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Characterizations of Inclined Curves According to Parallel Transport Frame in E^4 and Bishop Frame in E^3

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Abstract

The aim of this paper is to introduce inclined curves according to parallel transport frame. This paper begins by defined a vector field D called Darboux vector field of an inclined curve in E^4. It will then go on to an alternative characterization for the inclined curves

$$\text{“}\alpha : I \subset \mathbb{R} \longrightarrow E^4 \text{ is an inclined curve} \Leftrightarrow k_1(s)\int k_1(s)ds + k_2(s)\int k_2(s)ds + k_3(s)\int k_3(s)ds = 0\text{”}$$

where $k_1(s)$, $k_2(s)$, $k_3(s)$ are the principal curvature functions according to parallel transport frame of the curve α and also, similar characterization for the generalized helices according to Bishop frame in E^3 is given by

$$\alpha : I \subset \mathbb{R} \longrightarrow E^3 \text{ is a generalized helix} \Leftrightarrow k_1(s)\int k_1(s)ds + k_2(s)\int k_2(s)ds = 0\text{”}$$

where $k_1(s)$, $k_2(s)$ are the principal curvature functions according to Bishop frame of the curve α.
These curves have illustrated some examples and draw their figures with use of Mathematica programming language. Also, it is given an example for the inclined curve in E^4 and showed that the above condition is satisfied for this curve.

Keywords: *Parallel transport frame, Inclined curve, Bishop frame, Generalized helix*
2010 Mathematics Subject Classification: *14H45, 14H50, 53A04*

1. Introduction

The curves are a very important topic in all disciplines. They appear in physical applications as well as medical sciences heart chest film with X-ray curve, how to act is important to us. Curves give the movements of the particle in Physics.

Helical curves are the very important type of curves. Because, helices are among the simplest objects in the art, molecular structures, nature, etc. For example, the path, aroused by the climbing of beans and the orbit where the progressing of the screw is a helix curve. Also, in medicine DNA molecule is formed as two intertwined helices and many proteins have helical structures, known as alpha helices. So, helices are very important for understanding nature. Also, helices are called as inclined curves in higher dimensional Euclidean space E^n $(n \geq 4)$. Therefore, recently researchers have shown an increased interest in the helices in the Euclidean space E^3 (See for details: [1, 3, 6, 10, 11, 14]).

In 1802, M. A. Lancret first proposed a theorem and in 1845, B. de Saint Venant first proved this theorem: ”A necessary and sufficient condition of a curve to be a general helix is that the ratio of curvature to torsion should be a constant.” Another definition of the helix curve is that the tangent vector field at all points of the curve makes a constant angle with a fixed direction. Recently, many studies have been reported on generalized helices and inclined curves [1, 4, 9, 12].

The Frenet frame is constructed for the curve of third order continuously differentiable non-degenerate curves. Curvature of the curve may vanish on some points of the curve, that is, second derivative of the curve may be zero. In this situation, we need an alternative frame in E^3. Therefore in [2], Bishop defined a new frame for a curve and called as Bishop frame which is well defined even when the curve has vanishing second derivative in $3-$dimensional Euclidean space E^3.

Similarly, Gökçelik *et al.* defined a new frame for a curve and called parallel transport frame in E^4 [7]. The parallel transport frame is an alternative frame defined by a moving frame. They consider a regular curve $\alpha(s)$ parametrized by s and they defined a normal vector

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field $V(s)$ which is perpendicular to the tangent vector field $T(s)$ of the curve $\alpha(s)$ said to be relatively parallel vector field if its derivative is tangential along the curve $\alpha(s)$. They use the tangent vector $T(s)$ and three relatively parallel vector fields to construct this alternative frame. They choose any convenient arbitrary basis $\{M_1(s), M_2(s), M_3(s)\}$ of the frame, which are perpendicular to $T(s)$ at each point. The derivatives of $\{M_1(s), M_2(s), M_3(s)\}$ only depend on $T(s)$. It is called as parallel transport frame along a curve because the normal component of the derivatives of the normal vector field is zero. The advantages of the parallel frame and the comparable parallel frame with the Frenet frame in $3-$dimensional Euclidean space E^3 was given and studied by Bishop [2].

Spherical curves are characterized by the parallel transport frame. "The curve α is spherical curve if and only if $c_1k_1(s)+c_2k_2(s)+c_2k_3(s)+1=0$, where c_1, c_2, c_3 are constants and $k_1(s)$, $k_2(s)$, $k_3(s)$ are the principal curvatures according to parallel transport frame of the curve α" [7].

In this article, the inclined curves are studied according to parallel transport frame in terms of the harmonic curvature functions and are given characterizations of the inclined curves. Also, it is obtained some characterizations for the generalized helices according to Bishop frame in E^3. Finally, these curves are exemplified and their figures are plotted with the help of the Mathematica program.

