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On Some Sequence Spaces Related to a Sequence in a Normed space

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Abstract

In this paper, we introduce some new multiplier sequence spaces by using sequences in a normed space X and matrix domain of Cesáro summability method in ℓ_∞ and c_0. Then we obtain the characterizations of completeness and barrelledness of normed space X through its weakly and weakly* unconditionally Cauchy series.

Keywords: *Mulpilier spaces, Cesáro sequence spaces, weakly unconditionally Cauchy series, completeness, barrelledness.*
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1. Introduction

By w, we denote the space of all real sequences $x=(x_k)$. Any vector subspace of w is called a *sequence space*. Let ℓ_∞, c and c_0 be the spaces of all bounded, convergent and null sequences $x=(x_k)$, respectively, normed by $\|x\|_\infty = sup_k|x_k|$, where $k\in\mathbb{N}$, the set of positive integers. Also by cs and ℓ_1, we denote the spaces of all convergent and absolutely convergent series, respectively.
A sequence space λ with a linear topology is called a *K-space* provided each of the maps $p_i:\lambda\to\mathbb{R}$ defined by $p_i(x)=x_i$ is continuous for all $i\in\mathbb{N}$. A K-space λ is called an *FK-space* provided λ is a complete linear metric space. We say that an FK space $\lambda\supset c_{00}$ has AD if c_{00} is dense in λ, where $c_{00}=span\{e^n:n\in\mathbb{N}\}$, the set of all finitely non-zero sequences and e^n $(n\in\mathbb{N})$ the sequences with $e^n_n=1$ and $e^n_k=0$ for $k\neq n$.
Let $A=(a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n,\ k\in\mathbb{N}$. Then, we write $Ax=\left((Ax)_n\right)$, the A-transform of $x\in w$, if $(Ax)_n=\sum_k a_{nk}x_k$ converges for each $n\in\mathbb{N}$. For simplicity in notation, here and in what follows, the summation without limits runs from 1 to ∞. For a sequence space λ, the matrix domain λ_A of an infinite matrix A is defined by

$$\lambda_A=\left\{x=(x_k)\in w: Ax\in\lambda\right\},$$

which is a sequence space. The Cesàro matrix C with *Cesàro mean of order one,* which is a well–known method of summability and is defined by the matrix $C=(c_{nk})$ as follows;

$$c_{nk}=\begin{cases}\frac{1}{n}, & 1\leq k\leq n,\\ 0, & k>n.\end{cases}$$

The C-transform of a sequence $a=(a_k)$ is the sequence $\tau(a)=(\tau_n(a))$ defined by

$$\tau_n(a)=\frac{1}{n}\sum_{k=1}^{n}a_k \text{ for all } n\in\mathbb{N}.$$

The set of all sequences whose C-transforms are in the spaces ℓ_∞ and c_0 were defined by Shiue in [12], Ng and Lee in [10], and Şengönül and Başar in [13], respectively. Some other works about the study of sequence spaces are [3, 6, 7, 8, 9, 14].
Let X be a real Banach space, X^* is a dual space of X and $\sum_i x_i$ be a series in X. A series $\sum_i x_i$ is called weakly unconditionally Cauchy series ($wuCs$) if $\left(\sum_{i=1}^n x_{\pi_i}\right)_{n\in\mathbb{N}}$ is a weakly Cauchy for every permutation π of $\mathbb{N}$. It is known that $\sum_i x_i$ is a $wuCs$ if and only if $\sum_i|f(x_i)|<\infty$ for every $f\in X^*$. A series $\sum_i x_i$ is called unconditionally convergent series (ucs) if $\sum_i x_{\pi(i)}$ converges for every permutation π of $\mathbb{N}$. By $ucs(X)$, $uCs(X)$, $\ell_1(X)$, $cs(X)$, $wcs(X)$ and $wuCs(X)$, we denote the X-valued sequence spaces of unconditionally convergent, unconditionally Cauchy, absolutely convergent, convergent, weakly convergent and weakly unconditionally Cauchy series, respectively.
It is well known that [2, 4, 5]:

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(1) The sequence $x=(x_k)\in ucs(X)$ if and only if $(a_kx_k)\in cs(X)$ for every $a=(a_k)\in l_\infty$.
(2) The sequence $x=(x_k)\in wuCs(X)$ if and only if $(a_kx_k)\in cs(X)$ for every $a=(a_k)\in c_0$.
(3) Let X be a normed space. The sequence $x=(x_i)\in wuCs(X)$ if and only if the set

$$S=\left\{\sum_{i=1}^{n}a_ix_i : |a_i|\leq 1,\, i=1,2,\dots,n;\, n\in N\right\} \tag{1.1}$$

is bounded.

