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Research Article

Some commutative ring extensions defined by almost Bézout condition

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Abstract

In this paper, we study the almost Bézout property in different commutative ring extensions, namely, in bi-amalgamated algebras and pairs of rings. In Section 2, we deal with almost Bézout domains issued from bi-amalgamations. Our results capitalize well known results on amalgamations and pullbacks as well as generate new original class of rings satisfying this property. Section 3 investigates pairs of rings where all intermediate rings are almost Bézout domains. As an application of our results, we characterize pairs of rings (R, T), where R arises from a (T, M, D) construction to be an almost Bézout domain.

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1. Introduction

Throughout, all rings considered are commutative with unity and all modules are unital. In [2], Anderson and Zaffrullah enlarged the class of Bézout domains in the following way: they called a domain R an almost Bézout domain ($AB-$domain for short) if given any two elements $x, y \in R$, there is a positive integer n such that the ideal (x^n, y^n) is principal. Among other things, they proved that an integral domain R is an $AB-$domain if and only if the integral closure R' of R is a Prüfer domain with torsion class group and $R \subseteq R'$ is a root extension. Further, they proved that the theory of $AB-$domains is closed to the classical one of Bézout domains. In [4], the authors noticed that each $AB-$domain is nearly Bézout and a counter-example, using the classical pullback $K + XL[X]$ was given to disprove the converse. Moreover, They used the same example to illustrate that a Noetherian almost Bézout domain is not necessarily an almost principal ideal-domain (API-domain), although each Noetherian Bézout domain is a principal ideal-domain(PID). In [3], D.D. Anderson and M. Zaffrullah showed that a finite intersection of almost valuation domains with the same quotient field is an almost Bézout domain. This generalizes the result that

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a finite intersection of valuation domains with the same quotient field is a Bézout Domain and they gave a new characterization of Cohen-Kaplansky domains. In [18], Mimouni studied the transfer of the notion of AB-domain to pullbacks. Later, in [17], the authors extended the notion of $AB-$domain defined in [2], to class of rings with zero-divisors. In particular, they defined almost Bézout rings (AB-rings for short) and they investigated when this condition is satisfied by an amalgamated algebra and by an idealization (also called Nagata's ring). In this paper, we examine when a bi-amalgamation is an $AB-$ring. Our results capitalize previous well known results on amalgamations in [17] and on Anderson and Zafrullah's paper in [2] as well as generate new original class of rings satisfying this property. Among other things, we investigate pairs of integral domains where all intermediate rings are $AB-$domains. As a consequence of our results, we provide necessary and sufficient conditions for a pair (R, T) where R arises from a (T, M, D) construction, to be an $AB-$domain pair.

Section 2 is devoted to the study of $AB-$ring property in bi-amalgamated algebras. For this purpose, we recall the definition of bi-amalgamation of rings: Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $I_o := f^{-1}(J) = g^{-1}(J')$. The *bi-amalgamation* (or *bi-amalgamated algebra*) of A with (B, C) along (J, J') with respect to (f, g) is the subring of $B \times C$ given by

$$A \bowtie^{f,g} (J, J') := \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J'\}.$$

This construction was introduced in [15] as a natural generalization of duplications [9, 12, 13] and amalgamations [10, 11]. In [15], the authors provide original examples of bi-amalgamations and, in particular, show that Boisen-Sheldon's CPI-extensions [7] can be viewed as bi-amalgamations (Notice that [10, Example 2.7] shows that CPI-extensions can be viewed as quotient rings of amalgamated algebras). They also show how every bi-amalgamation can arise as a natural pullback (or even as a conductor square) and then characterize pullbacks that can arise as bi-amalgamations. Then, the last two sections of [15] deal, respectively, with the transfer of some basic ring theoretic properties to bi-amalgamations and the study of their prime ideal structures. All their results recover known results on duplications and amalgamations. Recently in [16], the authors established necessary and sufficient conditions for a bi-amalgamation to inherit the arithmetical property, with applications on the weak global dimension and transfer of the semihereditary property.