2. Preliminaries

Let $\alpha : I \subset \mathbb{R} \to E^4$ be an arbitrary curve in the Euclidean $4-$space E^4. Recall that the curve α is said to be a unit speed (or parameterized by arc length function s) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle , \rangle$ is the standard inner product of E^4 given by $\langle X, Y \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$ for each $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4) \in E^4$. In particular, the norm of a vector $X \in E^4$ is given by $\|X\|^2 = \langle X, X \rangle$. Let $\{T(s), N(s), B_1(s), B_2(s)\}$ be the Frenet frame along the unit speed curve α. Then $T(s), N(s), B_1(s)$ and $B_2(s)$ are the tangent, the principal normal, first and second binormal vectors of the curve α, respectively. If α is a space curve, then this set of orthogonal unit vectors is known as the Frenet-Serret frame. Derivatives of the Frenet frame vectors have the following properties

$$\begin{aligned} T'(s) &= \kappa(s)N(s) \\ N'(s) &= -\kappa(s)T(s) + \tau(s)B_1(s) \\ B_1'(s) &= -\tau(s)N(s) + \sigma(s)B_2(s) \\ B_2'(s) &= -\sigma(s)B_1(s) \end{aligned}$$

where $\kappa(s)$, $\tau(s)$ and $\sigma(s)$ denote principal curvature functions according to Frenet frame of the curve α.
The parallel transport frame is an alternative frame defined a moving frame. Curvature of the curve may vanish, that is, the $i-th$ $(1 < i < 4)$ derivative of the curve may be zero at some points on the curve. In this case, we can define a parallel transport frame along the curve by parallel transporting Frenet vectors N, B_1, B_2 except T. Newly formed frame $\{T(s), M_1(s), M_2(s), M_3(s)\}$ is called as parallel transport frame and $k_1(s) = \langle T'(s), M_1(s) \rangle$, $k_2(s) = \langle T'(s), M_2(s) \rangle$, $k_3(s) = \langle T'(s), M_3(s) \rangle$ called as parallel transport curvatures of the curve α. One of the properties of this frame is that the derivatives of the vectors $M_1(s), M_2(s)$ and $M_3(s)$ only depend on $T(s)$.

Theorem 2.1. *Let $\alpha : I \subset \mathbb{R} \to E^4$ be a unit speed curve with arc length parameter s. $\{T(s), N(s), B_1(s), B_2(s)\}$ and $\{T(s), M_1(s), M_2(s), M_3(s)\}$ are defined the Frenet frame and the parallel transport frame along the curve α, respectively. Using the Euler angles, the relations between the parallel transport vectors and the Frenet vectors are expressed as follows:*

$$\begin{aligned} T(s) &= T(s), \\ N(s) &= \cos\theta(s)\cos\psi(s)M_1(s) + (-\cos\phi(s)\sin\psi(s) + \sin\phi(s)\sin\theta(s)\cos\psi(s))M_2(s) \\ &\quad + (\sin\phi(s)\sin\psi(s) + \cos\phi(s)\sin\theta(s)\cos\psi(s))M_3(s), \\ \\ B_1(s) &= \cos\theta(s)\sin\psi(s)M_1(s) + (\cos\phi(s)\cos\psi(s) + \sin\phi(s)\sin\theta(s)\sin\psi(s))M_2(s) \\ &\quad + (-\sin\phi(s)\cos\psi(s) + \cos\phi(s)\sin\theta(s)\sin\psi(s))M_3(s), \\ \\ B_2(s) &= -\sin\theta(s)M_1(s) + \sin\phi(s)\cos\theta(s)M_2(s) + \cos\phi(s)\cos\theta(s)M_3(s) \end{aligned}$$

where θ, ϕ, ψ are the angles between the parallel transport vectors and the Frenet vectors which are shown in figure 2.1 [7].

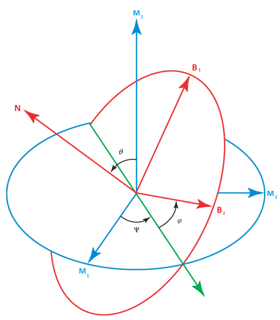


Figure 2.1: The relation between the vectors of the Frenet frame and the parallel transport frame in terms of the Euler angles.

Theorem 2.2. *The alternative parallel transport frame equations in E^4 are given by*

$$\begin{bmatrix} T'(s) \\ M_1'(s) \\ M_2'(s) \\ M_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & k_2(s) & k_3(s) \\ -k_1(s) & 0 & 0 & 0 \\ -k_2(s) & 0 & 0 & 0 \\ -k_3(s) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ M_1(s) \\ M_2(s) \\ M_3(s) \end{bmatrix} \tag{2.1}$$

where $k_1(s)$, $k_2(s)$ and $k_3(s)$ are the parallel transport curvatures of the curve α and their expressions are given by the Frenet curvature $\kappa(s)$ and Euler angles as follows:

$$\begin{aligned} k_1(s) &= \kappa(s)\cos\theta(s)\cos\psi(s), \\ k_2(s) &= \kappa(s)(-\cos\phi(s)\sin\psi(s)+\sin\phi(s)\sin\theta(s)\cos\psi(s)), \\ k_3(s) &= \kappa(s)(\sin\phi(s)\sin\psi(s)+\cos\phi(s)\sin\theta(s)\cos\psi(s)) \end{aligned} \tag{2.2}$$

$$\begin{aligned} \kappa(s) &= \sqrt{k_1^2(s)+k_2^2(s)+k_3^2(s)}, \\ \tau(s) &= -\psi'(s)+\phi'(s)\sin\theta(s), \\ \sigma(s)\sin\psi(s) &= \theta'(s), \end{aligned} \tag{2.3}$$

$$\phi'(s)\cos\theta(s)\sin\psi(s)+\theta'(s)\cos\psi(s)=0,$$

where $\kappa(s)$, $\tau(s)$ and $\sigma(s)$ are the Frenet curvature functions of the curve α [7].

