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APPROXIMATELY Γ-RINGS IN PROXIMAL RELATOR SPACES

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Abstract. This article introduces approximately Γ-rings, approximately Γ-ideals and approximately Γ-rings of all descriptive approximately cosets by considering new operations on the set of all descriptive approximately cosets. Afterwards, some properties of approximately Γ-rings and Γ-ideals were given.

1. Introduction

A proximal relator space is a pair $(X, \mathcal{R}_\delta)$ consists of a nonempty describable set X and set of proximity relations $\mathcal{R}_\delta$ defined on X, called proximal relator. There are different types of proximity relations such as Efremovič proximity, Wallman proximity, descriptive proximity, Lodato proximity [2, 3, 9, 15]. These proximity relations provide a very useful infrastructure for some applied sciences. In a proximal relator space, the sets are composed of non-abstract points instead of abstract points. These points describable with feature vectors in proximal relator spaces.

The aim of this article is to obtain some algebraic structures in proximal relator spaces that include descriptive upper approximations of the subsets of non-abstract points. The descriptive upper approximation of a nonempty set is obtained by using the set of points composed by the proximal relator space together with matching features of points. In the algebraic structures constructed on proximal relator spaces, the basic tool is consideration of these descriptively upper approximations of the subsets of non-abstract points.

There are two important differences between ordinary algebraic structures and approximately algebraic structures in proximal relator spaces. The first one is working with non-abstract points such as digital images, while the second one is considering of descriptively upper approximations of the subsets of non-abstract points for the closeness of binary operations. Using the theoretical background of this concept, it can be obtained more functional algorithms for applied sciences such as image processing.

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Nobusawa [12] introduced the notion of a Γ-ring, as more general than a ring. Barnes [1] weakened slightly the conditions in the definition of the Γ-ring in the sense of Nobusawa. Barnes [1], Kyuno [8] and Luh [10] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory.

Essentially, the aim is to obtain approximately Γ-rings, approximately Γ-ideals and approximately Γ-rings of all descriptive approximately cosets by considering new operations on the set of all descriptive approximately cosets. Furthermore, some properties of approximately Γ-rings and approximately Γ-ideals were introduced.

2. Preliminaries

Let X be a nonempty set. Family of relations $\mathcal{R}$ on a nonempty set X is called a *relator*. The pair $(X, \mathcal{R})$ (or $X(\mathcal{R})$) is a relator space which results from natural generalizations of uniform spaces [18]. If we consider a family of proximity relations on X, we have a proximal relator space $(X, \mathcal{R}_\delta)$ (also denoted by $X(\mathcal{R}_\delta)$). As in [15], $\mathcal{R}_\delta$ contains proximity relations, namely, Efremovič proximity δ_E [2, 3], Lodato proximity δ_L [9], Wallman proximity δ_W, descriptive proximity δ_Φ in defining $\mathcal{R}_{\delta_\Phi}$ [13, 17].

In this article, we consider the Efremovič proximity δ_E [3] and the descriptive proximity δ_Φ in defining a descriptive proximal relator space (denoted by $(X, \mathcal{R}_{\delta_\Phi})$).

An Efremovič proximity δ_E is a relation on 2^X that satisfies

1^o $A \ \delta_E \ B \Rightarrow B \ \delta_E \ A$,
2^o $A \ \delta_E \ B \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$,
3^o $A \cap B \neq \emptyset \Rightarrow A \ \delta_E \ B$,
4^o $A \ \delta_E \ (B \cup C) \Leftrightarrow A \ \delta_E \ B$ or $A \ \delta_E \ C$,
5^o $\{x\} \ \delta_E \ \{y\} \Leftrightarrow x = y$,
6^o EF axiom. $A \ \underline{\delta_E} \ B \Rightarrow \exists E \subseteq X$ such that $A \ \underline{\delta_E} \ E$ and $E^c \ \underline{\delta_E} \ B$.

