Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (1) (2020), 338 – 351
DOI : 10.15672/hujms.552213

Research Article

Basis properties of root functions of a regular fourth order boundary value problem

Ufuk Kaya*, Esma Kara Kuzu

Department of Mathematics, Faculty of Arts and Sciences, Bitlis Eren University, Bitlis 13000, Turkey

Abstract

In this paper, we consider the following boundary value problem

$$\begin{aligned} &y^{(4)} + q(x)\,y = \lambda y, \quad 0 < x < 1, \\ &y'''(1) - (-1)^{\sigma} y'''(0) + \alpha y(0) = 0, \\ &y^{(s)}(1) - (-1)^{\sigma} y^{(s)}(0) = 0, \quad s = \overline{0,2}, \end{aligned}$$

where λ is a spectral parameter, $q(x) \in L_1(0,1)$ is complex-valued function and $\sigma = 0, 1$. The boundary conditions of this problem are regular but not strongly regular. Asymptotic formulae for eigenvalues and eigenfunctions of the considered boundary value problem are established. When $\alpha \neq 0$, we proved that all the eigenvalues, except for finite number, are simple and the system of root functions of this spectral problem forms a Riesz basis in the space $L_2(0,1)$. Furthermore, we show that the system of root functions forms a basis in the space $L_p(0,1)$, $1 < p < \infty$ $(p \neq 2)$, under the conditions $\alpha \neq 0$ and $q(x) \in W_1^1(0,1)$.

Mathematics Subject Classification (2010). 34B05, 34L20, 34L10

Keywords. fourth order eigenvalue problem, not strongly regular boundary conditions, asymptotic behavior of eigenvalues and eigenfunctions, basis properties of the system of root functions

1. Introduction

Henceforth, L denotes the differential operator generated by the differential expression

$$l(y) = y^{(4)} + q(x)\,y, \quad x \in (0,1), \tag{1.1}$$

and boundary conditions

$$\begin{aligned} &U_3(y) \equiv y'''(1) - (-1)^{\sigma} y'''(0) + \alpha y(0) = 0, \\ &U_s(y) \equiv y^{(s)}(1) - (-1)^{\sigma} y^{(s)}(0) = 0, \end{aligned} \tag{1.2}$$

where $q(x) \in L_1(0,1)$ is complex-valued function, $s = \overline{0,2}$ and $\sigma = 0, 1$. It is easy to verify that boundary conditions (1.2) are regular, but not strongly regular.

*Corresponding Author.
Email addresses: mat-ufuk@hotmail.com (U. Kaya), esmakara_@hotmail.com (E. Kara Kuzu)
Received: 29.05.2017; Accepted: 29.11.2018

In [11, 14–16], Kerimov, Kaya and Gunes investigated the following problem

$$\begin{array}{l} y^{(4)} + p_2(x)\,y'' + p_1(x)\,y' + p_0(x)\,y = \lambda y, \qquad 0 < x < 1, \\ y'''(1) - (-1)^{\sigma} y'''(0) + \alpha_{3,2} y''(0) + \alpha_{3,1} y'(0) + \alpha_{3,0} y(0) = 0, \\ y''(1) - (-1)^{\sigma} y''(0) + \alpha_{2,1} y'(0) + \alpha_{2,0} y(0) = 0, \\ y'(1) - (-1)^{\sigma} y'(0) + \alpha_{1,0} y(0) = 0, \\ y(1) - (-1)^{\sigma} y(0) = 0 \end{array}$$

in various cases. However, the problems in [11, 14–16] cannot be reduced to eigenvalue problem for the operator (1.1)-(1.2).

In [8, 19, 27], it was proven that the system of root functions of a differential operator with strongly regular boundary conditions forms a basis. Besides, the basicity of root functions of a differential operator with non-strongly regular boundary conditions was investigated in [3–7, 9, 12, 17, 20–26, 29–33]. For more information about these papers, see [11, 14–16].

We define c_0 and ε_n as follows:

$$c_0 = \int_0^1 q(\xi)\,d\xi, \tag{1.3}$$

$$\varepsilon_n = \left|\int_0^1 q(\xi)\,.e^{2(2n-\sigma)\pi i\xi} d\xi\right| + \left|\int_0^1 q(\xi)\,.e^{-2(2n-\sigma)\pi i\xi} d\xi\right| + n^{-1}. \tag{1.4}$$

Now, we give two theorems and their corollary and we will prove them.

Theorem 1.1. *If $q(x) \in L_1(0,1)$ is a complex-valued function and $\alpha \neq 0$, all eigenvalues of differential operator (1.1)–(1.2), excluding a finite number, are simple and form two sequences $\{\lambda_{n,1}\}$ and $\{\lambda_{n,2}\}$ and these eigenvalues have the following asymptotic formulae for sufficiently large numbers n:*

$$\begin{aligned} \lambda_{n+n_1,1} &= ((2n-\sigma)\pi)^4 . \left\{1 + \frac{c_0}{((2n-\sigma)\pi)^4} + O\left(n^{-4}\varepsilon_n\right)\right\}, \\ \lambda_{n+n_2,2} &= ((2n-\sigma)\pi)^4 . \left\{1 + \frac{c_0 - 2(-1)^{\sigma}\alpha}{((2n-\sigma)\pi)^4} + O\left(n^{-4}\varepsilon_n\right)\right\}, \end{aligned} \tag{1.5}$$

where n_1, n_2 are certain integers. Moreover, for sufficiently large numbers n, the corresponding eigenfunctions $u_{n,1}(x)$ and $u_{n,2}(x)$ have the asymptotic formulae:

$$\begin{aligned} u_{n+n_1,1}(x) &= \sqrt{2}\sin(2n-\sigma)\pi x + O(\varepsilon_n), \\ u_{n+n_2,2}(x) &= \sqrt{2}\cos(2n-\sigma)\pi x + O(\varepsilon_n). \end{aligned} \tag{1.6}$$

Theorem 1.2. *If $q(x) \in L_1(0,1)$ is a complex-valued function and $\alpha \neq 0$, the root functions of differential operator (1.1)-(1.2) form a Riesz basis in the space $L_2(0,1)$. In addition, if $q(x) \in W_1^1(0,1)$, then the root functions form a basis in $L_p(0,1)$, $1 < p < \infty$, where*

$$L_p(0,1) = \left\{ f \middle| f : (0,1) \to \mathbb{C}, \int_0^1 |f(\xi)|^p\,d\xi < +\infty \right\},$$

$$W_p^n(0,1) = \left\{ f \middle| f : (0,1) \to \mathbb{C}, f^{(n)} \in L_p(0,1) \right\}.$$

Corollary 1.3. *If $q(x) \in L_2(0,1)$ is a complex-valued function and $\alpha \neq 0$, then $n_1 + n_2 = 1 - \sigma$. Hence, we can choose $n_1 = 0$, $n_2 = 1 - \sigma$.*