In special case, if $\sigma(s)=0$, then it is obtained the Bishop frame ([2]) in $3-$dimensional Euclidean space:

$$\begin{bmatrix} T'(s) \\ M_1'(s) \\ M_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ M_1(s) \\ M_2(s) \end{bmatrix}.$$

3. Characterizations of inclined curves according to parallel transport frame in E^4

In this section, it is given some characterizations for inclined curve by using its parallel transport frame and its harmonic curvature functions in $4-$dimensional Euclidean space.

Definition 3.1. *Let $\alpha : I \subset \mathbb{R} \to E^4$ be a unit speed curve in $4-$dimensional Euclidean space with the arc length parameter s and X be a unit and fixed vector of E^4. For all $s \in I$, if the condition*

$$\left\langle \alpha'(s), X \right\rangle = \cos\varphi = const.\ ,\ \varphi \neq \frac{\pi}{2}$$

is satisfied for the curve α then the curve α is called as an inclined curve in E^4. Here $\alpha'(s)$ is the unit tangent vector field to the curve α at its point $\alpha(s)$ and φ is a constant angle between the vectors α' and X.

Definition 3.2. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^4$ be the unit speed curve with the parallel transport curvatures $\{k_1(s), k_2(s), k_3(s)\}$. The harmonic curvature functions of the curve α are defined by*

$$H_i : I \longrightarrow \mathbb{R}\ ,\ (i=1,2,3)$$

$$H_1(s) = -\frac{1}{k_1(s)}(k_2(s)H_2(s)+k_3(s)H_3(s)),$$

$$H_2(s) = \frac{1}{\left(\frac{k_2(s)}{k_1(s)}\right)'}\left(\frac{k_1^2(s)+k_2^2(s)+k_3^2(s)}{k_1(s)}\right) - \left(\frac{k_3(s)}{k_1(s)}\right)' H_3(s),$$

$$H_3(s) = \frac{\left(k_1(s)k_2'(s)-k_1'(s)k_2(s)\right)\left(k_1(s)k_1'(s)+k_2(s)k_2'(s)+k_3(s)k_3'(s)+\left(1-k_1(s)k_2''(s)-k_1''(s)k_2(s)\right)\left(k_1^2(s)+k_2^2(s)+k_3^2(s)\right)\right)}{k_1(s)\left(k_2'(s)k_3''(s)-k_2''(s)k_3'(s)\right)+k_1''(s)\left(k_2(s)k_3'(s)-k_3(s)k_2'(s)\right)+k_1'(s)\left(k_2''(s)k_3(s)-k_3''(s)k_2(s)\right)}$$

with the conditions $k_1(s)\neq 0$, $\left(\frac{k_2}{k_1}\right)'\neq 0$ and
$\left(k_1(s)k_3''(s)-k_1''(s)k_3(s)\right)\left(k_1(s)k_2'(s)-k_1'(s)k_2(s)\right) \neq \left(k_1(s)k_3'(s)-k_1'(s)k_3(s)\right)\left(k_1(s)k_2''(s)-k_1''(s)k_2(s)\right)$.

Theorem 3.3. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^4$ be a regular curve with the parallel transport frame apparatus $\{k_1(s), k_2(s), k_3(s), T(s), M_1(s), M_2(s), M_3(s)\}$. If the curve α is the inclined curve in E^4 then it is satisfied the following equalities*

$$\begin{aligned} \langle M_1(s), X\rangle &= H_1(s)\langle T(s), X\rangle, \\ \langle M_2(s), X\rangle &= H_2(s)\langle T(s), X\rangle, \\ \langle M_3(s), X\rangle &= H_3(s)\langle T(s), X\rangle, \end{aligned} \tag{3.1}$$

where $H_i(s)$ $(i=1,2,3)$ are the harmonic curvature functions of the curve α.

Proof. Since the curve α is the inclined curve in E^4, the inner product $\langle T(s), X\rangle$ is a constant. Differentiating the last equation with respect to s, we obtain for $k_1(s) \neq 0$

$$\langle M_1(s), X\rangle = -\frac{1}{k_1(s)}\left(k_2(s)\langle M_2(s), X\rangle + k_3(s)\langle M_3(s), X\rangle\right)$$

is found. Again differentiating of the last equation

$$\langle M_2(s), X\rangle = \frac{1}{\left(\frac{k_2(s)}{k_1(s)}\right)'}\left(\frac{k_1^2(s)+k_2^2(s)+k_3^2(s)}{k_1(s)}\right)\langle T(s), X\rangle - \left(\frac{k_3(s)}{k_1(s)}\right)'\langle M_3(s), X\rangle .$$

From the Definition (3.2) it is easily obtained the Eq. (3.1). □

Theorem 3.4. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^4$ be the unit curve with the parallel transport frame $\{T(s), M_1(s), M_2(s), M_3(s)\}$ and the harmonic curvature functions $H_i(s), i = 1,2,3$. If the curve α is the inclined curve in E^4, the axis of the curve α is given by*

$$X = (T(s) + H_1(s)M_1(s) + H_2(s)M_2(s) + H_3(s)M_3(s))\langle T(s), X\rangle$$

or

$$X = (T(s) + H_1(s)M_1(s) + H_2(s)M_2(s) + H_3(s)M_3(s))\cos\varphi.$$

Proof. If the axis of the inclined curve α is X, then the vector X can be written in terms of the parallel transport vectors along the curve α as $X = \lambda_1(s)T(s) + \lambda_2(s)M_1(s) + \lambda_3(s)M_2(s) + \lambda_4(s)M_3(s)$. Since α is the inclined curve $\lambda_1 = \cos\varphi$ and then by using Theorem (3.3), we get

$$\lambda_2(s) = H_1(s)\langle T(s), X\rangle, \quad \lambda_3(s) = H_2(s)\langle T(s), X\rangle, \quad \lambda_4(s) = H_3(s)\langle T(s), X\rangle .$$