Section 3 is devoted to the study of pairs of $AB-$domains. When each intermediate ring T between R and S (that is for each $T \in [R, S]$) satisfies a ring theoretic property P, then (R, S) is said to be a $P-$pair. The notion of P-pairs was studied for different properties P (for instance $P :=$ Noetherian, Prüfer, almost valuation, treed see [5, 14, 19, 20]). To complete this circle of ideas, we deal with the property $P :=$ almost Bézout. Throughout, for a ring R, $Spec(R)$ (resp., $Max(R)$) will denote the set of all prime (resp., maximal) ideals of R. For an integral domain R, we denote by $qf(R)$ (resp., R') the quotient field of R (resp., the integral closure of R in $qf(R)$). For a ring extension $R \subseteq S$, we denote by $[R, S]$ (resp., $]R, S]$) the set of all rings T such that $R \subseteq T \subseteq S$ (resp., $R \subset T \subseteq S$). We shall call a ring T in $[R, S]$ an S-overring of R. Such a ring is said to be a proper S-overring of R if $T \neq S$. When $S = qf(R)$, then each ring $T \in [R, qf(R)]$ is called an overring of R. We denote by $Jac(R)$, the Jacobson radical of R. Recall that an extension of integral domains $R \subseteq S$ is said to be a root extension if for each $x \in S$, there exists a positive integer n such that $x^n \in R$.

2. Transfer of $AB-$ring property to bi-amalgamated algebras

Let A, B and C be three rings, $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J') = I_0$. All along this section, $A \bowtie^{f,g} (J, J')$ will denote the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g).
Our first result investigates the transfer of $AB-$ring property to bi-amalgamation $A \bowtie^{f,g} (J, J')$ in case J and J' are proper ideals ($J \neq B$ and $J' \neq C$).

Theorem 2.1. *Assume that B and C are integral domains and J (resp., J') is a proper ideal of B (resp., C). Then the following statements are equivalent:*

(1) *$A \bowtie^{f,g} (J, J')$ is an $AB-$ring.*
(2) *$f(A) + J$ and $g(A) + J'$ are $AB-$rings and $J = 0$ or $J' = 0$.*

Proof. (1) $\Rightarrow$ (2) Assume that $A \bowtie^{f,g} (J, J')$ is an $AB-$ring. Using the fact that the $AB-$ring property is stable under factor ring and in view of the isomorphisms:

$$\frac{A \bowtie^{f,g} (J, J')}{0 \times J'} \cong f(A) + J$$

and

$$\frac{A \bowtie^{f,g} (J, J')}{J \times 0} \cong g(A) + J'$$

given by [15, Proposition 4.1(2)]. It follows that $f(A) + J$ and $g(A) + J'$ are $AB-$rings. Next, we claim that $J = 0$ or $J' = 0$; otherwise, for nonzero elements $x \in J$ and $x' \in J'$, we would have $((0, x')^n, (x, 0)^n) = ((0, x'^n), (x^n, 0))$ is a principal ideal of $A \bowtie^{f,g} (J, J')$ for some integer $n \geq 1$. Therefore, there exists $(f(d) + j, g(d) + j') \in A \bowtie^{f,g} (J, J')$ such that

$$(x^n, 0)A \bowtie^{f,g} (J, J') + (0, x'^n)A \bowtie^{f,g} (J, J') = (f(d) + j, g(d) + j')A \bowtie^{f,g} (J, J').$$

And so there exist $(f(a) + k, g(a) + k')$, $(f(b) + l, g(b) + l')$, $(f(\alpha) + s, g(\alpha) + s')$ and $(f(\beta) + t, g(\beta) + t')$ elements of $A \bowtie^{f,g} (J, J')$ such that

$$\begin{cases} (0, x'^n) & = & (f(a) + k, g(a) + k')(f(d) + j, g(d) + j') \\ (x^n, 0) & = & (f(b) + l, g(b) + l')(f(d) + j, g(d) + j') \\ (f(d) + j, g(d) + j') & = & (x^n, 0)(f(\alpha) + s, g(\alpha) + s') + (0, x'^n)(f(\beta) + t, g(\beta) + t'). \end{cases}$$