Let $x=(x_i)$ be a sequence in normed space X and $f=(f_i)$ be a sequence in X^*. In this work we will study the following subspaces of $(\ell_\infty)_C$, which are defined by

$$\begin{aligned} SC(x) &= \left\{a=(a_i)\in(\ell_\infty)_C : \sum_i \tau_i(a)x_i \text{ converges in} X\right\},\\ SC_w(x) &= \left\{a=(a_i)\in(\ell_\infty)_C : \sum_i \tau_i(a)x_i \text{ weakly converges in} X\right\},\end{aligned}$$

and

$$SC_{w^*}(f) = \left\{a=(a_i)\in(\ell_\infty)_C : \sum_i \tau_i(a)f_i \text{ weak}^*-\text{converges in} X^*\right\}.$$

The sets $SC(x)$, $SC_w(x)$ and $SC_{w^*}(f)$ are linear spaces with the co-ordinatewise addition and scalar multiplication which are the normed spaces with the norm $\|a\|_{SC}=\|Ca\|_\infty$.

2. Main results

In the section, for $x\in wuCs(X)$ we will give necessary and sufficient conditions to be complete of a normed space X by means of the spaces $SC(x)$ and $SC_w(x)$. Also, for $f\in w^*uCs(X^*)$ we will characterize the barrelledness of a normed space X through the space $SC_{w^*}(f)$.
Firstly, we give a sufficient condition for equality between $SC(x)$ and $SC_w(x)$.

Lemma 2.1. *Let X be a normed space and $x\in uCs(X)$. Then, $SC(x)=SC_w(x)$.*

Proof. Since every convergent sequence is weakly convergent the inclusion $SC(x)\subseteq SC_w(x)$ holds.
We will prove that $SC_w(x)\subseteq SC(x)$. Let $a=(a_i)\in SC_w(x)$. Then there exists $x\in X$ such that for every $f\in X^*$,

$$\sum_{i=1}^{\infty}\tau_i(a)f(x_i)=f(x).$$

On the other hand, since $x\in uCs(X)$ the partial sums of the series $\sum_{i=1}^{\infty}\tau_i(a)x_i$ form a Cauchy sequence in X. Then, there exists $x^{**}\in X^{**}$ such that

$$\sum_{i=1}^{\infty}\tau_i(a)x_i=x^{**}.$$

Hence, from uniqueness of limit, $x^{**}=x$. That is $a=(a_i)\in SC(x)$. □

Now, we obtain necessary and sufficient condition for the space $SC(x)$ to be complete.

Theorem 2.2. *Let X be a Banach space. Then, $x=(x_k)$ is a sequence in $wuCs(X)$ if and only if the space $SC(x)$ is a Banach space.*

Proof. First, we will show that necessary condition holds.
Let $x\in wuCs(X)$. Then, since S defined be equation (1.1) is a bounded set, we suppose that $\|s\|\leq K$ for every $s\in S$. Let (a^m) be a Cauchy sequence in $SC(x)$. Since $SC(x)\subset(\ell_\infty)_C$ and $(\ell_\infty)_C$ is a Banach space, there exists $a=(a_i^0)\in(\ell_\infty)_C$ such that $a^m\to a^0$ in $(\ell_\infty)_C$ as $m\to\infty$. Therefore, for $\varepsilon>0$, there exists $m_0\in\mathbb{N}$ such that for every $m\geq m_0$ and $i\in\mathbb{N}$,

$$\left|\tau_i(a^m)-\tau_i(a^0)\right|<\frac{\varepsilon}{3K}.$$

Since $\frac{3K}{\varepsilon}|\tau_i(a^m)-\tau_i(a^0)|<1$, $\frac{3K}{\varepsilon}\sum_{i=1}^{n}(\tau_i(a^m)-\tau_i(a^0))x_i\in S$, and hence for $m>m_0$ we have

$$\left\|\sum_{i=1}^{n}\left(\tau_i(a^m)-\tau_i(a^0)\right)x_i\right\|<\frac{\varepsilon}{3}.$$