Thus, the axis of the inclined curve α can be expressed by

$$X = (T(s) + H_1(s)M_1(s) + H_2(s)M_2(s) + H_3(s)M_3(s))\cos\varphi.$$

□

Definition 3.5. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^4$ be an inclined curve and $H_i(s)$, $i = 1,2,3$ denote the harmonic curvature functions at the point $\alpha(s)$.*

$$D = T(s) + H_1(s)M_1(s) + H_2(s)M_2(s) + H_3(s)M_3(s)$$

is called as a Darboux vector field of the inclined curve α in E^4.

Theorem 3.6. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^4$ be a regular curve with the parallel transport frame apparatus $\{k_1(s), k_2(s), k_3(s), T(s), M_1(s), M_2(s), M_3(s)\}$ and the harmonic curvature functions $H_i(s)$ $(i = 1,2,3)$. The curve α is the inclined curve in E^4 if and only if the Darboux vector D is a constant vector field.*

Proof. Let α be the inclined curve in E^4 and the fixed vector field X is the axis of the curve α. From Theorem (3.4), we have

$$\begin{aligned} X &= (T(s) + H_1(s)M_1(s) + H_2(s)M_2(s) + H_3(s)M_3(s))\cos\varphi \\ &= D\cos\varphi \end{aligned}$$

where $\cos\varphi$ and X are the constants. Hence D is to be a constant vector field.
Assume that the Darboux vector D is the constant vector field, that is, $\langle D, D\rangle$ is the constant. Using the Theorem (3.4) we have

$$\|X\| = \|D\|\cos\varphi.$$

Since X is a unit vector and $\|D\|$ is the constant, we have $\cos\varphi = \frac{1}{\|D\|}$. So, the curve α is the inclined curve. □

Theorem 3.7. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^4$ be a regular curve with the parallel transport frame $\{T(s), M_1(s), M_2(s), M_3(s)\}$ and the harmonic curvature functions $H_i(s)$ $(i = 1,2,3)$. If the curve α is the inclined curve in E^4 then $\sum_{i=1}^{3} H_i^2(s)$ is a constant.*

Proof. Since the curve α is the inclined curve, then the axis $X = (T(s) + H_1M_1(s) + H_2M_2(s) + H_3M_3(s))\cos\varphi = D\cos\varphi$ is the unit vector field. So, it is satisfied

$$\begin{aligned} \|X\|^2 &= \langle X, X\rangle \\ &= \cos^2\varphi + \sum_{i=1}^{3} H_i^2(s)\cos^2\varphi \end{aligned}$$

and then

$$\sum_{i=1}^{3} H_i^2(s) = \tan^2\varphi = \text{constant}.$$

□

Theorem 3.8. (***Main Theorem***) *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^4$ be a curve with arc length parameter s. $\{k_1(s), k_2(s), k_3(s), T(s), M_1(s), M_2(s), M_3(s)\}$ denotes the parallel transport frame apparatus and $\{H_1(s), H_2(s), H_3(s)\}$ denotes the harmonic curvature functions of the curve α. The curve α is the inclined curve if and only if the following condition is satisfied*

$$k_1(s)\int k_1(s)ds + k_2(s)\int k_2(s)ds + k_3(s)\int k_3(s)ds = 0. \tag{3.2}$$

Proof. If we differentiate D along the curve α, we get

$$D' = -(k_1(s)H_1(s) + k_2(s)H_2(s) + k_3(s)H_3(s))T(s) + \left(k_1(s) + H_1'(s)\right)M_1(s) + \left(k_2(s) + H_2'(s)\right)M_2(s) + \left(k_3(s) + H_3'(s)\right)M_3(s).$$

If the curve α is the inclined curve, D is a constant vector field and so, the following conditions are obtained

$$k_1(s)H_1(s) + k_2(s)H_2(s) + k_3(s)H_3(s) = 0, \;\; H_1'(s) = -k_1(s), \quad H_2'(s) = -k_2(s) \;\; and \;\; H_3'(s) = -k_3(s). \tag{3.3}$$

From the above equations, the desired result in Eq. (3.2) is obtained. Conversely, if the Eq.(3.2) holds, we can easily see that $D' = 0$ and then from Theorem (3.6), we have an inclined curve α with the Darboux vector D in E^4. □

Remark 3.9. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^4$ be a curve with arc length parameter s. $\{k_1(s), k_2(s), k_3(s), T(s), M_1(s), M_2(s), M_3(s)\}$ denotes the parallel transport frame apparatus and $\{H_1, H_2, H_3\}$ denotes the harmonic curvature functions of the curve α. The curve α is the inclined curve if and only if the vector $D = (1, H_1, H_2, H_3) \in E^4$ satisfies the following equality*

$$\frac{d}{ds}\begin{bmatrix} 1 \\ H_1 \\ H_2 \\ H_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ H_1 \\ H_2 \\ H_3 \end{bmatrix}.$$