Lodato proximity δ_L [9] swaps the EF axiom 2 for the following condition:

$$A \ \delta_L \ B \text{ and } \{b\} \ \delta_L \ C \Rightarrow A \ \delta_L \ C \text{ for all } b \in B \ (\textit{Lodato Axiom}).$$

In a discrete space, a non-abstract point has a location and has features that can be measured [7, §3]. Let X be a nonempty set of non-abstract points in a proximal relator space $(X, \mathcal{R}_{\delta_\Phi})$ and let $\Phi = \{\phi_1, \cdots, \phi_n\}$ a set of probe functions that represent features of each $x \in X$.

A *probe function* $\Phi : X \to \mathbb{R}$ represents a feature of a sample point in a picture. Let $\Phi(x) = (\phi_1(x), \cdots, \phi_n(x))$ $(n \in \mathbb{N})$ be an object description, which is a feature vector of x, which provides a description of each $x \in X$. After the choosing a set of probe functions, one obtain a descriptive proximity relation δ_Φ.

Definition 1. [11] *Let X be a nonempty set of non-abstract points, Φ an object description and A a subset of X. Then the set description of A is defined as*

$$\mathcal{Q}(A) = \{\Phi(a) \mid a \in A\}.$$

Definition 2. [11, 14] *Let X be a nonempty set of non-abstract points, A and B any two subsets of X. Then the descriptive (set) intersection of A and B is defined as*

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B \mid \Phi(x) \in \mathcal{Q}(A) \text{ and } \Phi(x) \in \mathcal{Q}(B)\}.$$

Definition 3. [13] *Let X be a nonempty set of non-abstract points, A and B any two subsets of X. If $\mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset$, then A is called descriptively near B and denoted by $A\delta_\Phi B$. If $\mathcal{Q}(A) \cap \mathcal{Q}(B) = \emptyset$, then $A \ \underline{\delta}_\Phi \ B$ reads A is descriptively far from B.*

[16] Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and $A \subset X$, where A contains non-abstract objects. Let $(A, \cdot)$ and $(\mathcal{Q}(A), \circ)$ be groupoids. Let consider the object description Φ by means of a function

$$\Phi : A \subset X \longrightarrow \mathcal{Q}(A) \subset \mathbb{R},\ a \mapsto \Phi(a)\ (a \in A).$$

The object description Φ of A into $\mathcal{Q}(A)$ is an *object description homomorphism* if $\Phi(a \cdot b) = \Phi(a) \circ \Phi(b)$ for all $a, b \in A$.

Also, descriptive closure of a point $a \in A$ is defined by

$$cl_\Phi(a) = \{x \in X \mid \Phi(a) = \Phi(x)\}.$$

Definition 4. [5] *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and $A \subset X$. A descriptively upper approximation of A is defined as*

$$\Phi^* A = \{x \in X \mid x\delta_\Phi A\}.$$

Definition 5. [4] *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and $A \subset X$. A descriptively lower approximation of A is defined as*

$$\Phi_* A = \{a \in A \mid cl_\Phi(a) \subseteq A\}.$$

Let $A, B \subset X$. If $A \subseteq B$, then $\mathcal{Q}(A) \subseteq \mathcal{Q}(B)$. Thus, in Definition 5, we can consider $\mathcal{Q}(cl_\Phi(a)) \subseteq \mathcal{Q}(A)$ instead of $cl_\Phi(a) \subseteq A$.

Obviously, $\Phi_* A \subseteq A \subseteq \Phi^* A$ for all $A \subset X$.

Lemma 1. [5] *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and $A, B \subset X$. Then*

1. $\mathcal{Q}(A \cap B) = \mathcal{Q}(A) \cap \mathcal{Q}(B)$,
2. $\mathcal{Q}(A \cup B) = \mathcal{Q}(A) \cup \mathcal{Q}(B)$.