This implies that

$$\begin{cases} 0 & = & (f(a) + k)(f(d) + j) & (i) \\ x'^n & = & (g(a) + k')(g(d) + j') & (ii) \\ 0 & = & (g(b) + l')(g(d) + j') & (iii) \\ x^n & = & (f(b) + l)(f(d) + j) & (iv) \\ f(d) + j & = & x^n(f(\alpha) + s) & (v) \\ g(d) + j' & = & x'^n(g(\beta) + t') & (vi) \end{cases}$$

From equation (iv), we claim that $f(d) + j \neq 0$. Deny. It follows that $x^n = 0$, making $x = 0$, which is absurd. Since $f(d) + j \neq 0$ and B is an integral domain, then by equation (i), necessarily, $f(a) + k = 0$. Consequently, $a \in f^{-1}(J) = g^{-1}(J')$. By substituting equation (ii) in (vi), it follows that

$$(g(d) + j') = (g(d) + j')(g(a) + k')(g(\beta) + t') \quad (vii).$$

Since $x'^n \neq 0$, it follows from equation (ii) that $g(d) + j' \neq 0$. Hence, by equation (vii), $(g(a) + k')(g(\beta) + t') = 1$, which is a contradiction, since $g(a) + k' \in J'$ (as $a \in g^{-1}(J')$) and J' is a proper ideal of C. Thus $J = 0$ or $J' = 0$. The converse is straight via the isomorphisms given in [15, Proposition 4.1(2)]. $\square$

It is worth to mention that very recently, a similar result of Theorem 2.1 was independently obtained by Alqamoun and El Ouarrachi in [1, Theorem 4.1].

Recall that the amalgamation of A with B along J with respect to f is given by

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}.$$

Clearly, every amalgamation can be viewed as a special bi-amalgamation, since $A \bowtie^f J = A \bowtie^{id_A, f} (f^{-1}(J), J)$.

The following result is an immediate consequence of Theorem 2.1.

Corollary 2.2. *Under the above notation, assume that A and B are integral domains. Then $A \bowtie^f J$ is an $AB-$ring if and only if both A and $f(A) + J$ are $AB-$rings and $J = 0$ or $f^{-1}(J) = 0$.*

The following result is a particular case of Corollary 2.2 and is [2, Theorem 4.9].

Corollary 2.3. *Let D be an integral domain with quotient field K. Then $D + XK[X]$ is an $AB-$domain if and only if D is an $AB-$domain.*

Proof. Observe that $A = D$, $B = K[X]$, $f : D \hookrightarrow K[X]$ is the natural injection and $J = XK[X] \neq 0$. Clearly $f(A) \cap J = D \cap XK[X] = 0$ and so $f^{-1}(J) = 0$ and by [10, Proposition 5.1 (3)], $A \bowtie^f J = D \bowtie^f XK[X] \simeq f(A) + J = D + XK[X]$. Hence by Corollary 2.2, the conclusion is trivial. □

Remark 2.4. We recall here the following result proved by Mahdou and al. : "Let A and B be a pair of integral domains and $f : A \to B$ be a ring homomorphism. If J is a nonzero proper ideal of B. Then $A \bowtie^f J$ is an AB-ring if and only if f is injective, $f(A) + J$ is an $AB-$ring and $f(A) \cap J = (0)$." [17, Theorem 3.6]. Corollary 2.2 recovers the above result. Indeed, in the case J is a nonzero proper ideal of B, we show that the following statements are equivalent :

(1) "A and $f(A) + J$ are AB-rings and $f^{-1}(J) = 0$".
(2) "f is injective, $f(A) + J$ is an AB-ring and $f(A) \cap J = (0)$".

Indeed,
(1) $\Longrightarrow$ (2) Notice that $ker(f) \subseteq f^{-1}(J) = 0$. So f is injective, therefore the conclusion is trivial.
(2) $\Longrightarrow$ (1) Since $f(A) + J$ is an $AB-$ring and $f(A) \simeq \frac{f(A)+J}{J}$, then $f(A)$ is an $AB-$ring (as a factor ring of an $AB-$ring) and since f is injective, then $A(\simeq f(A))$ is also an AB-ring. Using the fact that $f(A) \cap J = (0)$, then $f^{-1}(J) = f^{-1}(\{0\}) = ker(f) = 0$, as f is injective.