Theorem 3.10. *Let α be a unit speed curve with the parallel transport frame apparatus $\{k_1(s), k_2(s), k_3(s), T(s), M_1(s), M_2(s), M_3(s)\}$. If the curve α is the inclined curve in E^4 then its spherical image curves $\alpha_{M_1}, \alpha_{M_2}, \alpha_{M_3}$ are the inclined curve and also, they have the same axis with the curve α.*

Proof. Let α be a unit speed non-degenerate curve in E^4. Its spherical image is the parametrized curve with arc length parametes s_{M_1}, then we have

$$\alpha_{M_1}(s_{M_1}) = M_1(s).$$

Differentiating the above equation with respect to s we obtain

$$\frac{d\alpha_{M_1}}{ds_{M_1}}\frac{ds_{M_1}}{ds} = \frac{dM_1}{ds}.$$

Using the Eq. (2.1), we obtain $T_{M_1}(s_{M_1}) = T(s)$ where $ds_{M1} = k_1(s)ds$. Consequently, if $\langle T(s), X\rangle$ is a constant, then $\langle T_{M_1}(s_{M_1}), X\rangle$, $\langle T_{M_2}(s_{M_2}), X\rangle$ and $\langle T_{M_3}(s_{M_3}), X\rangle$ must be a constant. Therefore, if the curve α is the inclined curve in E^4 then its spherical image curves $\alpha_{M_1}, \alpha_{M_2}, \alpha_{M_3}$ are the inclined curve with the fixed axis X. □

Example 3.11. *The unit speed curve*

$$\alpha(s) = \left(\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\cos s\right)$$

is a spherical curve with radius 1*. The Frenet curvature functions of the curve $\alpha(s)$ are given by $\kappa(s) = \frac{\sqrt{3}}{2}$, $\tau(s) = \frac{\sqrt{3}}{6}$ and $\sigma(s) = \frac{\sqrt{6}}{3}$ and Frenet frame vector fields are*

$$\begin{aligned}
T(s) &= \frac{1}{\sqrt{2}}\left(\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, \cos s, -\sin s\right),\\
N(s) &= -\frac{2}{\sqrt{3}}\left(\frac{1}{2}\sin\frac{s}{\sqrt{2}}, \frac{1}{2}\cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\cos s\right),\\
B_1(s) &= \frac{1}{\sqrt{2}}\left(\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, -\cos s, \sin s\right),\\
B_2(s) &= -\frac{8}{\sqrt{3}}\left(\frac{1}{4\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{4\sqrt{2}}\cos\frac{s}{\sqrt{2}}, -\frac{1}{8}\sin s, -\frac{1}{8}\cos s\right).
\end{aligned}$$

Also, the curve has the parallel transport frame vector fields

$$\begin{aligned}
T(s) &= \frac{1}{\sqrt{2}}\left(\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, \cos s, -\sin s\right),\\
M_1(s) &= \frac{8}{\sqrt{3}}\left(\frac{1}{4\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{4\sqrt{2}}\cos\frac{s}{\sqrt{2}}, -\frac{1}{8}\sin s, -\frac{1}{8}\cos s\right),\\
M_2(s) &= \sin\left(\frac{\sqrt{3}}{6}s\right)N(s) + \cos\left(\frac{\sqrt{3}}{6}s\right)B_1(s),\\
M_3(s) &= \cos\left(\frac{\sqrt{3}}{6}s\right)N(s) - \sin\left(\frac{\sqrt{3}}{6}s\right)B_1(s)
\end{aligned}$$

with parallel transport frame curvatures $k_1(s)=0$, $k_2(s)=\frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{6}s\right)$ *and* $k_3(s)=\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{6}s\right)$. *Since the integral characterization in Eq. (3.2) is satisfied for the curve* α, *the curve* α *is the inclined curve. The spherical image curves* $\alpha_{M_1},\alpha_{M_2},\alpha_{M_3}$ *are also inclined curves in* E^4. *We give the visualization of the inclined curves* α, $\alpha_{M_1},\alpha_{M_2},\alpha_{M_3}$ *with the parametrization* $\alpha(s)=(\alpha_1(s),\alpha_2(s),\alpha_3(s),\alpha_4(s))$ *in* E^4 *by use of Mathematica program language. We plot the graphs of the curves with plotting command*

$$ParametricPlot3D[\{\alpha_1(s),\alpha_2(s)+\alpha_3(s),\alpha_4(s)\},\{s,..,..\}]$$

In this example, since the α *and its spherical image curve* α_{M_1} *have the same graphics, we only presented various views of the curve* α *in Figure 3.1. Also, the same case is obtained for the spherical image curves* α_{M_2} *and* α_{M_3} *(see Figure 3.2).*