Definition 6. [5] *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and let "$\cdot$" a binary operation on X. $G \subset X$ is called an approximately groupoid in descriptive proximal relator space if $x \cdot y \in \Phi^* G$ for all $x, y \in G$.*

Definition 7. [4] *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and let "+" a binary operation on X. $G \subset X$ is called an approximately group in descriptive proximal relator space or shortly approximately group if the followings are true:*

($\mathcal{A}G_1$) *For all $x, y \in G$, $x + y \in \Phi^* G$,*
($\mathcal{A}G_2$) *For all $x, y, z \in G$, $(x + y) + z = x + (y + z)$ property holds in $\Phi^* G$,*
($\mathcal{A}G_3$) *There exists $e \in \Phi^* G$ such that $x + e = e + x = x$ for all $x \in G$ (e is called the approximately identity element of G),*
($\mathcal{A}G_4$) *There exists $y \in G$ such that $x + y = y + x = e$ for all $x \in G$ (y is called the inverse of x in G and denoted as $-x$).*

A subset G of the set of X is called an *approximately semigroup* in descriptive proximal relator space if ($\mathcal{A}G_1 - \mathcal{A}G_2$) properties are satisfied.

Theorem 1. [4] *Let G be an approximately group, H a nonempty subset of G and $\Phi^* H$ a groupoid. H is an approximately subgroup of G iff $-x \in H$ for all $x \in H$.*

Theorem 2. [4] *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and $G \subseteq X$ an approximately group. Then*

(1) *There is one and only one approximately identity element in G.*
(2) *There is only one $y \in G$ such that $x + y = y + x = e$ for all $x \in G$; we denote it by $-x$.*

Theorem 3. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and $G \subseteq X$ an approximately group. If either $x + z = y + z$ or $z + x = z + y$, then $x = y$ for all $x, y, z \in G$.*

Suppose that G is an approximately groupoid with the binary operation "$\cdot$" in $(X, \mathcal{R}_{\delta_\Phi})$, $g \in G$ and $A, B \subseteq G$. The subsets $g \cdot A$, $A \cdot g$, $A \cdot B \subseteq \Phi^* G \subseteq X$ are defined as follows:

$$g \cdot A = gA = \{ga : a \in A\},$$
$$A \cdot g = Ag = \{ag : a \in A\},$$
$$A \cdot B = AB = \{ab : a \in A, b \in B\}.$$

Lemma 2. [4] *Let (X, δ_Φ) be a descriptive proximity space and $A, B \subset X$. If $\Phi : X \longrightarrow \mathbb{R}$ is an object descriptive homomorphism, then*

$$\mathcal{Q}(A)\mathcal{Q}(B) = \mathcal{Q}(AB).$$

Theorem 4. [6] *Let G be an additive approximately group, H an approximately subgroup of G and $G/_{\rho_l}$ a set of all approximately left cosets of G by H. If $(\Phi^* G)/_{\rho_l} \subseteq \Phi^* \left(G/_{\rho_l}\right)$, then $G/_{\rho_l}$ is an approximately group under the operation given by $xH \oplus yH = (x + y) H$ for all $x, y \in G$.*

Definition 8. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and "+", "$\cdot$" binary operations defined on X. A $R \subseteq X$ is called an approximately ring in descriptive proximal relator space if the following properties are satisfied:*

($\mathcal{A}R_1$) *R is an abelian approximately group with the binary operation "+",*

($\mathcal{A}R_2$) *R is an approximately semigroup with the binary operation "$\cdot$",*
($\mathcal{A}R_3$) *For all $x, y, z \in R$,*

$x \cdot (y+z) = (x \cdot y) + (x \cdot z)$, $(x+y) \cdot z = (x \cdot z) + (y \cdot z)$
properties hold in $\Phi^* R$.
If in addition:

($\mathcal{A}R_4$) $x \cdot y = y \cdot x$ *for all $x, y \in R$,*

then R is said to be a commutative approximately ring.

($\mathcal{A}R_5$) *If $\Phi^* R$ contains an element 1_R such that $1_R \cdot x = x \cdot 1_R = x$ for all $x \in R$,*

then R is said to be an approximately ring with identity.