2. Some auxiliary formulae

We denote the set

$$\left\{\rho \in \mathbb{C} : 0 \leq \arg \rho \leq \frac{\pi}{4}\right\} \tag{2.1}$$

by S_0 and the different four roots of the algebraic equation $\omega^4 + 1 = 0$ by ω_k, $k = \overline{1,4}$. The numbers ω_k, $k = \overline{1,4}$, can be ordered so that the inequalities

$$\Re(\rho\omega_1) \leq \Re(\rho\omega_2) \leq \Re(\rho\omega_3) \leq \Re(\rho\omega_4) \tag{2.2}$$

hold for all $\rho \in S_0$, where $\Re(z)$ denotes the real parts of a complex number z (see [28, Chapter II, §4.2]). From now on, the numbers ω_k, $k = \overline{1,4}$, will be chosen by satisfying the inequalities (2.2) for all $\rho \in S_0$. Then, we get by [28, Chapter II, §4.8] that the numbers $\omega_1, \omega_2, \omega_3, \omega_4$ are determined as

$$\omega_1 = e^{\frac{3\pi i}{4}}, \qquad \omega_2 = e^{-\frac{3\pi i}{4}}, \qquad \omega_3 = e^{\frac{\pi i}{4}}, \qquad \omega_4 = e^{-\frac{\pi i}{4}}. \tag{2.3}$$

One can easily see that

$$\omega_1 = -\omega_4, \qquad \omega_2 = -\omega_3. \tag{2.4}$$

Lemma 2.1 ([16]). *For all $\rho \in S_0$, the inequalities*

$$\Re(\rho\omega_1) \leq -\frac{\sqrt{2}}{2}|\rho|, \qquad \Re(\rho\omega_4) \geq \frac{\sqrt{2}}{2}|\rho|. \tag{2.5}$$

are valid.

Let

$$T_0 = \{\rho - c : \rho \in S_0\},$$

where c is a complex number. The inequalities (2.2) and (2.5) will be rewritten in the forms

$$\Re((\rho+c)\omega_1) \leq \Re((\rho+c)\omega_2) \leq \Re((\rho+c)\omega_3) \leq \Re((\rho+c)\omega_4), \tag{2.6}$$

$$\Re((\rho+c)\omega_1) \leq -\frac{\sqrt{2}}{2}|\rho+c|, \qquad \Re((\rho+c)\omega_4) \geq \frac{\sqrt{2}}{2}|\rho+c| \tag{2.7}$$

for all $\rho \in T_0$.

For each $\rho \in T_0$, the equation

$$l(y) + \rho^4 y = 0 \tag{2.8}$$

has four solutions $y_1(x,\rho)$, $y_2(x,\rho)$, $y_3(x,\rho)$, $y_4(x,\rho)$. These solutions are linearly independent and analytic when $|\rho| \geq M_0$, where M_0 is a positive constant [28, Chapter II, §4.5-4.6]. Besides, the derivatives of these functions satisfy the following integro-differential equations

$$\begin{aligned}\frac{d^s y_k(x,\rho)}{dx^s} &= \rho^s \omega_k^s e^{\rho\omega_k x} + \frac{1}{4\rho^3}\int_0^x \frac{\partial^s K_1(x,\xi,\rho)}{\partial x^s} q(\xi)\, y_k(\xi,\rho)\, d\xi - \\ &-\frac{1}{4\rho^3}\int_x^1 \frac{\partial^s K_2(x,\xi,\rho)}{\partial x^s} q(\xi)\, y_k(\xi,\rho)\, d\xi, \quad s = \overline{0,3},\end{aligned} \tag{2.9}$$

where

$$K_1(x,\xi,\rho) = \sum_{\alpha=1}^{k} \omega_\alpha e^{\rho\omega_\alpha(x-\xi)}, \qquad K_2(x,\xi,\rho) = \sum_{\alpha=k+1}^{4} \omega_\alpha e^{\rho\omega_\alpha(x-\xi)}. \tag{2.10}$$

Let $z_{k,s}(x,\rho)$, $k = \overline{1,4}$, $s = \overline{0,3}$, be functions that satisfy the equations

$$\frac{d^s y_k(x,\rho)}{dx^s} = \rho^s e^{\rho\omega_k x} z_{k,s}(x,\rho). \tag{2.11}$$

By [28, Chapter II, §4.5], the functions $z_{k,s}(x,\rho)$ are analytic with respect to ρ and satisfy

$$z_{k,s}(x,\rho)=\omega_k^s+O\left(\rho^{-1}\right),\quad s=\overline{0,3},\quad k=\overline{1,4}. \tag{2.12}$$

By (2.9)-(2.11), we have

$$\begin{gathered}
z_{k,s}(x,\rho)=\omega_k^s+\frac{\omega_k^{s+1}}{4\rho^3}\int_0^x q(\xi)\,z_{k,0}(\xi,\rho)\,d\xi+\\
+\frac{1}{4\rho^3}\sum_{\alpha=1}^{k-1}\omega_\alpha^{s+1}\int_0^x e^{\rho(\omega_\alpha-\omega_k)(x-\xi)}q(\xi)\,z_{k,0}(\xi,\rho)\,d\xi-\\
-\frac{1}{4\rho^3}\sum_{\alpha=k+1}^{4}\omega_\alpha^{s+1}\int_x^1 e^{\rho(\omega_\alpha-\omega_k)(x-\xi)}q(\xi)\,z_{k,0}(\xi,\rho)\,d\xi.
\end{gathered} \tag{2.13}$$

Note that, by (2.6), we get

$$\Re\left(\rho\left(\omega_\alpha-\omega_\beta\right)\right)=\Re\left(\left(\rho+c\right)\left(\omega_\alpha-\omega_\beta\right)\right)-\Re\left(c\left(\omega_\alpha-\omega_\beta\right)\right)\le 2\left|c\right|,$$

where $1\le\alpha\le\beta\le 4$. By using the above inequality and (2.12), we obtain for $k=\overline{1,4}$

$$\int_0^x q(\xi)\,z_{k,0}(\xi,\rho)\,e^{\rho(\omega_\alpha-\omega_k)(x-\xi)}d\xi=O(1),\quad \alpha\le k,$$

$$\int_x^1 q(\xi)\,z_{k,0}(\xi,\rho)\,e^{\rho(\omega_\alpha-\omega_k)(x-\xi)}d\xi=O(1),\quad \alpha> k.$$

By using the last relations and the formulae (2.12)-(2.13), we get

$$z_{k,s}(x,\rho)=\omega_k^s+O\left(\rho^{-3}\right),\quad s=\overline{0,3},\quad k=\overline{1,4}. \tag{2.14}$$

If we now put (2.14) in (2.13), then (2.13) takes the form

$$\begin{gathered}
z_{k,s}(x,\rho)=\omega_k^s+\frac{\omega_k^{s+1}}{4\rho^3}\int_0^x q(\xi)\,d\xi+\frac{1}{4\rho^3}\sum_{\alpha=1}^{k-1}\omega_\alpha^{s+1}\int_0^x q(\xi)\,e^{\rho(\omega_\alpha-\omega_k)(x-\xi)}d\xi-\\
-\frac{1}{4\rho^3}\sum_{\alpha=k+1}^{4}\omega_\alpha^{s+1}\int_x^1 q(\xi)\,e^{\rho(\omega_\alpha-\omega_k)(x-\xi)}d\xi+O\left(\rho^{-6}\right).
\end{gathered}$$