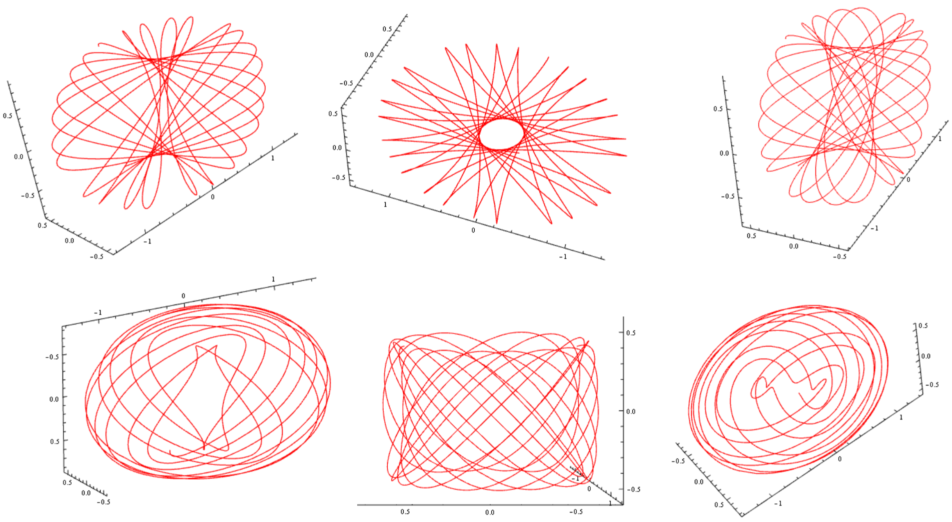


Figure 3.1: The projections of the inclined curves α or α_{M_1} in E^3 obtained for $s\in[-15\pi,15\pi]$.

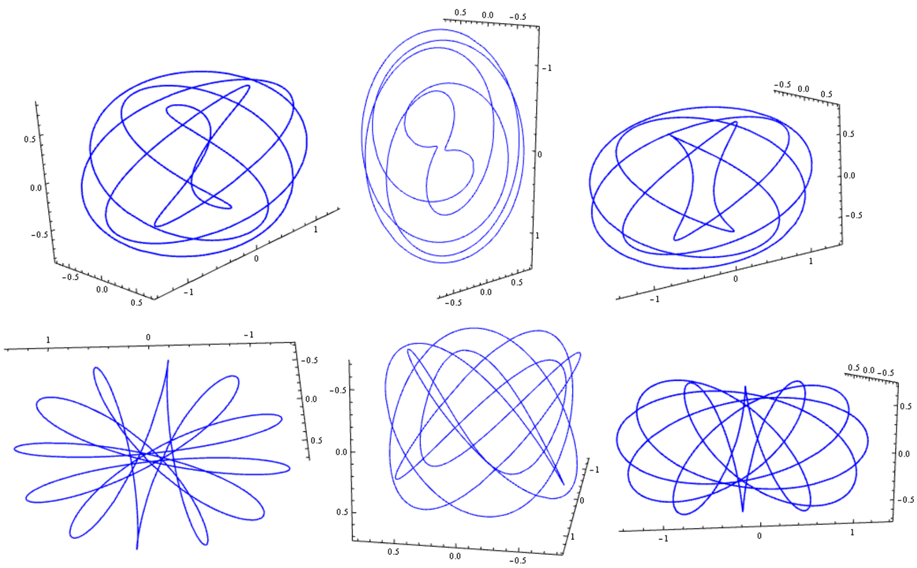


Figure 3.2: The projections of the inclined curves α_{M_2} or α_{M_3} in E^3 obtained for $s\in[-7\pi,7\pi]$.

4. Characterizations of generalized helices according to Bishop frame in E^3

In this section, we will give some theorems without proofs of the generalized helices according to Bishop frame. These proofs can be shown using the similar method of the proofs of the above theorems for inclined curves in E^4. Finally, we give the some examples of the generalized helices according to Bishop frame and plot their graphs with the help of Mathematica program.

Definition 4.1. *Let* $\alpha: I\subset\mathbb{R}\to E^3$ *be a unit speed curve with the arc length parameter s and X be a unit constant vector of* E^3. *If the following condition is satisfied for all* $s\in I$

$$\langle\alpha'(s),X\rangle=\cos\theta,\quad \theta\neq\frac{\pi}{2}$$

then the curve α *is called as a generalized helix in* E^3 *where* θ *is a constant angle between the vectors* α' *and* X.

Definition 4.2. *Let α be a unit speed curve with arc length parameter s in E^3. Harmonic curvature functions of the curve α are defined as*

$$H_i : I \longrightarrow \mathbb{R}\ ,\ (i = 1,2)$$

$$H_1(s) = \frac{k_2(s)(1+f^2(s))}{f'(s)},$$

$$H_2(s) = -\frac{k_1(s)(1+f^2(s))}{f'(s)}$$

where $f(s) = \frac{k_1(s)}{k_2(s)}$ and $\{k_1(s), k_2(s)\}$ are the curvatures of the curve α according to Bishop frame.

Theorem 4.3. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^3$ be a unit speed curve with the Bishop frame apparatus $\{k_1(s), k_2(s), T(s), M_1(s), M_2(s)\}$. If the curve α is the generalized helix then the following equations are satisfied*

$$\begin{aligned} \langle M_1(s), X\rangle &= H_1(s)\langle T(s), X\rangle, \\ \langle M_2(s), X\rangle &= H_2(s)\langle T(s), X\rangle \end{aligned}$$

where $H_i(s)$ $(i = 1,2)$ are the harmonic curvature functions of the curve α.

Theorem 4.4. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^3$ be a unit speed curve with the Bishop frame apparatus $\{k_1(s), k_2(s), T(s), M_1(s), M_2(s)\}$. If the curve α is the generalized helix in E^3, the axis of α is given by*

$$X = (T(s) + H_1(s)M_1(s) + H_2(s)M_2(s))\cos\theta.$$

where $H_i(s)$ $(i = 1,2)$ are the harmonic curvature functions of the curve α.