Definition 9. [1] *A Γ-ring (in the sense of Barnes) is a pair (M, Γ) where M and Γ are (additive) abelian groups for which exists a $(-,-,-) : M \times \Gamma \times M \to M$, the image of (a, α, b) being denoted by $a\alpha b$ for $a, b \in M$ and $\alpha \in \Gamma$, satisfying for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$:*

- $(a+b)\alpha c = a\alpha c + b\alpha c$,
- $a(\alpha+\beta)b = a\alpha b + a\beta b$,
- $a\alpha(b+c) = a\alpha b + a\alpha c$,
- $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Definition 10. [1] *Let M be a Γ-ring. A left (right) ideal of M is an additive subgroup U of M such that $M\Gamma U \subseteq U$ ($U\Gamma M \subseteq U$). If U is both a left and a right ideal, then we say that U is an ideal of M.*

Definition 11. [1] *A mapping $\theta : M \to N$ of Γ-rings is called a Γ-homomorphism if $\theta(a+b) = \theta(a) + \theta(b)$ and $\theta(a\alpha b) = \theta(a)\alpha\theta(b)$ for all $a, b \in M$ and $\alpha \in \Gamma$.*

3. Approximately Γ-rings

Definition 12. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and $M, \Gamma \subseteq X$ be additive abelian approximately groups in $(X, \mathcal{R}_{\delta_\Phi})$. If for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ the conditions*

($\mathcal{A}\Gamma_1$) $a\alpha b \in \Phi^* M$,
($\mathcal{A}\Gamma_2$) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$ *properties verify on $\Phi^* M$,*
($\mathcal{A}\Gamma_3$) $(a\alpha b)\,\beta c = a\alpha\,(b\beta c)$ *property verify on $\Phi^* M$*

are satisfied, then M is called an approximately Γ-ring in descriptive proximal relator space or shortly approximately Γ-ring.

In addition, if $a\alpha b = b\alpha a$ for all $a, b \in M$ and $\alpha \in \Gamma$, then M is called a commutative approximately Γ-ring.

Example 1. *Let X be a digital image endowed with descriptive proximity relation δ_Φ and consists of 25 pixels as in Figure 1.*

FIGURE 1. Digital image X and subimage M

A pixel x_{ij} *is an element at position* (i, j) *(row and column) in digital image* X*. Let* ϕ *be a probe function that represent RGB colour of each pixel are given in Table 1.*

	Red	*Green*	*Blue*
x_{00}	255	230	150
x_{01}	180	255	200
x_{02}	255	230	150
x_{03}	204	245	185
x_{04}	204	245	185
x_{10}	255	230	150
x_{11}	180	255	200
x_{12}	204	245	185
x_{13}	100	160	145
x_{14}	130	182	167
x_{20}	100	160	145
x_{21}	204	245	185
x_{22}	181	232	231

	Red	*Green*	*Blue*
x_{23}	100	160	145
x_{24}	200	200	250
x_{30}	204	245	185
x_{31}	100	160	145
x_{32}	200	200	250
x_{33}	170	240	200
x_{34}	200	230	255
x_{40}	204	245	185
x_{41}	100	160	145
x_{42}	255	230	150
x_{43}	200	230	255
x_{44}	130	182	167

Table 1. RGB colour of each pixel

Let

$$+_1 \quad : \quad \begin{array}{ll} X \times X & \longrightarrow X \\ (x_{ij}, x_{mn}) & \longmapsto x_{ij} +_1 x_{mn} \end{array} \quad ,$$

$$x_{ij} +_1 x_{mn} = x_{pr} \quad , i + m \equiv p \pmod 2 \text{ and } j + n \equiv r \pmod 2$$

be a binary operation (first addition) on X*. Let* $M = \{x_{01}, x_{10}\}$ *a subimage (subset) of* X*.*

We can compute the descriptively upper approximation of M*, that is,* $\Phi^* M = \{x_{ij} \in X \mid x_{ij} \delta_\phi M\}$ *by using the Definition 4. Then* $\phi(x_{ij}) \cap Q(M) \neq \emptyset$ *such that*

$x_{ij} \in X$, *where* $Q(M) = \{\phi(x_{ij}) \mid x_{ij} \in M\}$. *From Table 1, we obtain*

$$\begin{aligned} \mathcal{Q}(M) &= \{\phi(x_{01}), \phi(x_{10})\} \\ &= \{(180, 255, 200), (255, 230, 150)\}. \end{aligned}$$