By the last relation, we have

$$
\begin{aligned}
z_{2,s}(0,\rho) &= \omega_2^s - \frac{\omega_3^{s+1}}{4\rho^3}\int_0^1 q(\xi)\, e^{2\rho\omega_2\xi} d\xi \\
&\quad - \frac{\omega_4^{s+1}}{4\rho^3}\int_0^1 q(\xi)\, e^{\rho(\omega_2-\omega_4)\xi} d\xi + O\left(\rho^{-6}\right), \\
z_{3,s}(0,\rho) &= \omega_3^s - \frac{\omega_4^{s+1}}{4\rho^3}\int_0^1 q(\xi)\, e^{\rho(\omega_3-\omega_4)\xi} d\xi + O\left(\rho^{-6}\right), \\
z_{2,s}(1,\rho) &= \omega_2^s + \frac{\omega_1^{s+1}}{4\rho^3}\int_0^1 q(\xi)\, e^{\rho(\omega_1-\omega_2)(1-\xi)} d\xi + O\left(\rho^{-6}\right), \\
z_{3,s}(1,\rho) &= \omega_3^s + \frac{\omega_1^{s+1}}{4\rho^3}\int_0^1 q(\xi)\, e^{\rho(\omega_1-\omega_3)(1-\xi)} d\xi + \\
&\quad + \frac{\omega_2^{s+1}}{4\rho^3}\int_0^1 q(\xi)\, e^{2\rho\omega_2(1-\xi)} d\xi + O\left(\rho^{-6}\right),
\end{aligned}
\tag{2.15}
$$

where we assume that $c_0 = 0$. The case $c_0 \neq 0$ will be investigated later.

3. Proof of Theorem 1.1

Let

$$
\Delta(\rho) = \begin{vmatrix} U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) \\ U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) \\ U_0(y_1) & U_0(y_2) & U_0(y_3) & U_0(y_4) \end{vmatrix}. \tag{3.1}
$$

If the vertex $-c$ in the domain T_0 is properly chosen, then eigenvalues λ of the operator (1.1)-(1.2) whose absolute values are sufficiently large have the form $\lambda = -\rho^4$, where the numbers ρ are the zeros of the following equation

$$
\Delta(\rho) = 0 \tag{3.2}
$$

and in T_0. Conversely, the set of such numbers ρ contains all the zeros of (3.2) in T_0 excluding a finite number [28, Chapter II. § 4.9]. By (2.11), we have

$$
\begin{aligned}
U_s(y_k) &= \rho^s \left\{ e^{\rho\omega_k} z_{k,s}(1,\rho) - (-1)^\sigma z_{k,s}(0,\rho) \right\}, \\
U_3(y_k) &= \rho^3 \left\{ e^{\rho\omega_k} z_{k,3}(1,\rho) - (-1)^\sigma z_{k,3}(0,\rho) \right\} + \alpha z_{k,0}(0,\rho)
\end{aligned}
\tag{3.3}
$$

for $s = \overline{0,2}$ and $k = \overline{1,4}$. By (2.7), $e^{\rho\omega_1}$ exponentially tends to zero and $e^{\rho\omega_4}$ exponentially tends to infinity. So, the relations

$$
\begin{aligned}
U_s(y_1) &= -(-1)^\sigma \rho^s \left\{ z_{1,s}(0,\rho) + O\left(\rho^{-7}\right) \right\}, \ s = \overline{0,2}, \\
U_3(y_1) &= -(-1)^\sigma \rho^3 \left\{ z_{1,3}(0,\rho) - (-1)^\sigma \frac{\alpha}{\rho^3} z_{1,0}(0,\rho) + O\left(\rho^{-7}\right) \right\}, \\
U_s(y_4) &= \rho^s e^{\rho\omega_4} \left\{ z_{4,s}(1,\rho) + O\left(\rho^{-7}\right) \right\}, \ s = \overline{0,3}
\end{aligned}
\tag{3.4}
$$

are valid by (2.14) and (3.3).

Let

$$A_{s,k}(\rho)=\begin{cases} z_{1,s}(0,\rho), & \text{if } k=1,\\ e^{\rho\omega_k}z_{k,s}(1,\rho)-(-1)^{\sigma}z_{k,s}(0,\rho), & \text{if } k=2,3,\\ z_{4,s}(1,\rho), & \text{if } k=4,\end{cases}$$

$$A_{3,k}(\rho)=\begin{cases} z_{1,3}(0,\rho)-(-1)^{\sigma}\dfrac{\alpha}{\rho^3}z_{1,0}(0,\rho), & \text{if } k=1,\\ e^{\rho\omega_k}z_{k,3}(1,\rho)-(-1)^{\sigma}z_{k,3}(0,\rho)+\dfrac{\alpha}{\rho^3}z_{k,0}(0,\rho), & \text{if } k=2,3,\\ z_{4,3}(1,\rho), & \text{if } k=4,\end{cases} \tag{3.5}$$

where $s=\overline{0,2}$. By the formulae (3.3)-(3.5), it is obvious that

$$\begin{aligned} U_s(y_1)&=-(-1)^{\sigma}\rho^s\left\{A_{s,1}(\rho)+O\left(\rho^{-7}\right)\right\},\\ U_s(y_k)&=\rho^s A_{s,k}(\rho),\\ U_s(y_4)&=\rho^s e^{\rho\omega_4}\left\{A_{s,4}(\rho)+O\left(\rho^{-7}\right)\right\},\end{aligned} \tag{3.6}$$

where $k=2,3$ and $s=\overline{0,3}$. We put these formulae of boundary conditions in the equation (3.2). If we divide out the common multipliers ρ^3, ρ^2, ρ of the rows and also divide out the common multipliers $-(-1)^{\sigma}$ and $e^{\rho\omega_4}$ of the columns of the determinant $\Delta(\rho)$, then we get that the equation (3.2) is equivalent to

$$\Delta_1(\rho)+O\left(\rho^{-7}\right)=0, \tag{3.7}$$

where

$$\Delta_1(\rho)=\begin{vmatrix} A_{3,1}(\rho) & A_{3,2}(\rho) & A_{3,3}(\rho) & A_{3,4}(\rho)\\ A_{2,1}(\rho) & A_{2,2}(\rho) & A_{2,3}(\rho) & A_{2,4}(\rho)\\ A_{1,1}(\rho) & A_{1,2}(\rho) & A_{1,3}(\rho) & A_{1,4}(\rho)\\ A_{0,1}(\rho) & A_{0,2}(\rho) & A_{0,3}(\rho) & A_{0,4}(\rho)\end{vmatrix}. \tag{3.8}$$

We now rewrite the formulae (35)-(36) in [14]. If ρ is a root of equation (3.7), we get that the equalities

$$e^{\rho\omega_2}-(-1)^{\sigma}=O\left(\rho^{-3}\right),\quad e^{\rho\omega_3}-(-1)^{\sigma}=O\left(\rho^{-3}\right) \tag{3.9}$$

are valid.