Definition 4.5. *Let α be a generalized helix with the harmonic curvature functions H_i $(i = 1,2)$ at the point $\alpha(s)$.*

$$D = T(s) + H_1(s)M_1(s) + H_2(s)M_2(s)$$

is defined as a Darboux vector field of the generalized helix α in E^3.

Theorem 4.6. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^3$ be a unit speed curve with Bishop frame apparatus $\{k_1(s), k_2(s), T(s), M_1(s), M_2(s)\}$ and the harmonic curvature functions $H_i(s)$ $(i = 1,2)$. The curve α is the generalized helix in E^3 if and only if D is a constant vector field.*

Theorem 4.7. *Let α be a unit speed curve with Bishop frame $\{T(s), M_1(s), M_2(s)\}$ and the harmonic curvature functions $H_i(s)$ $(i = 1,2)$. If the curve α is the generalized helix in E^3 then $H_1^2(s) + H_2^2(s)$ is a constant.*

Remark 4.8. *Let α be a unit speed curve in E^3. $\{k_1(s), k_2(s)\}$ and $\{H_1(s), H_2(s)\}$ denote the curvature functions and the harmonic curvature functions according to Bishop frame of the curve α, respectively. The curve α is the generalized helix if and only if the following equality is obtained*

$$\frac{d}{ds}\begin{bmatrix} 1 \\ H_1(s) \\ H_2(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ H_1(s) \\ H_2(s) \end{bmatrix}$$

for the vector $D = (1, H_1(s), H_2(s)) \in E^3$.

Theorem 4.9. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^3$ be a curve with arc length parameter s. $\{k_1(s), k_2(s), T(s), M_1(s), M_2(s)\}$ denotes the Bishop frame apparatus and $\{H_1(s), H_2(s)\}$ denotes the harmonic curvature functions of the curve α. The curve α is the generalized helix if and only if the following equation is obtained*

$$k_1(s)\int k_1(s)ds + k_2(s)\int k_2(s)ds = 0.$$

Theorem 4.10. *Let α be a unit speed curve with Bishop frame apparatus $\{k_1(s), k_2(s), T(s), M_1(s), M_2(s)\}$. If the curve α is the generalized helix with the axis X then its spherical image curves α_{M_1}, α_{M_2} are the generalized helix with the axis X.*

Example 4.11. *Let us consider the spherical nephroid curve $\beta(s^*) = (\beta_1(s), \beta_2(s), \beta_3(s))$ of E^3 with arc length parameter s^* and $ds^* = \dfrac{2\cos s}{\sqrt{3}}ds$. It is a helix that makes a half-turn between two cuspidal points, also its horizontal projection is a nephroid.*

$$\begin{cases} \beta_1(s) = \dfrac{3}{4}\sin(s) - \dfrac{1}{12}\sin(3s) + \dfrac{1}{2}\sin(s)\cos(2s) \\ \beta_2(s) = -\dfrac{3}{4}\cos(s) + \dfrac{1}{12}\cos(3s) + \dfrac{1}{2}\sin(s)\sin(2s) \\ \beta_3(s) = \dfrac{1}{\sqrt{3}}\sin(s) \end{cases}$$

The Frenet-Serret frame apparatus $\{\kappa(s),\tau(s),T(s),N(s),B(s)\}$ *of the curve* $\beta=\beta(s^*)$ *are calculated with the help of Mathematica program as follows*

$$\begin{aligned} T(s) &= \left(\frac{\sqrt{3}}{2}\cos(2s),\frac{\sqrt{3}}{2}\sin(2s),\frac{1}{2}\right), \\ N(s) &= \left(-\sin(2s),\cos(2s),0\right), \\ B(s) &= \left(-\frac{1}{2}\cos(2s),-\frac{1}{2}\sin(2s),\frac{\sqrt{3}}{2}\right) \end{aligned}$$

$$\kappa(s)=\frac{3}{2\cos(s)} \quad and \quad \tau(s)=\frac{\sqrt{3}}{2\cos(s)}.$$

Also, the curve β *has the Bishop frame apparatus* $\{k_1(s),k_2(s),T(s),M_1(s),M_2(s)\}$ *with the angle of rotation* $\theta(s)=-\int\tau(s)ds^*=-s+c_0$ *where* c_0 *is a constant.*

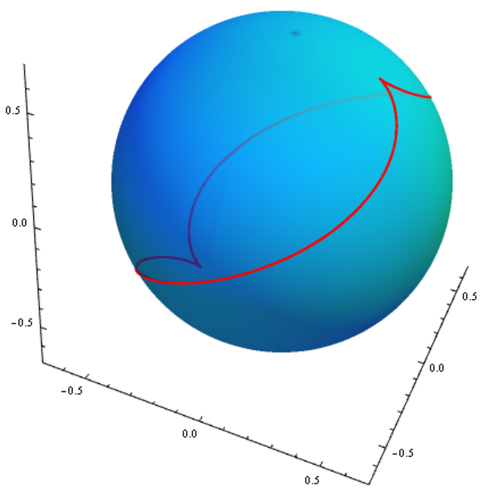
$$\begin{aligned} T(s) &= T(s), \\ M_1(s) &= \cos(-s+c_0)N(s)+\sin(-s+c_0)B(s), \\ M_2(s) &= -\sin(-s+c_0)N(s)+\cos(-s+c_0)B(s), \end{aligned}$$

$$k_1(s)=\frac{3}{2\cos(s)}\cos(-s+c_0) \quad and \quad k_2(s)=-\frac{3}{2\cos s}\sin(-s+c_0).$$