Hence we get $\Phi^* M = \{x_{00}, x_{01}, x_{02}, x_{10}, x_{11}, x_{42}\}$. *Consequently,* M *is an additive abelian approximately group in* $(X, \mathcal{R}_{\delta_\Phi})$ *from Definition 7. Furthermore, let*

$$+_2 \;:\; \begin{array}{ll} X \times X & \longrightarrow X \\ (x_{ij}, x_{mn}) & \longmapsto x_{ij} +_2 x_{mn} \end{array},$$

$$x_{ij} +_2 x_{mn} = x_{st} \quad, i + m \equiv s \pmod 4 \text{ and } j + n \equiv t \pmod 4$$

be a binary operation (second addition) on X. *Let* $\Gamma = \{x_{42}\}$ *a subimage (subset) of* X.

We can calculate the descriptively upper approximation of Γ, *that is,* $\Phi^*\Gamma = \{x_{ij} \in X \mid x_{ij}\delta_\phi \Gamma\}$ *by using the Definition 4. Then* $\phi(x_{ij}) \cap Q(\Gamma) \neq \emptyset$ *such that* $x_{ij} \in X$, *where* $Q(\Gamma) = \{\phi(x_{ij}) \mid x_{ij} \in \Gamma\}$. *From Table 1, we have*

$$\begin{aligned} \mathcal{Q}(\Gamma) &= \{\phi(x_{42})\} \\ &= \{(255, 230, 150)\}. \end{aligned}$$

Hence we get $\Phi^*\Gamma = \{x_{00}, x_{02}, x_{10}, x_{42}\}$. *As a result,* Γ *is an additive abelian approximately group in* $(X, \mathcal{R}_{\delta_\Phi})$ *from Definition 7.*

Also, let

$$\begin{array}{ll} X \times \Gamma \times X & \longrightarrow X \\ (x_{ij}, x_{kl}, x_{mn}) & \longmapsto x_{ij} x_{kl} x_{mn} \end{array},$$

$$x_{ij} x_{kl} x_{mn} = x_{uv} \quad, u = \min\{i, k, m\} \text{ and } v = \min\{j, l, n\}$$

be an operation on X. *In this case, for all* $a, b, c \in M$ *and* $\alpha, \beta \in \Gamma$, *since*

($\mathcal{A}\Gamma_1$) $a\alpha b \in \Phi^* M$,

($\mathcal{A}\Gamma_2$) $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$ *properties verify on* $\Phi^* M$,

($\mathcal{A}\Gamma_3$) $(a\alpha b)\beta c = a\alpha(b\beta c)$ *property verify on* $\Phi^* M$,

M *is an approximately* Γ*-ring.*

Definition 13. *Let* $(X, \mathcal{R}_{\delta_\Phi})$ *be a descriptive proximal relator space,* $M, \Gamma \subseteq X$, M *be an approximately* Γ*-ring and and* $K \subseteq M$. *If* K *additive abelian approximately group and satisfy the conditions* $(\mathcal{A}\Gamma_1 - \mathcal{A}\Gamma_3)$, K *is called an approximately* Γ*-subring of* M.

Theorem 5. *Let* $(X, \mathcal{R}_{\delta_\Phi})$ *be a descriptive proximal relator space,* $M, \Gamma \subseteq X$, M *be an approximately* Γ*-ring,* $K \subseteq M$ *and* $\Phi^* K$ *be an additive groupoid and* Γ*-groupoid. Then* K *is an approximately* Γ*-subring of* M *iff* $-k \in K$ *for all* $k \in K$.

Proof. It obvious from Theorem 1. □

Some elementary properties of elements in approximately Γ-rings are not always provided as in ordinary Γ-rings.