By using the relations (2.14), (2.15) and (3.9) for $s=\overline{0,3}$, we have

$$\begin{aligned} A_{s,k}(\rho)&=A^{(k)}_{s,k}(\rho)+B^{(k)}_{s,k}(\rho)+O\left(\rho^{-6}\right),\quad k=2,3,\\ A_{s,k}(\rho)&=\omega_k^s+O\left(\rho^{-3}\right),\quad k=1,4,\end{aligned} \tag{3.10}$$

where

$$\begin{aligned} A^{(2)}_{s,2}(\rho)&=\omega_2^s\left(e^{\rho\omega_2}-(-1)^{\sigma}\right)+\frac{(-1)^{\sigma}\omega_3^{s+1}}{4\rho^3}\int_0^1 q(\xi)e^{2\rho\omega_2\xi}d\xi,\quad s=\overline{0,2},\\ A^{(3)}_{s,3}(\rho)&=\omega_3^s\left(e^{\rho\omega_3}-(-1)^{\sigma}\right)+\frac{(-1)^{\sigma}\omega_2^{s+1}}{4\rho^3}\int_0^1 q(\xi)e^{2\rho\omega_2(1-\xi)}d\xi,\quad s=\overline{0,2},\\ A^{(2)}_{3,2}(\rho)&=\omega_2^3\left(e^{\rho\omega_2}-(-1)^{\sigma}\right)+\frac{(-1)^{\sigma}\omega_3^{4}}{4\rho^3}\int_0^1 q(\xi)e^{2\rho\omega_2\xi}d\xi+\frac{\alpha}{\rho^3},\\ A^{(3)}_{3,3}(\rho)&=\omega_3^3\left(e^{\rho\omega_3}-(-1)^{\sigma}\right)+\frac{(-1)^{\sigma}\omega_2^{4}}{4\rho^3}\int_0^1 q(\xi)e^{2\rho\omega_2(1-\xi)}d\xi+\frac{\alpha}{\rho^3},\end{aligned} \tag{3.11}$$

and

$$B_{s,k}^{(k)}(\rho)=\frac{(-1)^{\sigma}\omega_1^{s+1}}{4\rho^3}\int\limits_0^1 q(\xi)\,e^{\rho(\omega_1-\omega_k)(1-\xi)}d\xi$$
$$+\frac{(-1)^{\sigma}\omega_4^{s+1}}{4\rho^3}\int\limits_0^1 q(\xi)\,e^{\rho(\omega_k-\omega_4)\xi}d\xi,\quad s=\overline{0,3},\quad k=2,3. \tag{3.12}$$

By the relations (3.9), (3.11) and (3.12), we have

$$A_{s,k}(\rho)=O\left(\rho^{-3}\right),\quad k=2,3,\quad s=\overline{0,3}. \tag{3.13}$$

If we put the equalities (3.10) in the determinant (3.8), then, by using (3.13), we get that the equation (3.7) is equivalent to

$$\Delta_2(\rho)+O\left(\rho^{-7}\right)=0, \tag{3.14}$$

where

$$\Delta_2(\rho)=\begin{vmatrix} \omega_1^3 & A_{3,2}^{(2)}(\rho)+B_{3,2}^{(2)}(\rho) & A_{3,3}^{(3)}(\rho)+B_{3,3}^{(3)}(\rho) & \omega_4^3 \\ \omega_1^2 & A_{2,2}^{(2)}(\rho)+B_{2,2}^{(2)}(\rho) & A_{2,3}^{(3)}(\rho)+B_{2,3}^{(3)}(\rho) & \omega_4^2 \\ \omega_1 & A_{1,2}^{(2)}(\rho)+B_{1,2}^{(2)}(\rho) & A_{1,3}^{(3)}(\rho)+B_{1,3}^{(3)}(\rho) & \omega_4 \\ 1 & A_{0,2}^{(2)}(\rho)+B_{0,2}^{(2)}(\rho) & A_{0,3}^{(3)}(\rho)+B_{0,3}^{(3)}(\rho) & 1 \end{vmatrix}.$$

By the definition of $B_{s,k}^{(k)}(\rho)$ (see: (3.12)), it can be easily proven that the columns

$$\left(B_{3,2}^{(2)}(\rho),B_{2,2}^{(2)}(\rho),B_{1,2}^{(2)}(\rho),B_{0,2}^{(2)}(\rho)\right)^T$$

and

$$\left(B_{3,3}^{(3)}(\rho),B_{2,3}^{(3)}(\rho),B_{1,3}^{(3)}(\rho),B_{0,3}^{(3)}(\rho)\right)^T$$

are two linear combinations of the first and fourth columns of the determinant $\Delta_2(\rho)$. Consequently, the determinant $\Delta_2(\rho)$ can be rewritten as follows:

$$\Delta_2(\rho)=\begin{vmatrix} \omega_1^3 & A_{3,2}^{(2)}(\rho) & A_{3,3}^{(3)}(\rho) & \omega_4^3 \\ \omega_1^2 & A_{2,2}^{(2)}(\rho) & A_{2,3}^{(3)}(\rho) & \omega_4^2 \\ \omega_1 & A_{1,2}^{(2)}(\rho) & A_{1,3}^{(3)}(\rho) & \omega_4 \\ 1 & A_{0,2}^{(2)}(\rho) & A_{0,3}^{(3)}(\rho) & 1 \end{vmatrix}. \tag{3.15}$$

If we put (3.11) in the determinant (3.15) and calculate it, then we get that the equation (3.14) is reduced to

$$-16\left(e^{\rho\omega_2}-(-1)^{\sigma}\right)\left(e^{\rho\omega_3}-(-1)^{\sigma}\right)$$
$$-\frac{4\omega_2\alpha\left(e^{\rho\omega_2}-(-1)^{\sigma}\right)}{\rho^3}+\frac{4\omega_2\alpha\left(e^{\rho\omega_3}-(-1)^{\sigma}\right)}{\rho^3}+O\left(\rho^{-6}\varepsilon(\rho)\right)=0, \tag{3.16}$$

where

$$\varepsilon(\rho)=\left|\int\limits_0^1 q(\xi)\,e^{2\rho\omega_2\xi}d\xi\right|+\left|\int\limits_0^1 q(\xi)\,e^{2\rho\omega_2(1-\xi)}d\xi\right|+\left|\rho^{-1}\right|. \tag{3.17}$$

Note that the formula

$$\varepsilon(\rho)=o(1)$$

can be easily proved by using the proof of Riemann-Lebesque Lemma.

After some calculations, the equation (3.16) splits into the following two equations:

$$e^{\rho\omega_2}=(-1)^{\sigma}+O\left(\rho^{-3}\varepsilon(\rho)\right), \tag{3.18}$$

$$e^{\rho\omega_2} = (-1)^\sigma + \frac{\omega_2\alpha}{2\rho^3} + O\left(\rho^{-3}\varepsilon(\rho)\right). \tag{3.19}$$

Consider the equation (3.18). By Rouche's theorem, we can get that the roots of the equation (3.18) in T_0 with sufficiently large absolute values lie in the sets $G_n \subset T_0$, where G_n is $O(n^{-1})$-neighborhood of $-(2n-\sigma)\pi i/\omega_2$, $n = n_0, n_0+1, \dots$ and n_0 is sufficiently large positive integer [28, Chapter II, § 4.9]. Besides, the equation (3.18) has a unique root in G_n. Assume that $\widetilde{\rho}$ is the unique root of (3.18) in G_n. By the equalities (40) and (41) in [14], we obtain

$$\widetilde{\rho} = -\frac{(2n-\sigma)\pi i}{\omega_2} + r, \qquad r = O\left(n^{-3}\right). \tag{3.20}$$

If we use the formulae (3.20) in (3.17), we obtain

$$\varepsilon(\rho) = O(\varepsilon_n), \tag{3.21}$$

where ε_n is the sequence defined in (1.4).

