frac{f(u) + f(v)}{2}$$

are known as the H-H type inequalities.

Some refinements of the H-H inequalities have been extensively studied by a number of authors (Hadamard, 1893; Dragomir and Pearce, 2002; Dragomir, 2002) and they have obtained some new refinements of the H-H inequalities.

Definition 1.3. (İşcan, 2014) Given $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. $f: I \to \mathbb{R}$ is called harmonically convex function, if

$$f\left(\frac{uv}{\lambda u + (1 - \lambda)v}\right) \leq \lambda f(v) + (1 - \lambda)f(u)$$

for all $u, v \in I$ and $\lambda \in [0,1]$. If the above inequality is reversed, then f is called harmonically concave function.

Definition 1.4. (İşcan, 2016) Given $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. $f: I \to \mathbb{R}$ is called a p-convex function, if

$$f\left([\lambda u^p + (1 - \lambda)v^p]^{\frac{1}{p}}\right) \leq \lambda f(u) + (1 - \lambda)f(v)$$

for all $u, v \in I$ and $\lambda \in [0,1]$. If this inequality is reversed, then the function f is said to be p-concave.

From this definition, we can easily see that for $p = 1$ and $p = -1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on the interval $I \subset (0, \infty)$, respectively.

H-H type inequalities for the p-convex function are following:

Theorem 1.5. (İşcan, 2016) Given $f: I \subset (0, \infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\backslash\{0\}$, $u, v \in I, u < v$. If $f \in L[u, v]$, then we obtain

$$f\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{v^p - u^p}\int_u^v \frac{f(x)}{x^{1-p}}dx$$
$$\leq \frac{f(u) + f(v)}{2}.$$

These inequalities are sharp (Kunt and İşcan, 2017a,b,c). f is said to be p-concave, if these inequalities are reversed. We refer the reader to the recent papers related to p-convexity (Kunt and İşcan, 2017d; Latif et al, 2015; Niculescu, 2000; Yang and Tseng, 1999) and references therein.

Definition 1.6. (Roberts and Varberg, 1973) $f: I \to \mathbb{R}$ is said to satisfy the Lipschitz condition if there is a constant $M > 0$ such that

$$|f(u) - f(v)| \leq M|u - v|, \quad \forall u, v \in I.$$

Theorem 1.7. (Roberts and Varberg, 1973) If the function $f: I \to \mathbb{R}$ is convex, then f satisfies a Lipschitz condition on any closed interval $[a, b]$ contained in the interior $I°$ of I. Consequently, the function f is absolutely continuous on the interval $[a, b]$ and continuous on $I°$.

Theorem 1.8. (Dragomir et al, 2000) Given $f: I \subset (0, \infty) \to \mathbb{R}$ be an M-Lipschitzian mapping on I and $\alpha, \beta \in I$, $\alpha < \beta$. H is defined on the interval $[0,1]$ by

$$H(t) = \frac{1}{\beta - \alpha}\int_\alpha^\beta f\left(tx + (1 - t)\frac{\alpha + \beta}{2}\right)dx$$

is a Lipschitzian function for all $t \in [0,1]$. $H(t)$ is an $\left(\frac{M(\beta-\alpha)}{4}\right)$-Lipschitzian function.

Theorem 1.9. (Dragomir et al, 2000) Given $f: I \subset (0, \infty) \to \mathbb{R}$ be an M-Lipschitzian mapping on the interval I and $\alpha, \beta \in I, \alpha < \beta$. Then

$$\left|\frac{f\left(t\beta + (1-t)\frac{\alpha+\beta}{2}\right) + f\left(t\alpha + (1-t)\frac{\alpha+\beta}{2}\right)}{2} - H(t)\right|$$

$$\leq \frac{Mt}{3}(\beta - \alpha)$$

for all $t \in [0,1]$.

In (İşcan et al., 2018), the authors established the following H-H type inequalities for Lipschitzian functions.