Let I be a *proper* ideal of A. The (amalgamated) duplication of A along I is a special amalgamation given by

$$A \bowtie I := A \bowtie^{id_A} I = \{(a, a + i) \mid a \in A, i \in I\}.$$

The next corollary is an immediate consequence of Corollary 2.2 on the transfer of $AB-$ring property to duplications.

Corollary 2.5. *Let A be an integral domain and I be a proper ideal of A. Then $A \bowtie I$ is an AB-ring if and only if so is A and $I = 0$.*

We recall an important characterization of a local Gaussian ring A. Namely, for any two elements a and b in the ring A, we have $(a, b)^2 = (a^2)$ or (b^2); moreover if $ab = 0$ and $(a, b)^2 = (a^2)$, then $b^2 = 0$ (see, [6, Theorem 2.2]).

The following proposition is a partial result about when a bi-amalgamation is an $AB-$ring, in case B and C are not integral domains.

Proposition 2.6. *Assume (A, m) is local Gaussian, $J \times J' \subseteq Jac(B \times C)$, $J^2 = 0$, $J'^2 = 0$, and for all $a \in m$, $f(a)J = f(a)^2J$, $g(a)J' = g(a)^2J'$. Then $A \bowtie^{f,g} (J, J')$ is an $AB-$ring.*

Proof. Assume (A, m) is local Gaussian, $J^2 = 0$, $J'^2 = 0$ and, for all $a \in m$, $f(a)J = f(a)^2J$, $g(a)J' = g(a)^2J'$. From [15, Proposition 5.3 (b)], $(A \bowtie^{f,g} (J, J'), m \bowtie^{f,g} (J, J'))$ is local since A is local and $J \times J' \subseteq Jac(B \times C)$. Let $(f(a)+i, g(a)+i')$ and $(f(b)+j, g(b)+j')$ be elements of $A \bowtie^{f,g} (J, J')$. Two cases are then possible:
Case 1: a or $b \notin m$. Then a or b is invertible in A. Assume without loss of generality that a is invertible, then
$(f(a) + i, g(a) + i')(f(a^{-1}) - f(a^{-1})^2 i, g(a^{-1}) - g(a^{-1})^2 i') = (1, 1)$. So $(f(a) + i, g(a) + i')$ is invertible in $A \bowtie^{f,g} (J, J')$. Therefore, $((f(a) + i, g(a) + i'), (f(b) + j, g(b) + j'))^2 = ((f(a) + i, g(a) + i')^2) = A \bowtie^{f,g} (J, J')$. Thus, it follows that there exists an integer $n = 2$ such that $(f(a) + i, g(a) + i')^n$ and $(f(b) + j, g(b) + j')^n$ are comparable, as desired.
Case 2: a and $b \in m$. Using the fact that A is local Gaussian, then $(a, b)^2 = (a^2)$ or (b^2). We may assume that $(a, b)^2 = (a^2)$. So we have, $b^2 = a^2x$ and $ab = a^2y$ for some x and $y \in A$, and so $f(b)^2 = f(a)^2f(x)$, $g(b)^2 = g(a)^2g(x)$ and $f(a)f(b) = f(a)^2f(y)$, $g(a)g(b) = g(a)^2g(y)$. By assumption, $2f(b)j \in f(b)^2J$ and $2f(a)if(x) \in f(a)^2J$. Therefore, there exist $j_1, i_1 \in J$ such that $2f(b)j = f(a)^2f(x)j_1$, $2f(a)if(x) = f(a)^2i_1$, and similarly, there exist $j'_1, i'_1 \in J'$ such that $2g(b)j' = g(a)^2g(x)j'_1$, $2g(a)i'g(x) = g(a)^2i'_1$. In view of the fact that $J^2 = 0$ and $J'^2 = 0$, one can easily check that $(f(b) + j, g(b) + j')^2 = (f(a) + i, g(a) + i')^2(f(x) + f(x)j_1 - i_1, g(x) + g(x)j'_1 - i'_1)$. Hence, there exists an integer $n = 2$ such that $(f(a) + i, g(a) + i')^n$ and $(f(b) + j, g(b) + j')^n$ are comparable, as desired. Thus, in all cases, $((f(a) + i, g(a) + i')^2, (f(b) + j, g(b) + j')^2) = ((f(a) + i, g(a) + i')^2)$ which is principal, making $A \bowtie^{f,g} (J, J')$ an $AB-$ring. □

Proposition 2.6 recovers the special case of amalgamated algebra, as recorded in the next corollary.