Lemma 3. *Let (X, δ_Φ) be a descriptive proximity space, $M \subseteq X$ be an approximately Γ-ring and $0_M \in M$ be an additive approximately identity element of M. If $0_M\alpha b, a0_\Gamma b, a\alpha 0_M \in M$, then $0_M\alpha b = a0_\Gamma b = a\alpha 0_M = 0_M$ for all $a, b \in M$ and $\alpha \in \Gamma$.*

Proof. From Definition 12 $(\mathcal{A}\Gamma_2)$,

$$\begin{aligned} 0_M\alpha b &= (0_M + 0_M)\,\alpha b \\ &= (0_M\alpha b) + (0_M\alpha b). \end{aligned}$$

Since $0_M \in M$ is unique, $0_M\alpha b = 0_M$. Similarly, $a0_\Gamma b = a\alpha 0_M = 0_M$ for all $a, b \in M$ and $\alpha \in \Gamma$. □

Equalities $a\alpha\,(-b) = (-a)\,\alpha b = -\,(a\alpha b)$ and $(-a)\,\alpha\,(-b) = a\alpha b$ are not provide in general. But, if $a\alpha b, a\alpha\,(-b)\,, (-a)\,\alpha b \in M$, then $a\alpha\,(-b) = (-a)\,\alpha b = -\,(a\alpha b)$ and $(-a)\,\alpha\,(-b) = a\alpha b$ for all $a, b \in M$ and $\alpha \in \Gamma$.

Definition 14. *A subset U of the approximately Γ-ring M is a left (right) approximately Γ-ideal of M if U is an additive approximately subgroup of M and*

$$M\Gamma U = \{a\alpha u : a \in M, \alpha \in \Gamma, u \in U\} \subseteq \Phi^* U \quad (U\Gamma M \subseteq \Phi^* U).$$

If U is both a left and a right approximately Γ-ideal, then U is a two-sided approximately Γ-ideal, or simply an approximately Γ-ideal of M.

Remark 1. *Every approximately Γ-ideal of M is also approximately Γ-subring of M in (X, δ_Φ).*

Let U and V are both left approximately Γ-ideals of M. Then

$$U + V = \{u + v : u \in U, v \in V\}$$

called the sum of U and V.

Lemma 4. *Let (X, δ_Φ) be a descriptive proximity space, $M \subseteq X$ be an approximately Γ-ring and $K, L \subseteq M$. If $\Phi : X \longrightarrow \mathbb{R}$ is an object descriptive homomorphism, then*

(1) *$cl_\Phi\,(k) + cl_\Phi\,(l) = cl_\Phi\,(k + l)$ for all $k \in K$ and $l \in L$,*
(2) *$\mathcal{Q}\,(K + L) = \mathcal{Q}\,(K) + \mathcal{Q}\,(L)$.*

Proof. (1) Since Φ is an object descriptive homomorphism,

$$\begin{aligned} cl_\Phi\,(k) + cl_\Phi\,(l) &= \{a \in M : \Phi\,(k) = \Phi\,(a)\} + \{b \in M : \Phi\,(l) = \Phi\,(b)\} \\ &= \{a + b : \Phi\,(k) = \Phi\,(a)\,, \Phi\,(l) = \Phi\,(b)\} \\ &= \{a + b : \Phi\,(k) + \Phi\,(l) = \Phi\,(a) + \Phi\,(b)\} \\ &= \{a + b : \Phi\,(k + l) = \Phi\,(a + b)\} \\ &= \{c : \Phi\,(k + l) = \Phi\,(c)\,, c = a + b\} \\ &= cl_\Phi\,(k + l). \end{aligned}$$

(2) Since Φ is an object descriptive homomorphism,

$$\begin{aligned}\mathcal{Q}(K+L) &= \{\Phi(k+l) : k \in K, l \in L\}\\ &= \{\Phi(k)+\Phi(l) : k \in K, l \in L\}\\ &= \{\Phi(k) : k \in K\} + \{\Phi(l) : l \in L\}\\ &= \mathcal{Q}(K)+\mathcal{Q}(L).\end{aligned}$$ □

Lemma 5. *Let (X, δ_Φ) be a descriptive proximity space, $M \subseteq X$ be an approximately Γ-ring and $K, L \subseteq M$. If $\Phi : X \longrightarrow \mathbb{R}$ is an object descriptive monomorphism, then $(\Phi^* K) + (\Phi^* L) = \Phi^*(K+L)$.*

Theorem 6. *Let (X, δ_Φ) be a descriptive proximity space, $M \subseteq X$ be an approximately Γ-ring and $K, L \subseteq M$. If $\Phi : X \longrightarrow \mathbb{R}$ is an object descriptive homomorphism, then*

(1) $(\Phi_* K) + (\Phi_* L) \subseteq \Phi_*(K+L)$,
(2) $(\Phi^* K) + (\Phi^* L) \subseteq \Phi^*(K+L)$.