Since for each $m\in\mathbb{N}$ the sequence (a^m) is in $SC(x)$, there exists a sequence $(y_m)\subset X$ such that for $n\geq n_0$

$$\left\|\sum_{i=1}^{n}\tau_i(a^m)x_i-y_m\right\|<\frac{\varepsilon}{3}.$$

Then for $p > q > m_0$ and $n \in \mathbb{N}$,

$$\begin{aligned} \|y_p - y_q\| &\leq \left\| \sum_{i=1}^{n} \tau_i(a^p) x_i - y_p \right\| + \left\| \sum_{i=1}^{n} \tau_i(a^q) x_i - y_q \right\| + \left\| \sum_{i=1}^{n} (\tau_i(a^q) - \tau_i(a^p)) x_i \right\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

and thus (y_m) is a Cauchy sequence in X. Hence, for $\varepsilon > 0$ there exists $y_0 \in X$ such that for each $m > m_1$

$$\|y_m - y_0\| < \frac{\varepsilon}{3}.$$

Take $m_2 = \max\{m_0, m_1\}$. Then for $n \geq n_0$ and $m > m_2$, we have

$$\begin{aligned} \left\| \sum_{i=1}^{n} \tau_i(a^0) x_i - y_0 \right\| &\leq \left\| \sum_{i=1}^{n} (\tau_i(a^0) - \tau_i(a^m)) x_i \right\| + \left\| \sum_{i=1}^{n} \tau_i(a^m) x_i - y_m \right\| + \|y_m - y_0\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

As a consequence, $(a^0) \in SC(x)$ and hence $SC(x)$ is complete.

Now, suppose that $SC(x)$ is complete, but x is not in $wuCs(X)$. Then there exists a sequence $a^0 = (a_i^0)$ in c_0 such that $\sum_{i=1}^{n} a_i^0 x_i$ is not convergent. Therefore there exists $b^0 = (b_i^0) \in (c_0)_C$ such that $\tau_i(b^0) = a_i^0$ and hence $\sum_{i=1}^{n} \tau_i(b^0) x_i$ is not convergent. That is, $b^0 = (b_i^0) \notin SC(x)$ and so $(c_0)_C \nsubseteq SC(x)$. On the other hand, since $(c_0)_C$ is a AD-space by [13, Theorem 2.4], there exists a Cauchy sequence $y = (y_i^m)$ in c_{00} (also in $SC(x)$) such that

$$\lim_{m\to\infty} y_i^m = b_i^0.$$

Consequently, $SC(x)$ is not complete. □

The following theorem gives us a characterization of completeness of normed spaces.

Theorem 2.3. *The normed space X is a Banach space if and only if $SC(x)$ is a Banach space for every $x = (x_k)$ in $wuCs(X)$.*

Proof. Necessary condition is obtained from Theorem 2.2.
Suppose that X is not a Banach space. Then there exists a sequence $x = (x_k) \in \ell_1(X) \setminus cs(X)$ such that for every $k \in \mathbb{N}$

$$\|x_k\| < \frac{1}{k2^k}.$$

We define the sequence $y = (y_k)$ by

$$y_k = \begin{cases} kx_k, & \text{if } k \text{ is odd}, \\ -kx_k, & \text{if } k \text{ is even}, \end{cases}$$

and consider the sequence $b = (b_k) \in (c_0)_C$ defined by

$$b_k = \begin{cases} \frac{1}{2}, & \text{if } k = 1, \\ \frac{2k+1}{k+1}, & \text{if } k \neq 1 \text{ and } k \text{ is odd}, \\ -\frac{2k-1}{k}, & \text{if } k \text{ is even}, \end{cases}$$

then $y = (y_k) \in wuCs(X)$ and $\sum_k \tau_k(b) y_k$ does not converge. This proves $(c_0)_C \nsubseteq SC(y)$. Hence the space $SC(y)$ is not complete. □

Now, we will extend some of the above results to weak topology. First, let us start with the following lemma.