Corollary 2.7. *Let (A, m) be a local Gaussian ring, $f : A \to B$ be an injective ring homomorphism and J be an ideal of B such that $J \subseteq Jac(B)$. If $f(a)J = f(a)^2J$, $af^{-1}(J) = a^2f^{-1}(J)$ for all $a \in m$ and $J^2 = 0$, then $A \bowtie^f J$ is an $AB-$ring.*

Proof. It is easy to show that $(f^{-1}(J))^2 \subset f^{-1}(J^2)$. Since f is injective and $J^2 = 0$, it follows that $(f^{-1}(J))^2 = 0$. Therefore, by using Proposition 2.6, $A \bowtie^f J = A \bowtie^{id_A, f} (f^{-1}(J), J)$ is an $AB-$ring. □

As an application of Proposition 2.6, we give an explicit example showing the failure of Theorem 2.1, beyond the context B and C are integral domains. Recall that for a ring A and an $A-$module E, the trivial ring extension of A by E (also called idealization of E over A) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + ea')$.

Example 2.8. Let $(A, m) := (Z/9Z, 3Z/9Z)$ be a local Gaussian Bézout ring with $m^2 = 0$, $K := A/m$, E be a $K-$vector space and $B := A \propto E$ be the trivial ring extension of A by E. Consider $f : A \hookrightarrow B$ be the natural injection (defined by $f(a) = (a, 0)$). Let $C := A$ and $g = id_A$ be the identity of A. Let $J = m \propto E$ be the maximal ideal of B. Let $J' = m$ be the maximal ideal of C. Clearly, $f^{-1}(J) = g^{-1}(J') = m$, $J^2 = J'^2 = 0$, $f(a)J = f(a)^2J = 0$, and $g(a)J' = g(a)^2J' = 0$, for all $a \in m$. Then:

(1) $A \bowtie^{f,g} (J, J')$ is an $AB-$ring.
(2) $f(A) + J$ and $g(A) + J'$ are $AB-$rings.
(3) $J \neq 0$ and $J' \neq 0$.

Proof. (1) Since $Jac(B \times C) = (m \propto E) \times (m)$, then by Proposition 2.6, $A \bowtie^{f,g} (J, J')$ is an $AB-$ring.

(2) $f(A) + J = A \propto 0 + m \propto E = A \propto E$ which is an $AB-$ring by [17, Theorem 3.1 (2)] and $g(A) + J' = A + m = A$ which is an $AB-$ring.
(3) Trivial. □

Theorem 2.1 enriches the current literature with new original class of $AB-$rings that are not Bézout rings. Recall that if A is a Bézout ring and I is an ideal of A, then A/I is a Bézout ring.

Example 2.9. Let F be a field of characteristic $p > 0$ (for instance $F = \mathbb{Z}_p$) and let $F \subseteq L$ be a purely inseparable field extension. Consider $A = F + XL[X]$. Observe that the integral closure A' of A in its quotient field $L(X)$ is $L[X]$. One can easily check that for each $q(x)$ in $L[X]$ there exists $n \geq 0$ such that $(q(x))^{p^n} \in A$. Therefore, $A \subset A'$ is a root extension. Since $A' = L[X]$ is a principal ideal domain, then A' is a Bézout domain and so is Prüfer. From [2, Corollary 4.8 (1)], it follows that A is an AB-domain. Since A is not integrally closed, then A is not a Bézout domain. Let $f : A \hookrightarrow L(X)[Y]$ be an injective ring homomorphism and let $J := YL(X)[Y]$ be a maximal ideal of $L(X)[Y]$. Consider the injective ring homomorphism $g : A \hookrightarrow A \times A$, given by $g(a) = (a, 0)$ and $J' := 0$ be an ideal of $A \times A$. Clearly, $f^{-1}(J) = 0 = I_0 = g^{-1}(J')$. Then:

(1) $A \bowtie^{f,g} (J, J')$ is $AB-$ring.
(2) $A \bowtie^{f,g} (J, J')$ is not a Bézout ring.