Now, we find more accurate formula for the number r. The following formulae

$$\frac{1}{\widetilde{\rho}^3} = \frac{\omega_2}{(2n-\sigma)^3\pi^3} + O\left(n^{-7}\right), \tag{3.22}$$

$$e^{\widetilde{\rho}\omega_2} = (-1)^\sigma\left\{1 + r\omega_2 + O\left(n^{-6}\right)\right\} \tag{3.23}$$

can be easily obtained by using (3.20). By putting $\rho = \widetilde{\rho}$ in (3.18) and using the relations (3.21) and (3.23), we have

$$r = O\left(n^{-3}\varepsilon_n\right). \tag{3.24}$$

Thus, the equation (3.18) has the unique root

$$\widetilde{\rho}_{n,1} = -\frac{(2n-\sigma)\pi i}{\omega_2} + O\left(n^{-3}\varepsilon_n\right) \tag{3.25}$$

in $O(n^{-1})$-neigbourhood G_n of $z_n = -(2n-\sigma)\pi i/\omega_2$, $n = n_0, n_0+1, \dots$ by (3.20) and (3.24).

Similarly, we conclude that the equation (3.19) has the unique root

$$\widetilde{\rho}_{n,2} = -\frac{1}{\omega_2}\left\{(2n-\sigma)\pi i - \frac{(-1)^\sigma i\alpha}{2(2n-\sigma)^3\pi^3}\right\} + O\left(n^{-3}\varepsilon_n\right) \tag{3.26}$$

in $O(n^{-1})$-neigbourhood G_n of the point z_n, $n = n_0, n_0+1, \dots$ by the formulae (3.20)-(3.23).

Now, we investigate the eigenfunction $\widetilde{u}_{n,1}(x)$ corresponding to the eigenvalue $\lambda = -(\widetilde{\rho}_{n,1})^4$. We use the following determinant for this eigenfunction

$$\widetilde{u}_{n,1}(x) = \frac{(-1)^\sigma e^{-\rho\omega_4}\sqrt{2}}{4\omega_2 i\alpha\rho^3}\begin{vmatrix} y_1(x,\rho) & y_2(x,\rho) & y_3(x,\rho) & y_4(x,\rho) \\ U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) \\ U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) \end{vmatrix},$$

where $\rho = \widetilde{\rho}_{n,1}$ and n is sufficiently large positive integer. Easily, we can rewrite

$$\widetilde{u}_{n,1}(x) = -\frac{\rho^3\sqrt{2}}{4\omega_2 i\alpha}$$

$$\times\begin{vmatrix} -(-1)^\sigma y_1(x,\rho) & y_2(x,\rho) & y_3(x,\rho) & e^{-\rho\omega_4}y_4(x,\rho) \\ -(-1)^\sigma\rho^{-3}U_3(y_1) & \rho^{-3}U_3(y_2) & \rho^{-3}U_3(y_3) & \rho^{-3}e^{-\rho\omega_4}U_3(y_4) \\ -(-1)^\sigma\rho^{-2}U_2(y_1) & \rho^{-2}U_2(y_2) & \rho^{-2}U_2(y_3) & \rho^{-2}e^{-\rho\omega_4}U_2(y_4) \\ -(-1)^\sigma\rho^{-1}U_1(y_1) & \rho^{-1}U_1(y_2) & \rho^{-1}U_1(y_3) & \rho^{-1}e^{-\rho\omega_4}U_1(y_4) \end{vmatrix}, \tag{3.27}$$

By (2.11)-(2.12), we can obtain

$$y_k(x,\rho) = O(1), \quad k = 1,2,3, \qquad e^{-\rho\omega_4} y_4(x,\rho) = O(1), \tag{3.28}$$

where $\rho = \widetilde{\rho}_{n,1}$. Putting the formulae (3.6) in (3.27) and using (3.28), we get that the formulae (3.27) has the form

$$\widetilde{u}_{n,1}(x) = -\frac{\rho^3\sqrt{2}}{4\omega_2 i\alpha}\begin{vmatrix} -(-1)^\sigma y_1(x,\rho) & y_2(x,\rho) & y_3(x,\rho) & e^{-\rho\omega_4} y_4(x,\rho) \\ A_{3,1}(\rho) & A_{3,2}(\rho) & A_{3,3}(\rho) & A_{3,4}(\rho) \\ A_{2,1}(\rho) & A_{2,2}(\rho) & A_{2,3}(\rho) & A_{2,4}(\rho) \\ A_{1,1}(\rho) & A_{1,2}(\rho) & A_{1,3}(\rho) & A_{1,4}(\rho) \end{vmatrix}, \tag{3.29}$$

where $\rho = \widetilde{\rho}_{n,1}$. If we calculate the determinant in (3.29) by using (3.10), (3.13) and (3.28), then we have

$$\widetilde{u}_{n,1}(x) = -\frac{\rho^3\sqrt{2}}{4\omega_2 i\alpha}\{y_3(x,\rho) E_2(\rho) - y_2(x,\rho) E_3(\rho)\} + O\left(\rho^{-3}\right), \tag{3.30}$$

where $\rho = \widetilde{\rho}_{n,1}$ and

$$E_k(\rho) = \begin{vmatrix} \omega_1^3 & A_{3,k}(\rho) & \omega_4^3 \\ \omega_1^2 & A_{2,k}(\rho) & \omega_4^2 \\ \omega_1 & A_{1,k}(\rho) & \omega_4 \end{vmatrix}, \quad k = 2,3.$$

By the last formula and (3.10), we get that the determinant $E_k(\rho)$ can be rewritten as follows

$$E_k(\rho) = \begin{vmatrix} \omega_1^3 & A_{3,k}^{(k)}(\rho) & \omega_4^3 \\ \omega_1^2 & A_{2,k}^{(k)}(\rho) & \omega_4^2 \\ \omega_1 & A_{1,k}^{(k)}(\rho) & \omega_4 \end{vmatrix} + \begin{vmatrix} \omega_1^3 & B_{3,k}^{(k)}(\rho) & \omega_4^3 \\ \omega_1^2 & B_{2,k}^{(k)}(\rho) & \omega_4^2 \\ \omega_1 & B_{1,k}^{(k)}(\rho) & \omega_4 \end{vmatrix} + O\left(\rho^{-6}\right), \quad k = 2,3,$$

where $\rho = \widetilde{\rho}_{n,1}$. By (3.12), the second determinant above is zero, i.e.,

$$E_k(\rho) = \begin{vmatrix} \omega_1^3 & A_{3,k}^{(k)}(\rho) & \omega_4^3 \\ \omega_1^2 & A_{2,k}^{(k)}(\rho) & \omega_4^2 \\ \omega_1 & A_{1,k}^{(k)}(\rho) & \omega_4 \end{vmatrix} + O\left(\rho^{-6}\right), \quad k = 2,3, \tag{3.31}$$

where $\rho = \widetilde{\rho}_{n,1}$. The following formulae

$$A_{1,k}^{(k)}(\rho) = A_{2,k}^{(k)}(\rho) = O\left(\rho^{-3}\varepsilon\right), \quad A_{3,k}^{(k)}(\rho) = \frac{\alpha}{\rho^3} + O\left(\rho^{-3}\varepsilon\right)$$

are directly obtained by using (3.11) and (3.18), where $k = 2,3$, $\rho = \widetilde{\rho}_{n,1}$ and $\varepsilon = \varepsilon_n$. If we calculate the determinant in (3.31) by using the last relations, we get

$$E_k(\rho) = -\frac{2\omega_2\alpha}{\rho^3} + O\left(\rho^{-3}\varepsilon\right),$$

where $k = 2,3$ and $\rho = \widetilde{\rho}_{n,1}$ and $\varepsilon = \varepsilon_n$. Consequently, we have

$$\widetilde{u}_{n,1}(x) = \frac{\sqrt{2}}{2i}\left(y_3(x,\widetilde{\rho}_{n,1}) - y_2(x,\widetilde{\rho}_{n,1})\right) + O(\varepsilon_n)$$

by (3.30). On the other hand, we can write

$$y_2(x,\widetilde{\rho}_{n,1}) = e^{-(2n-\sigma)\pi i x} + O\left(n^{-1}\right), \qquad y_3(x,\widetilde{\rho}_{n,1}) = e^{(2n-\sigma)\pi i x} + O\left(n^{-1}\right),$$

$$(\widetilde{\rho}_{n,1})^{-1} = O\left(n^{-1}\right),$$

by (2.11), (2.12) and (3.25). Finally, we have the expression

$$\widetilde{u}_{n,1}(x) = \sqrt{2}\sin(2n-\sigma)\pi x + O(\varepsilon_n). \tag{3.32}$$