Lemma 2.4. *Let X be a Banach space. Then $x = (x_k)$ in $wuCs(X)$ if and only if $(c_0)_C \subseteq SC_w(x)$.*

Proof. Let x be a sequence in $wuCs(X)$. Then for $a = (a_i) \in c_0$, the series $\sum_{i=1}^{n} a_i x_i$ is convergent. If we take $\tau_i(b) = a_i$ for $b = (b_i) \in (c_0)_C$, the series $\sum_{i=1}^{n} \tau_i(b) x_i$ is convergent, and hence weakly convergent. Therefore $b = (b_i) \in SC_w(x)$.
Conversely, suppose that $(c_0)_C \subseteq SC_w(x)$. Then for every sequence $b = (b_i) \in (c_0)_C$, the sequence $(\tau_i(b) x_k)$ is in $wcs(X)$. We define the sequence

$$z_n = \begin{cases} \tau_n(b), & \text{if } n = n_k, \\ 0, & \text{if } n \neq n_k \end{cases}$$

for an increasing sequence of positive integers (n_k). Then the series $\sum_{i=1}^{n} z_i x_i = \sum_{k=1}^{n} \tau_{i_k}(b) x_{i_k}$ is weakly convergent, and thus $\sum_{i=1}^{n} \tau_i(b) x_i$ is subseries weakly convergent. From Orlicz-Pettis Theorem, $(\tau_i(b) x_i)$ is in $ucs(X)$. Then the series $(\tau_i(b) x_i)$ belongs to $cs(X)$, and hence x is in $wuCs(X)$. □

Theorem 2.5. *Let X be a Banach space and $x = (x_k)$ is a sequence in X. $SC_w(x)$ is complete if and only if $x \in wuCs(X)$.*

Proof. Necessary condition can be easily obtained from Lemma 2.4.
Let $x \in wuCs(X)$. Then, since S is a bounded set, let $\|s\| \leq K$ for every $s \in S$. Let $a = (a^m)$ be a Cauchy sequence in $SC_w(x)$ such that $a^m \to a^0$ in $(\ell_\infty)_C$ as $m \to \infty$. Therefore, for $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$ and $i \in \mathbb{N}$

$$\left|\tau_i(a^m) - \tau_i(a^0)\right| < \frac{\varepsilon}{3K}.$$

Since $3K/\varepsilon|\tau_i(a^m) - \tau_i(a^0)| < 1$, $3K/\varepsilon \sum_{i=1}^n \left(\tau_i(a^m) - \tau_i(a^0)\right) x_i \in S$, and hence for $m > m_0$

$$\left\|\sum_{i=1}^n \left(\tau_i(a^m) - \tau_i(a^0)\right) x_i\right\| < \frac{\varepsilon}{3}.$$

On the other hand, there exists a sequence $(y_m) \subset X$ such that for $n \geq n_0$ and for all $f \in X^*$

$$\left|\sum_{i=1}^n \tau_i(a^m) f(x_i) - f(y_m)\right| < \frac{\varepsilon}{3}.$$

Also there exists $f \in B_{X^*}$ such that $\|y_p - y_q\| = |f(y_p - y_q)|$. Then for $p > q > m_0$ and $n \in \mathbb{N}$

$$\|y_p - y_q\| = |f(y_p - y_q)| < \varepsilon,$$

and thus (y_m) is a Cauchy sequence in X. Hence, for $\varepsilon > 0$ there exists $y_0 \in X$ such that for $m > m_1$

$$\|y_m - y_0\| < \frac{\varepsilon}{3}.$$

Take $m_2 = \max\{m_0, m_1\}$. Then for $n \geq n_0$ and $m > m_2$ we have

$$\begin{aligned} \left|\sum_{i=1}^n \tau_i(a^0) f(x_i) - f(y_0)\right| &\leq \left|\sum_{i=1}^n \left(\tau_i(a^0) - \tau_i(a^m)\right) f(x_i)\right| + \left|\sum_{i=1}^n \tau_i(a^m) f(x_i) - f(y_m)\right| \\ &\quad + |f(y_m) - f(y_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore $(a^0) \in SC_w(x)$ and hence $SC_w(x)$ is complete. □