Proof. (1) By Theorem 2.1, $A \bowtie^{f,g} (J, J')$ is $AB-$ring since $f(A) + J = A + YL(X)[Y]$ which is $AB-$ring by Corollary 2.3 and $g(A) + J' = A \times 0 \simeq A$ which is $AB-$domain and $J' = 0$.
(2) Since A is not a Bézout domain, and so by [15, Proposition 4.1 (3)], $A \simeq \frac{A}{I_0} \simeq \frac{A \bowtie^{f,g} (J,J')}{J \times J'}$ is not a Bézout ring. Hence, $A \bowtie^{f,g} (J, J')$ is not a Bézout ring (as its quotient is not a Bézout ring). □

3. Almost Bézout domain pairs

In this section, we characterize almost Bézout domain pairs.

Definition 3.1. Let $R \subset S$ be an extension of integral domains. We say that (R, S) is an almost Bézout domain pair (for short $AB-$domain pair) if each ring $T \in [R, S]$ is an $AB-$domain.

It is worth to mention that the proof of the statement (1) of the following theorem is straightforward from [2, Lemma 4.5]. We are very grateful to Anderson and Zafrullah for their result.

Theorem 3.2. *Let $R \subset S$ be an extension of integral domains. Then the following statements hold:*
(1) If S is an overring of R, then (R, S) is an $AB-$domain pair if and only if R is an $AB-$domain.
(2) If (R, S) is an $AB-$domain pair, then R is an $AB-$domain and S is algebraic over R. The converse holds if R is integrally closed in S.

Proof. (1) If (R, S) is an $AB-$domain pair, then R is an $AB-$domain. Conversely, let $T \in [R, S]$. Then T is an overring of R since S is an overring of R. By [2, Lemma 4.5], it follows that T is an $AB-$domain.

(2) Assume that (R, S) is an $AB-$domain pair. Then it is clear that R is an $AB-$domain. We claim that $R \subset S$ is an algebraic extension. Deny. there exists $t \in S$ such that t is transcendental over R.
Claim 1. R is a field

Indeed, let $0 \neq \alpha \in R$. Using the fact that $R \subset R[t] \subset S$, then $R[t]$ is an AB-domain. Therefore, there exists a positive integer $l \geq 1$ such that the ideal (α^l, t^l) is principal generated by some $u(t) \in R[t]$. In particular, $u(t)$ divides α^l. Consequently, $u(t) = u$ for some constant $u \in R$. Observe that t^l is a multiple of u. Then there exists $u' \in R$ such that $t^l = uu't^l$ and so $uu' = 1$. Hence, u is a unit of R. On the other hand, $(\alpha^l, t^l) = (u) = R[t]$. In particular, there exist $p(t), q(t) \in R[t]$ such that $1 = \alpha^l p(t) + t^l q(t)$. By identification, we get $q(t) = 0$ and there exists $\beta \in R$ such that $p(t) = \beta$ and so it follows $\alpha^l \beta = 1$. Hence, α is a unit of R. Thus, R is a field, denoted by k.

Claim 2. If t is a transcendental element over k, then there exists a domain T that is not an AB-domain with $k \subset$T$\subset k[t]$

Consider the ring $T = k + t^3k[t]$. Clearly, $k \subset T \subset k[t]$. Consider the elements $1 + t^3$ and t^4 in T. We claim that T is not an AB-domain. Deny. Then there exist an integer $n \geq 1$ and $w(t) \in T$ such that the ideal $((1 + t^3)^n, t^{4n}) = w(t)T$. Since $w(t)$ divides t^{4n}, then there exist an element $w \in k$ and a positive integer $j \leq 4n$ such that $w(t) = wt^j$. Next, the fact that $w(t)$ divides $(1 + t^3)^n$, then one can easily check that j must be equal to 0. And so $w(t) = w$ for some nonzero $w \in k$. Observe that w is a unit of T. Consequently, $((1 + t^3)^n, t^{4n}) = T$. In particular, there exist $p(t)$, $q(t)$ in T such that $t^3 = p(t)(1 + t^3)^n + q(t)t^{4n}$. This implies that $q(t) = 0$ and $p(t) = ct^i$, for some positive integer $i \leq 3$ and $c \in k$, which is absurd, since $n \geq 1$. Hence, T is not an AB-domain, a contradiction since each proper S-overring of $R = k$ is an AB-domain. It follows that $R \subset S$ is an algebraic extension. Conversely, assume that R is an $AB-$domain and S is algebraic over R and R is integrally closed in S. We claim that $S \subset qf(R)$. Indeed, let $u \in S$. Then u is algebraic over R. Therefore, there is a in R such that $v = au$ is integral over R. Hence, v is an element of R, as R is integrally closed in S. Consequently, $u = \frac{v}{a}$ is an element of $qf(R)$. Hence, $S \subset qf(R)$, making S an overring of R. Finally, by statement (1) above, it follows that (R, S) is an $AB-$domain pair, as desired. □