Theorem 1.10. Given $f: I \subset (0, \infty) \to \mathbb{R}$ be an M-Lipschitzian function on the interval I of real numbers and $\alpha, \beta \in I$, $\alpha < \beta$. Then following inequalities hold for $p \geq 1$:

i)

$$\left|\frac{f(\alpha) + f(\beta)}{2} - \frac{p}{\beta^p - \alpha^p}\int_\alpha^\beta \frac{f(x)}{x^{1-p}}dx\right|$$

$$\leq 2M|\beta^p - \alpha^p|^{\frac{1}{p}}\frac{p^2}{(p+1)(2p+1)} \qquad (1)$$

ii)

$$\left|\frac{p}{\beta^p - \alpha^p}\int_\alpha^\beta \frac{f(x)}{x^{1-p}}dx - f\left(\left[\frac{\alpha^p + \beta^p}{2}\right]^{\frac{1}{p}}\right)\right|$$

$$\leq 2M\left(\frac{1}{2}\right)^{\frac{1}{p}}|\beta^p - \alpha^p|^{\frac{1}{p}}\frac{p}{p+1}. \qquad (2)$$

2. Main Results

Theorem 2.1. Given $f: I \subset (0, \infty) \to \mathbb{R}$ be an M-Lipschitzian function on the interval I, $a, b \in I$, $a < b$. For $p \geq 1$, the function defined by

$$H_p(t)$$

$$= \frac{p}{b^p - a^p}\int_a^b \frac{f\left(\left[tx^p + (1-t)\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right)}{x^{1-p}}dx,$$

fulfills the following Hölder condition

$$|H_p(t_2) - H_p(t_1)| \leq \frac{pM}{p+1}\left(\frac{b^p - a^p}{2}\right)^{\frac{1}{p}}|t_2 - t_1|^{\frac{1}{p}}$$

for $t_1, t_2 \in [0,1]$.

Proof. Let $t_1, t_2 \in [0,1]$. We have the following inequality:

$$|H_p(t_2) - H_p(t_1)|$$

$$= \frac{p}{b^p - a^p}\left|\int_a^b \frac{f\left(\left[t_2x^p + (1-t_2)\left(\frac{a^p + b^p}{2}\right)\right]^{\frac{1}{p}}\right)}{x^{1-p}}dx\right.$$

$$\left. - \int_a^b \frac{f\left(\left[t_1x^p + (1-t_1)\left(\frac{a^p + b^p}{2}\right)\right]^{\frac{1}{p}}\right)}{x^{1-p}}dx\right|$$

$$\leq \frac{pM}{b^p - a^p}$$

$$\times \int_a^b \left|\frac{\left[t_2x^p + (1-t_2)\frac{a^p + b^p}{2}\right]^{\frac{1}{p}} - \left[t_1x^p + (1-t_1)\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}}{x^{1-p}}\right|dx.$$

Set $A_p := \frac{a^p + b^p}{2}$. For $0 \leq x < y$ and $0 < \alpha \leq 1$ if we use the following well known inequality,

$$y^\alpha - x^\alpha \leq (y - x)^\alpha,$$

we have

$$|H_p(t_1) - H_p(t_2)| \leq \frac{pM}{b^p - a^p}$$

$$\times \int_a^b \left|\frac{[t_2x^p + (1-t_2)A_p]^{\frac{1}{p}} - [t_1x^p + (1-t_1)A_p]^{\frac{1}{p}}}{x^{1-p}}\right|dx$$

$$\leq \frac{pM}{b^p - a^p}\int_a^b \frac{|t_2x^p + (1-t_2)A_p - (t_1x^p + (1-t_1)A_p)|^{\frac{1}{p}}}{x^{1-p}}dx$$

$$= \frac{pM}{b^p - a^p}|t_2 - t_1|^{\frac{1}{p}}\int_a^b \frac{|x^p - A_p|^{\frac{1}{p}}}{x^{1-p}}dx. \qquad (3)$$

Now, let us calculate the integral in inequality (3). Since

$$x^p - A_p = 0 \Rightarrow x = M_p = \left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}},$$

One can write the following equality:

$$\int_a^b \frac{|x^p - A_p|^{\frac{1}{p}}}{x^{1-p}}dx$$

$$= \int_a^{M_p} \frac{(A_p - x^p)^{\frac{1}{p}}}{x^{1-p}}dx + \int_{M_p}^b \frac{(x^p - A_p)^{\frac{1}{p}}}{x^{1-p}}dx. \quad (4)$$