Now, we also investigate the eigenfunction $\widetilde{u}_{n,2}(x)$ corresponding to the eigenvalue $\lambda = -(\widetilde{\rho}_{n,2})^4$ by using the following determinant

$$\widetilde{u}_{n,2}(x) = \frac{(-1)^{\sigma} e^{-\rho\omega_4}\sqrt{2}}{4i\alpha} \begin{vmatrix} y_1(x,\rho) & y_2(x,\rho) & y_3(x,\rho) & y_4(x,\rho) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) \\ U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) \\ U_0(y_1) & U_0(y_2) & U_0(y_3) & U_0(y_4) \end{vmatrix}$$

where $\rho = \widetilde{\rho}_{n,2}$. In a similar way, we get

$$\widetilde{u}_{n,2}(x) = \sqrt{2}\cos(2n-\sigma)\pi x + O(\varepsilon_n). \tag{3.33}$$

We now prove the formulae (1.5) and (1.6). By the relation $\lambda = -\rho^4$, we have

$$\widetilde{\lambda}_{n,1} = -(\widetilde{\rho}_{n,1})^4 = ((2n-\sigma)\pi)^4\left\{1 + O\left(n^{-4}\varepsilon_n\right)\right\},$$

$$\widetilde{\lambda}_{n,2} = -(\widetilde{\rho}_{n,2})^4 = ((2n-\sigma)\pi)^4\left\{1 - \frac{2(-1)^{\sigma}\alpha}{((2n-\sigma)\pi)^4} + O\left(n^{-4}\varepsilon_n\right)\right\}.$$

The above formulae are valid in case of $c_0 = 0$. Now, assume that $c_0 \neq 0$ (see (1.3)). Consider the eigenvalue problem with the differential expression

$$y^{(4)} + q(x)y = \lambda y$$

(see (1.1)). We can rewrite this problem as

$$y^{(4)} + (q(x) - c_0)y = (\lambda - c_0)y.$$

One can easily see that the integral of $q(x) - c_0$ on the line $[0,1]$ is zero. Then, by the above proof, for the eigenvalues $\lambda - c_0$, the formulae

$$\begin{aligned} \widetilde{\lambda}_{n,1} - c_0 &= ((2n-\sigma)\pi)^4\left\{1 + O\left(n^{-4}\varepsilon_n\right)\right\}, \\ \widetilde{\lambda}_{n,2} - c_0 &= ((2n-\sigma)\pi)^4\left\{1 - \frac{2(-1)^{\sigma}\alpha}{((2n-\sigma)\pi)^4} + O\left(n^{-4}\varepsilon_n\right)\right\}. \end{aligned} \tag{3.34}$$

are valid and the eigenfunctions y do not change. On the other hand, the construction of the integers n_1 and n_2 is similar to the way in [11, 14–16]. Hence, the formulae (1.5) and (1.6) can be obtained by (3.32), (3.33) and (3.34).

4. Proofs of Theorem 1.2 and Corollary 1.3

First, we prove that the root functions of the operator L form a Riesz basis in $L_2(0,1)$ provided $q(x) \in L_1(0,1)$.

Let

$$v_{1,1}(x), v_{1,2}(x), \ldots, v_{n,1}(x), v_{n,2}(x), \ldots \tag{4.1}$$

be the biorthogonal system of the following system

$$u_{1,1}(x), u_{1,2}(x), \ldots, u_{n,1}(x), u_{n,2}(x), \ldots, \tag{4.2}$$

i.e. $(u_{n,j}, v_{m,s}) = \delta_{n,m}.\delta_{j,s}$, $n,m = 1,2,\ldots, j,s = 1,2$. By [19, p.84] or [28, p.99], (4.1) is the root functions of the adjoint differential operator L^*. L^* consists of the differential expression and boundary conditions

$$\begin{aligned} l^*(z) &= z^{\text{iv}} + \overline{q(x)}z, \\ U_0^*(z) &\equiv z(1) - (-1)^{\sigma} z(0) = 0, \\ U_1^*(z) &\equiv z'(1) - (-1)^{\sigma} z'(0) = 0, \\ U_2^*(z) &\equiv z''(1) - (-1)^{\sigma} z''(0) = 0, \\ U_3^*(z) &\equiv z'''(1) - (-1)^{\sigma} z'''(0) + \overline{\alpha} z(0) = 0. \end{aligned} \tag{4.3}$$

(4.3) shows that the differential operator L^* provides the conditions of Theorem 1.1. So, the formulae

$$\begin{aligned} \overline{v_{n+n_1,1}(x)} &= r_{n+n_1,1}\left(\sin(2n-\sigma)\pi x + O(\varepsilon_n)\right), \\ \overline{v_{n+n_2,2}(x)} &= r_{n+n_2,2}\left(\cos(2n-\sigma)\pi x + O(\varepsilon_n)\right) \end{aligned} \tag{4.4}$$

are valid for sufficiently large numbers n, where the numbers $r_{n_j+n,j}$, $j=1,2$ are determined by the inner product $\left(u_{n_j+n,j}, v_{n_j+n,j}\right)=1$. By these equality and (1.6), (4.4), we have

$$r_{n+n_j,j} = \sqrt{2} + O(\varepsilon_n), \quad j=1,2,$$

for sufficiently large numbers n. Consequently, if we put the last equality in (4.4), we get

$$\begin{aligned} \overline{v_{n+n_1,1}(x)} &= \sqrt{2}\sin(2n-\sigma)\pi x + O(\varepsilon_n), \\ \overline{v_{n+n_2,2}(x)} &= \sqrt{2}\cos(2n-\sigma)\pi x + O(\varepsilon_n). \end{aligned} \tag{4.5}$$

Each of the systems (4.1) and (4.2) is complete in $L_2(0,1)$ [2]. Furthermore, by (1.6) and (4.5), we get that the sequence of the multiplication of the norms of the elements of the systems (4.1) and (4.2) is bounded i.e. $\|u_n\|\|v_n\| \le M$ for all $n\in\mathbb{N}$, where M is a constant. On the other hand, since all the eigenvalues, excluding a finite number, are simple, then there are at most finitely many associate functions in the root functions of L. Hence, the system (4.2) is a Riesz basis in $L_2(0,1)$ by the main theorem in [18].