Combining Theorem 3.2 and Corollary 2.3 of Section 2, we give a new characterization of almost Bézout property in pairs of rings.

Corollary 3.3. *Let D be an integral domain with quotient field K. Then the following statements are equivalent:*

(1) *D is an $AB-$domain.*
(2) *$S := D + XK[X]$ is an $AB-$domain.*
(3) *(D, K) is an $AB-$domain pair.*
(4) *$(S, K(X))$ is an $AB-$domain pair.*

Proof. (1) $\iff$ (3) and (2) $\iff$ (4) By statement (1) of Theorem 3.2, we have the desired results.
(1) $\iff$ (2) By Corollary 2.3, we have the desired result. □

Recall that $R := (T, M, D)$ is a pullback of canonical homomorphisms

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is an integral domain, M is a maximal ideal of T, $\phi : T \to T/M$ is the natural projection, D is a domain contained in $K = T/M$ and $R = \phi^{-1}(D)$. Note that $R := (T, M, D)$ if and only if R is contained in T and shares the maximal ideal M with T (see [8] for more details).

As an application of Theorem 3.2, we give necessary and sufficient conditions for (R, T) to be an $AB-$domain pair when R arises from a (T, M, D) construction, where M is a maximal ideal of T.

Corollary 3.4. *Let $R := (T, M, D)$. Then the following statements are equivalent:*

(i) *(R, T) is an $AB-$domain pair.*
(ii) *R is an $AB-$domain.*

Proof. By [2, Lemma 4.5], T is an $AB-$domain as an overring of R and so by using statement (1) of Theorem 3.2, the conclusion is trivial. □

Corollary 3.5. *Assume that (T, M) is a local domain, $k := qf(D)$ and $K := T/M$ and $R := (T, M, D)$. Then the following statements are equivalent:*

(i) *(R, T) is an $AB-$domain pair.*
(ii) *R is an $AB-$domain.*
(iii) *T and D are $AB-$domains, and the extension $k \subseteq K$ is a root extension.*

Proof. $(i) \iff (ii)$ From Corollary 3.4.
$(ii) \iff (iii)$ From [18, Theorem 2.9], the conclusion is trivial. □

As a consequence of our results, we give a necessary and sufficient condition for the polynomial ring and the power series ring to be an AB-domain.

Corollary 3.6. *Let R be an integral domain. The following statements are equivalent:*

(i) *R is a field.*
(ii) *$R[X]$ is an AB-domain.*
(iii) *$R[[X]]$ is an AB-domain.*
(iv) *$(R[X], R(X))$ is an AB-domain pair.*
(v) *$(R[[X]], R((X)))$ is an AB-domain pair.*

Proof. $(ii) \iff (iv)$ and $(iii) \iff (v)$ From statement (1) of Theorem 3.2.
$(ii) \implies (i)$ and $(iii) \implies (i)$ Using similar argument as in the proof of claim 1, statement (2) of Theorem 3.2 by replacing $R[t]$ by $R[X]$ (resp., $R[[X]]$), it follows that R is a field.
$(i) \implies (ii)$ Assume that R is a field. Then $R[X]$ is a Bézout domain and so is an $AB-$domain.
$(i) \implies (iii)$ If R is a field, then $R[[X]]$ is a valuation domain and so is an $AB-$domain. □

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