It is easily seen that

$$\int_a^{M_p} \frac{(A_p - x^p)^{\frac{1}{p}}}{x^{1-p}} dx$$

$$= \int_{M_p}^b \frac{(x^p - A_p)^{\frac{1}{p}}}{x^{1-p}} dx$$

$$= \frac{\left(\frac{b^p - a^p}{2}\right)^{\frac{p+1}{p}}}{p+1} \qquad (5)$$

Finally, if we put equality (5) in inequality (3), we obtain the desired result:

$$|H_p(t_2) - H_p(t_1)| \le \frac{pM}{p+1} \left(\frac{b^p - a^p}{2}\right)^{\frac{1}{p}} |t_2 - t_1|^{\frac{1}{p}}.$$

Corollary 2.2. For $p = 1$, the inequality

$$|H_1(t_2) - H_1(t_1)| \le M \frac{b-a}{4} |t_2 - t_1|$$

holds. This coincides with the Theorem 1.8.

Theorem 2.3. Given the function $f: I \subset (0, \infty) \to \mathbb{R}$ M-Lipschitzian on the interval I, $a, b \in I$, $a < b$ and $p \ge 1$. Then, the following inequality

$$\left| \frac{f\left(\left[ta^p + (1-t)\left(\frac{a^p + b^p}{2}\right)\right]^{\frac{1}{p}}\right)}{2} \right.$$

$$\left. + \frac{f\left(\left[tb^p + (1-t)\left(\frac{a^p + b^p}{2}\right)\right]^{\frac{1}{p}}\right)}{2} - H_p(t) \right|$$

$$\le 2Mt^{\frac{1}{p}} |b^p - a^p|^{\frac{1}{p}} \frac{p^2}{(p+1)(2p+1)}$$

inequality holds for all $t \in [0,1]$.

Proof. Let

$$u = \left[ta^p + (1-t)\left(\frac{a^p + b^p}{2}\right)\right]^{\frac{1}{p}}$$

and

$$v = \left[tb^p + (1-t)\left(\frac{a^p + b^p}{2}\right)\right]^{\frac{1}{p}}.$$

If we make the changes of variable

$$z = \left[tx^p + (1-t)\left(\frac{a^p + b^p}{2}\right)\right]^{\frac{1}{p}}$$

with $\frac{a^p + b^p}{2} = A_p$ in the equality

$$H_p(t) = \frac{p}{b^p - a^p} \int_a^b \frac{f\left(\left[tx^p + (1-t)A_p\right]^{\frac{1}{p}}\right)}{x^{1-p}} dx,$$

then we have

$$H_p(t)$$

$$= \frac{p}{t(b^p - a^p)} \int_a^b \frac{f\left(\left[tx^p + (1-t)A_p\right]^{\frac{1}{p}}\right)}{x^{1-p}} t dx$$

$$= \frac{p}{v^p - u^p} \int_u^v \frac{f(z)}{z^{1-p}} dz.$$

Now, using inequality (1) with $\alpha = \left[ta^p + (1-t)A_p\right]^{\frac{1}{p}} = u$, $\beta = \left[tb^p + (1-t)A_p\right]^{\frac{1}{p}} = v$ and $v^p - u^p = t(b^p - a^p)$, we obtain the following inequality:

$$\left| \frac{f(u) + f(v)}{2} - \frac{p}{v^p - u^p} \int_u^v \frac{f(z)}{z^{1-p}} dz \right|$$

$$\le 2Mt^{\frac{1}{p}} |b^p - a^p|^{\frac{1}{p}} \frac{p^2}{(p+1)(2p+1)}.$$

Corollary 2.4. The results obtained for $p = 1$ coincides with Theorem 1.9.

Proposition 2.5. Given $f: I \subset (0, \infty) \to \mathbb{R}$ a p-convex on I and $\alpha, \beta \in I$, $\alpha < \beta$. Then, for all $t \in [0,1]$, the following inequalities

i)

$$H_p(t) = \frac{p}{\beta^p - \alpha^p} \int_\alpha^\beta \frac{f\left(\left[tx^p + (1-t)\left(\frac{\alpha^p + \beta^p}{2}\right)\right]^{\frac{1}{p}}\right)}{x^{1-p}} dx$$

$$\le \frac{p}{\beta^p - \alpha^p} \int_\alpha^\beta \frac{f(x)}{x^{1-p}} dx,$$

ii)

$$H_p(t) = \frac{p}{\beta^p - \alpha^p} \int_\alpha^\beta \frac{f\left(\left[tx^p + (1-t)\left(\frac{\alpha^p + \beta^p}{2}\right)\right]^{\frac{1}{p}}\right)}{x^{1-p}} dx$$