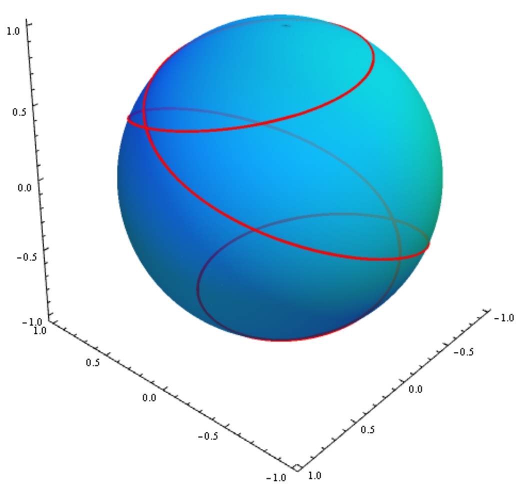
Using the above curvatures, it is obtained

$$k_1(s)\int k_1(s)ds^*+k_2(s)\int k_2(s)ds^*=0.$$

So, the curve β *is the generalized helix. From Theorem* (4.10)*, we know that spherical images* β_{M_1}, β_{M_2} *of the curve* β *are the generalized helices (see Figure 4.2). Since the spherical image curves* β_{M_1}, β_{M_2} *have the same graphs, only one's graph is shown in figure 4.2* ***(b)****.*



(a) The spherical nephroid curve β on sphere with radius $\frac{2}{3}$.



(b) Bishop spherical image curve β_{M_1} on S^2.

Figure 4.1: The spherical helices.

Example 4.12. *Let us consider the Euler Spiral* $\gamma(s)=(\frac{3}{5}\int\sin(s^2+1)ds,\frac{3}{5}\int\cos(s^2+1)ds,\frac{4}{5}s)$ *of* E^3*. The Frenet-Serret frame apparatus of the curve* $\gamma=\gamma(s)$ *are calculated as follows:*

$$\begin{aligned} T(s) &= \left(\frac{3}{5}\sin(s^2+1),\frac{3}{5}\cos(s^2+1),\frac{4}{5}\right), \\ N(s) &= (\cos(s^2+1),-\sin(s^2+1),0), \\ B(s) &= (\frac{4}{5}\sin(s^2+1),\frac{4}{5}\cos(s^2+1),-\frac{3}{5}). \end{aligned}$$

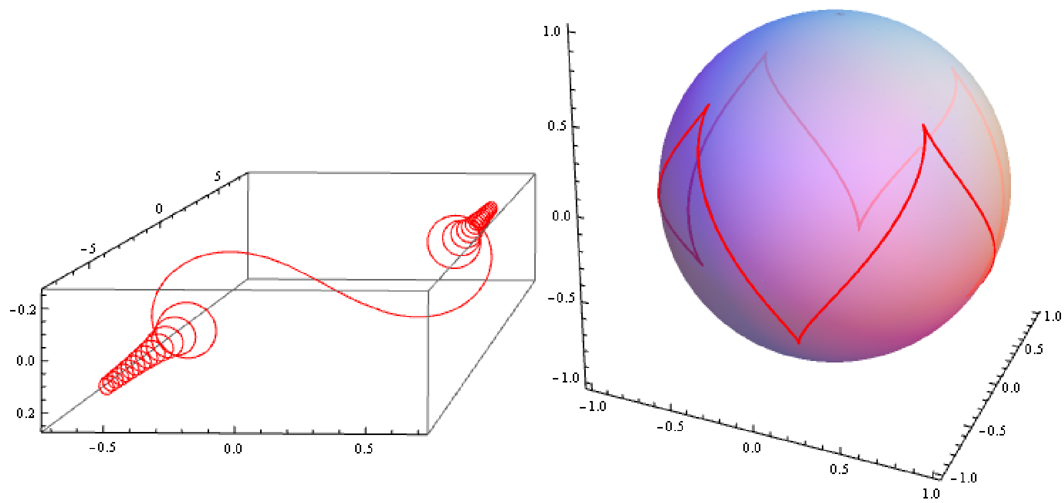
$$\kappa(s)=\frac{6s}{5} \quad and \quad \tau(s)=-\frac{8s}{5}.$$

To construct the Bishop frame, we calculate the angle of rotation $\theta(s) = -\int \tau ds = \frac{4s^2}{5}$*. So, the Bishop frame apparatus of the curve* γ *are*

$$\begin{aligned} T(s) &= T(s), \\ M_1(s) &= \cos\left(\frac{4s^2}{5}\right) N(s) + \sin\left(\frac{4s^2}{5}\right) B(s), \\ M_2(s) &= -\sin\left(\frac{4s^2}{5}\right) N(s) + \cos\left(\frac{4s^2}{5}\right) B(s). \end{aligned}$$

$$k_1(s) = \frac{6s}{5}\cos\left(\frac{4s^2}{5}\right) \quad \textit{and} \quad k_2(s) = -\frac{6s}{5}\sin\left(\frac{4s^2}{5}\right).$$

We obtain the equality $k_1(s)\int k_1(s)ds + k_2(s)\int k_2(s)ds = 0$*, so the Euler Spiral* $\gamma = \gamma(s)$ *is the generalized helix according to Bishop frame.*



(a) The Euler spiral γ. **(b)** Bishop spherical image curve γ_{M_1} on S^2.

Figure 4.2: Helices.

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