Now, we prove Corollary 1.3 by the assumption $q(x)\in L_2(0,1)$. Let

$$g_0(x)=1, \qquad g_{2n-1}(x)=\sqrt{2}\sin 2n\pi x, \qquad g_{2n}(x)=\sqrt{2}\cos 2n\pi x, \tag{4.6}$$

$$\widetilde{g}_{2n-1}=\sqrt{2}\sin(2n-1)\pi x, \qquad \widetilde{g}_{2n}=\sqrt{2}\cos(2n-1)\pi x, \tag{4.7}$$

where $n=1,2,\ldots$. The systems (4.6) and (4.7) are seperately orthonormal bases in $L_2(0,1)$. Since $q(x)\in L_2(0,1)$, then the sum of the squares of the absolute values of Fourier coefficients is convergent. Then, we can easily obtain the following

$$\sum_{n=1}^{\infty}\varepsilon_n^2 < +\infty. \tag{4.8}$$

Now, we assume $\sigma=0$. In the case $\sigma=1$, proof can be obtained in a similar method by using (4.7). Let $n_1\ge 0$ and $n_2\ge 0$. By (1.6), (4.6) and (4.8), we obtain

$$\sum_{n=1}^{\infty}\left(\|u_{n+n_1,1}-g_{2n-1}\|^2 + \|u_{n+n_2,2}-g_{2n}\|^2\right) \le \text{const}\sum_{n=1}^{\infty}\varepsilon_n^2 < +\infty. \tag{4.9}$$

One can easily see that n_1+n_2 root functions of L and one function in the system (4.6) are absent in (4.9). Let $n_1+n_2>1$. By (4.9), the system S generated by all functions excluding n_1+n_2-1 functions in the system (4.2) is quadratically close to the system (4.6). Since (4.6) is a Riesz basis in $L_2(0,1)$, then S is also a Riesz basis in $L_2(0,1)$ [10]. This contradicts the basicity of the system (4.2). Similarly, let $n_1=n_2=0$. Since (4.2) forms a Riesz basis in $L_2(0,1)$, then again by (4.9), the system $\{g_k(x)\}_{k=1}^{\infty}$ is a Riesz basis in $L_2(0,1)$. Obviously, the latter contradicts the basicity of $\{g_k(x)\}_{k=0}^{\infty}$ in $L_2(0,1)$. All other cases can be checked in a similar method.

Hence, the equality $n_1+n_2=1$ is valid. So, we can assume that $n_1=0$, $n_2=1-\sigma$ without loss of generality. Then, we obtain

$$\begin{aligned} u_{n,1}(x) &= \sqrt{2}\sin(2n-\sigma)\pi x + O(\varepsilon_n), \\ u_{n+1-\sigma,2}(x) &= \sqrt{2}\cos(2n-\sigma)\pi x + O(\varepsilon_n), \\ \overline{v_{n,1}(x)} &= \sqrt{2}\sin(2n-\sigma)\pi x + O(\varepsilon_n), \\ \overline{v_{n+1-\sigma,2}(x)} &= \sqrt{2}\cos(2n-\sigma)\pi x + O(\varepsilon_n). \end{aligned} \tag{4.10}$$

by (1.6) and (4.5).

Now, we show that the root functions of L form a basis in the Lebesgue space $L_p(0,1)$ when $q(x)\in W_1^1(0,1)$, where $1<p<\infty$, $p\ne 2$. We prove the basicity in $L_p(0,1)$ in the case $\sigma = 0$. In the case $\sigma = 1$, the proof is similar. Since the function $q(x)$ is in the space $W_1^1(0,1)$, then it is differentiable and its derivative is integrable. So, we get

$$\varepsilon_n = O\left(n^{-1}\right)$$

by using (1.3). Thus, the formulae (4.10) turn into

$$\begin{aligned} u_{n,1}(x) &= \sqrt{2}\sin(2n-\sigma)\pi x + O\left(n^{-1}\right), \\ \underline{u_{n+1-\sigma,2}(x)} &= \sqrt{2}\cos(2n-\sigma)\pi x + O\left(n^{-1}\right), \\ \underline{v_{n,1}(x)} &= \sqrt{2}\sin(2n-\sigma)\pi x + O\left(n^{-1}\right), \\ \overline{v_{n+1-\sigma,2}(x)} &= \sqrt{2}\cos(2n-\sigma)\pi x + O\left(n^{-1}\right). \end{aligned} \tag{4.11}$$

For each $p \in (1,\infty)$, (4.6) is a basis in $L_p(0,1)$ [1, Chapter VIII, §20, Theorem 2]. Then, there exists $M_p > 0$ such that the inequality

$$\left\| \sum_{n=0}^{N} (f, g_n) g_n \right\|_p \le M_p \|f\|_p, \qquad N = 1, 2, \ldots, \tag{4.12}$$

holds for each function $f(x) \in L_p(0,1)$, where $\|\cdot\|_p$ is the norm of the normed space $L_p(0,1)$ [13, Chapter I, §4, Theorem 6]. Let $p \in (1,2)$. Since (4.2) is a complete system in $L_2(0,1)$, then it is also complete in $L_p(0,1)$. Besides, one can easily see that the inequality

$$\|(f, v_{n,j}) u_{n,j}\|_p \le \text{const} \|f\|_p,$$

where $j = 1, 2$ and $n = 1, 2, \ldots$.

By theorem 6 in [13, Chapter VIII, §4], for the basicity of this system in $L_p(0,1)$, we must prove that there exists a constant $M > 0$ such that the inequality

$$\left\| \sum_{n=1}^{m} \sum_{j=1}^{2} (f, v_{n,j}) u_{n,j} \right\|_p \le M \|f\|_p \qquad m = 1, 2, \ldots,$$

holds for $f(x) \in L_p(0,1)$. Instead of the above inequality, it is enough to prove the following

$$J_m(f) = \left\| \sum_{n=1}^{m} \left\{ (f, v_{n,1}) u_{n,1} + (f, v_{n+1,2}) u_{n+1,2} \right\} \right\|_p \le M' \|f\|_p, \tag{4.13}$$

where M' is a positive constant and $m = 1, 2, \ldots$.

By (4.6) and (4.11), we have

$$J_m(f) \le J_{m,1}(f) + J_{m,2}(f) + J_{m,3}(f) + J_{m,4}(f), \tag{4.14}$$

where

$$J_{m,1}(f) = \left\| \sum_{n=1}^{2m} (f, g_n) g_n \right\|_p, \quad J_{m,2}(f) = \left\| \sum_{n=1}^{2m} (f, g_n) O\left(n^{-1}\right) \right\|_p,$$

$$J_{m,3}(f) = \left\| \sum_{n=1}^{2m} \left(f, O\left(n^{-1}\right)\right) g_n \right\|_p, \quad J_{m,4}(f) = \left\| \sum_{n=1}^{2m} \left(f, O\left(n^{-1}\right)\right) O\left(n^{-1}\right) \right\|_p.$$

By (4.12),

$$J_{m,1}(f) \le \text{const} \|f\|_p. \tag{4.15}$$

By Theorem 2.8 (Riesz theorem) [34, Chapter XII, §2,], the relations

$$J_{m,2}(f) \le \text{const} \sum_{n=1}^{2m} |(f,g_n)|\, n^{-1} \le \text{const} \left(\sum_{n=1}^{2m} |(f,g_n)|^q \right)^{1/q} \left(\sum_{n=1}^{2m} n^{-p} \right)^{1/p} \le \text{const} \, \|f\|_p , \tag{4.16}$$

holds, where $1/p + 1/q = 1$. Moreover,

$$J_{m,3}(f) \le \left\| \sum_{n=1}^{2m} \left(f, O\left(n^{-1}\right)\right) g_n \right\|_2 = \left(\sum_{n=1}^{2m} \left| \left(f, O\left(n^{-1}\right)\right) \right|^2 \right)^{1/2} \le \text{const} \, \|f\|_1 \left(\sum_{n=1}^{2m} n^{-2} \right)^{1/2} \le \text{const} \, \|f\|_p . \tag{4.17}$$

Further,

$$J_{m,4} \le \text{const} \, \|f\|_1 \sum_{n=1}^{2m} n^{-2} \le \text{const} \, \|f\|_p . \tag{4.18}$$

The inequalities (4.14)-(4.18) prove the inequality (4.13). The basicity of (4.2) in $L_p(0,1)$ is obtained when $1 < p < 2$.