Theorem 2.6. *X is a Banach space if and only if $SC_w(x)$ is a Banach space for every $x \in wuCs(X)$.*

Proof. As in the proof of Theorem 2.2, suppose that X is not a Banach space. Then we can find a sequence $y = (y_k) \in wuCs(X)$ and $(c_0)_C \not\subseteq SC(y)$. Thus $SC(y)$ is not a Banach space. Since $y = (y_k) \in uCs(X)$, by Lemma 2.1 we obtain that $SC(y) = SC_w(y)$, and hence $SC_w(y)$ is not a Banach space. □

Finally, we give a characterization of barrelledness of normed spaces.

Theorem 2.7. *Let X be a normed space and $f = (f_i)$ be a sequence in X^*. Consider the following statements:*

(1) $f \in wuCs(X^*)$.
(2) $SC_{w^*}(f) = (l_\infty)_C$.
(3) $f \in w^*uCs(X^*)$; that is, $\sum_i |f_i(x)| < \infty$ for every $x \in X$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). The normed space X is a barrelled space if and only if (3) $\Rightarrow$ (1).

Proof. ((1) $\Rightarrow$ (2)). We consider $b = (b_i) \in (l_\infty)_C$. Then there exists $a = (a_i) \in l_\infty$ such that $\tau_i(b) = (a_i)$. Since $f \in wuCs(X^*)$, $(a_i f_i) \in wuCs(X^*)$, and hence $(\tau_i(b) f_i) \in wuCs(X^*)$. Therefore (s_n) is a bounded sequence in X^* and a Cauchy sequence for the weak* topology on X^*, where $s_n = \sum_{i=1}^n \tau_i(b) f_i$. Thus, $\sum_i \tau_i(b) f_i$ is weak* convergent.

((2) $\Rightarrow$ (3)). Let $S_{w^*} = (\ell_\infty)_C$. Then for every $x \in X$ and $b = (b_i) \in (\ell_\infty)_C$ the sequence $(\tau_i(b)_i f_i(x))$ in $cs(X)$. If we take $\tau_i(b) = \mathrm{sgn} f_i(x)$, then we have $\sum_i |f_i(x)| < \infty$.

Now, let X be a barrelled space. We will show (3) implies (1).
We define the set S' by

$$S' = \left\{\sum_{i=1}^n a_i f_i : |a_i| \leq 1, i = 1, 2, \ldots, n; n \in \mathbb{N}\right\}.$$

It can be easily seen that the set S' is pointwise bounded and hence S' is bounded for the norm topology of X^*. So, $(f_i) \in wuCs(X^*)$.
Assume that (3) implies (1) holds but X is not a barrelled space. Then there exists a weak*-bounded set $A \subseteq X^*$ that is not bounded. Let $(f_i) \in A$ such that $\|f_i\| > 2^{2i}$ for $i \in \mathbb{N}$. If we take $g_i = \frac{1}{2^i} f_i$ for $i \in \mathbb{N}$, then it is obvious that for every $x \in X$, $(g_i(x)) \in \ell_1$.
On the other hand, since $\|g_i\| > 2^i$ for every $i \in \mathbb{N}$, the series $\sum_i \frac{1}{2^i} g_i$ does not converge. Hence, $(g_i) \notin wuCs(X^*)$. □

Corollary 2.8. *X is a barrelled normed space if and only if $wuCs(X^*) = w^*uCs(X^*)$.*

3. Conclusion

In [1], some new spaces were defined and, by means of these spaces conditionally and weakly unconditionally Cauchy series were characterized. Also using these spaces, Pérez-Fernández et al. [11] obtained new characterizations of completeness and barrelledness of a normed space via the behaviour of its weakly and weak* unconditionally Cauchy series.
In this paper, we will characterize the completeness and barrelledness of a normed space X in terms of $SC(x)$, $SC_w(x)$ and $SC_{w^*}(f)$.

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