$$\ge f(M_p)$$

and

$$f(M_p) \leq H_p(t)$$

$$= \frac{p}{\beta^p - \alpha^p} \int_\alpha^\beta \frac{f\left(\left[tx^p + (1-t)\left(\frac{\alpha^p + \beta^p}{2}\right)\right]^{\frac{1}{p}}\right)}{x^{1-p}} dx$$

$$\leq \frac{f\left(\left[t\alpha^p + (1-t)\left(\frac{\alpha^p + \beta^p}{2}\right)\right]^{\frac{1}{p}}\right)}{2}$$

$$- \frac{-f\left(\left[t\beta^p + (1-t)\left(\frac{\alpha^p + \beta^p}{2}\right)\right]^{\frac{1}{p}}\right)}{2} \qquad (6)$$

hold with $M_p = \left(\frac{\alpha^p + \beta^p}{2}\right)^{\frac{1}{p}}$.

Proof. (i) Since f is a p-convex function, we have

$$H_p(t)$$

$$= \frac{p}{\beta^p - \alpha^p} \int_\alpha^\beta \frac{f\left(\left[tx^p + (1-t)\left(\frac{\alpha^p + \beta^p}{2}\right)\right]^{\frac{1}{p}}\right)}{x^{1-p}} dx$$

$$\leq \frac{p}{\beta^p - \alpha^p} \int_\alpha^\beta \left[\frac{tf(x)}{x^{1-p}} + \frac{(1-t)f(M_p)}{x^{1-p}}\right] dx$$

$$= \frac{p}{\beta^p - \alpha^p}\left[t\int_\alpha^\beta \frac{f(x)}{x^{1-p}} dx + (1-t)f(M_p)\int_\alpha^\beta \frac{1}{x^{1-p}} dx\right]$$

$$= \frac{p}{\beta^p - \alpha^p}\left[t\int_\alpha^\beta \frac{f(x)}{x^{1-p}} dx + (1-t)f(M_p)\frac{\beta^p - \alpha^p}{p}\right].$$

By using the H-H inequality for p-convex functions, we get the following inequality:

$$H_p(t) \leq t\frac{p}{\beta^p - \alpha^p} \int_\alpha^\beta \frac{f(x)}{x^{1-p}} dx$$

$$+ (1-t)\frac{p}{\beta^p - \alpha^p} \int_\alpha^\beta \frac{f(x)}{x^{1-p}} dx$$

$$= \frac{p}{\beta^p - \alpha^p} \int_\alpha^\beta \frac{f(x)}{x^{1-p}} dx.$$

(ii) Let

$$u = \left[t\beta^p + (1-t)\left(\frac{\alpha^p + \beta^p}{2}\right)\right]^{\frac{1}{p}}$$

and

$$v = \left[t\alpha^p + (1-t)\left(\frac{\alpha^p + \beta^p}{2}\right)\right]^{\frac{1}{p}}.$$

If we make the change of variable

$$z = \left[t\alpha^p + (1-t)\left(\frac{\alpha^p + \beta^p}{2}\right)\right]^{\frac{1}{p}}$$

with $\frac{\alpha^p + \beta^p}{2} = A_p$ in the equality

$$H_p(t)$$

$$= \frac{p}{\beta^p - \alpha^p} \int_\alpha^\beta \frac{f\left(\left[tx^p + (1-t)\left(\frac{\alpha^p + \beta^p}{2}\right)\right]^{\frac{1}{p}}\right)}{x^{1-p}} dx,$$

then we have

$$H_p(t) = \frac{p}{v^p - u^p} \int_u^v \frac{f(z)}{z^{1-p}} dz.$$

By using the H-H inequality for p-convex function

$$H_p(t) = \frac{p}{v^p - u^p} \int_u^v \frac{f(z)}{z^{1-p}} dz$$

$$\geq f\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right)$$

$$= \left(\left[\frac{\alpha^p + \beta^p}{2}\right]^{\frac{1}{p}}\right)$$

$$= f(M_p)$$

is obtained. Consequently one gets

$$f(M_p) \leq \frac{p}{v^p - u^p} \int_u^v \frac{f(z)}{z^{1-p}} dz \leq \frac{f(u) + f(v)}{2}$$

which is equivalent inequality (6).

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