Assume that the relations $2 < p < \infty$ and $1/p + 1/q = 1$ hold. Then, $1 < q < 2$ and the biorthogonal system (4.1) is the root functions of the adjoint operator L^*. Above, we show that the system of root functions of such operator is a basis of $L_q(0,1)$. So, the system (4.2) being biorthogonal system of (4.1) is a basis in $L_p(0,1)$.

References

[1] N.K. Bari, *Trigonometric Series* (Russian), Fizmatgiz, Moscow, 1961. English transl.: N.K. Bary, *A Treatise on Trigonometric Series. Vols. I, II,* MacMillan, New York, 1964.

[2] G.D. Birkhoff, *Boundary value and expansion problems of ordinary linear differential equations,* Trans. Amer. Math. Soc. **9** (4), 373-395, 1908.

[3] N. Dernek and O. Veliev, *On the Riesz basisness of the root functions of the nonself-adjoint Sturm-Liouville operator,* Israel J. Math. **145** (1), 113-123, 2005.

[4] P. Djakov and B. Mityagin, *Convergence of spectral decompositions of Hill operators with trigonometric polynomials as potentials,* Dokl. Akad. Nauk. **436**, 11-13, 2011.

[5] P. Djakov and B. Mityagin, *Convergence of spectral decompositions of Hill operators with trigonometric polynomial potentials,* Math. Ann. **351** (3), 509-540, 2011.

[6] P. Djakov and B. Mityagin, *Criteria for existence of Riesz bases consisting of root functions of Hill and 1D Dirac operators,* J. Funct. Anal. **263** (8), 2300-2332, 2012.

[7] P. Djakov and B. Mityagin, *Instability zones of periodic 1-dimensional Schrodinger and Dirac operators,* Uspekhi Mat. Nauk **61** (4), 77-182, 2006.

[8] N. Dunford, J.T. Schwartz, W.G. Bade and R.G. Bartle, *Linear Operators,* Wiley-Interscience, New York, 1971.

[9] F. Gesztesy and V. Tkachenko, *A Schauder and Riesz basis criterion for non-self-adjoint Schrodinger operators with periodic and antiperiodic boundary conditions,* J. Differential Equations **253** (2), 400-437, 2012.

[10] I. Gohberg and M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators,* American Mathematical Society, Providence, 1969.

[11] H. Gunes, N.B. Kerimov and U. Kaya, *Spectral properties of fourth order differential operators with periodic and antiperiodic boundary conditions,* Results Math. **68** (3-4), 501-518, 2015.

[12] N. Ionkin and E. Moiceev, *Solution of boundary value problem in heat conduction theory with nonlocal boundary conditions,* Differ. Equ. **13** (2), 294-304, 1977.

[13] B.S. Kashin and A. Saakyan, *Orthogonal Series, volume 75,* American Mathematical Society, Providence, 2005.

[14] N.B. Kerimov and U. Kaya, *Some problems of spectral theory of fourth-order differential operators with regular boundary conditions,* Arab. J. Math. **3** (1), 49-61, 2014.

[15] N.B. Kerimov and U. Kaya, *Spectral asymptotics and basis properties of fourth order differential operators with regular boundary conditions,* Math. Methods Appl. Sci. **37** (5), 698-710, 2014.

[16] N.B. Kerimov and U. Kaya, *Spectral properties of some regular boundary value problems for fourth order differential operators,* Cent. Eur. J. Math. **11** (1), 94-111, 2013.

[17] N.B. Kerimov and K.R. Mamedov, *On the Riesz basis property of the root functions in certain regular boundary value problems,* Math. Notes **64** (4), 483-487, 1998.

[18] N.B. Kerimov, *Unconditional basis property of a system of eigen and associated functions of a fourth-order differential operator,* Dokl. Akad. Nauk **286**, 803-808, 1986.

[19] G. Keselman, *On the unconditional convergence of eigenfunction expansions of certain differential operators,* Izv. Vyssh. Uchebn. Zaved. Mat. **2**, 82-93, 1964.

[20] A.A. Kirac, *Riesz basis property of the root functions of non-selfadjoint operators with regular boundary conditions,* Int. J. Math. Anal. (Ruse) **3** (21-24), 1101-1109, 2009.

[21] A.S. Makin, *On a class of boundary value problems for the SturmLiouville operator,* Differ. Uravn. **35** (8), 1067-1076, 1999.

[22] A.S. Makin, *Asymptotics of the spectrum of the Sturm-Liouville operator with regular boundary conditions,* Differ. Equ. **44** (5), 645-658, 2008.

[23] A.S. Makin, *Characterization of the spectrum of regular boundary value problems for the Sturm-Liouville operator,* Differ. Equ. **44** (3), 341-348, 2008.

[24] A.S. Makin, *Convergence of expansions in the root functions of periodic boundary value problems,* Dokl. Math. **73**, 71-76, 2006.

[25] A.S. Makin, *On spectral decompositions corresponding to non-self-adjoint Sturm-Liouville operators,* Dokl. Math. **73**, 15-18, 2006.

[26] A.S. Makin, *On the basis property of systems of root functions of regular boundary value problems for the Sturm-Liouville operator,* Differ. Equ. **42** (12), 1717-1728, 2006.

[27] V. Mikhailov, *On Riesz bases in $L_2(0,1)$,* Dokl. Akad. Nauk **144** (5), 981-984, 1962.

[28] M.A. Naimark, *Linear Differential Operators, 2nd ed.* Nauka, Moskow, 1969.

[29] A.A. Shkalikov, *Bases formed by eigenfunctions of ordinary differential operators with integral boundary conditions,* Moscow Univ. Math. Bull. **6**, 12-21, 1982.

[30] A.A. Shkalikov and O.A. Veliev, *On the Riesz basis property of the eigen and associated functions of periodic and antiperiodic Sturm-Liouville problems,* Math. Notes **85** (5-6), 647-660, 2009.

[31] O.A. Veliev, *Asymptotic analysis of non-self-adjoint Hill operators,* Open Math. J. **11** (12), 2234-2256, 2013.

[32] O.A. Veliev and M.T. Duman, *The spectral expansion for a nonself-adjoint Hill operator with a locally integrable potential,* J. Math. Anal. Appl. **265** (1), 76-90, 2002.

[33] O.A. Veliev, *On the nonself-adjoint ordinary differential operators with periodic boundary conditions,* Israel J. Math. **176** (1), 195-207, 2010.

[34] A. Zygmund, *Trigonometric Series, volume 2,* Cambridge University Press, Cambridge, 2002.