Proof. (1) Let $x \in (\Phi_* K) + (\Phi_* L)$. In this case, $x = k + l$ for some $k \in \Phi_* K$, $l \in \Phi_* K$. Then $cl_\Phi(k) \subseteq K$ and $cl_\Phi(l) \subseteq L$. From Lemma 4 (1), $cl_\Phi(k) + cl_\Phi(l) = cl_\Phi(k+l) \subseteq K + L$. Thus $x = k + l \in \Phi_*(K+L)$. Therefore $(\Phi_* K) + (\Phi_* L) \subseteq \Phi_*(K+L)$.

(2) Let $x \in (\Phi^* K) + (\Phi^* L)$. In this case, $x = k + l$ for some $k \in \Phi^* K$, $l \in \Phi^* L$. Then $\Phi(k) \in Q(K)$ and $\Phi(l) \in Q(L)$. Hence $\Phi(k) + \Phi(l) \in \mathcal{Q}(K) + \mathcal{Q}(L)$ and from Lemma 4 (2) $\Phi(k+l) \in \mathcal{Q}(K+L)$. Thus $x = k + l \in \Phi^*(K+L)$. Consequently, $(\Phi^* K) + (\Phi^* L) \subseteq \Phi^*(K+L)$. □

Theorem 7. *Let (X, δ_Φ) be a descriptive proximity space, $M \subseteq X$ be an approximately Γ-ring and $U, V \subseteq M$. If U, V are both left (resp. right, two-sided) approximately Γ-ideals of M and $\Phi : X \longrightarrow \mathbb{R}$ is an object descriptive homomorphism, then $U + V$ is also a left (resp. right, two-sided) approximately Γ-ideal of M.*

Proof. Since U and V are both left approximately Γ-ideals of M, $M\Gamma U \subseteq \Phi^* U$ and $M\Gamma V \subseteq \Phi^* V$. Then, from Theorem 6 (2)

$$\begin{aligned}M\Gamma(U+V) &= \{a\alpha(u+v) : a \in M, \alpha \in \Gamma, u \in U, v \in V\}\\ &= \{a\alpha u + a\alpha v : a \in M, \alpha \in \Gamma, u \in U, v \in V\}\\ &= \{a\alpha u : a \in M, \alpha \in \Gamma, u \in U\} + \{a\alpha v : a \in M, \alpha \in \Gamma, v \in V\}\\ &= M\Gamma U + M\Gamma V\\ &\subseteq (\Phi^* U) + (\Phi^* V)\\ &\subseteq \Phi^*(U+V).\end{aligned}$$

Therefore $M\Gamma(U+V) \subseteq \Phi^*(U+V)$, that is, $U + V$ is a left approximately Γ-ideal of M. The other cases can be seen in a similar way. □

Corollary 1. *Let (X, δ_Φ) be a descriptive proximity space, $M \subseteq X$ be an approximately Γ-ring and $U_i \subseteq M$ $(1 \leq i \leq n,\ n \geqslant 2)$. If U_i are left (resp. right, two-sided) approximately Γ-ideals of M, $\Phi : X \longrightarrow \mathbb{R}$ is an object descriptive homomorphism* *and Φ^*U_i are additive groupoids and Φ^*U_i are Γ-groupoids, then $\sum\limits_{1\leq i\leq n} U_i$ is also a left (resp. right, two-sided) approximately Γ-ideal of M.*

Theorem 8. *Let (X, δ_Φ) be a descriptive proximity space, $M \subseteq X$ be an approximately Γ-ring and $U, V \subseteq M$. If U, V are both left (resp. right, two-sided) approximately Γ-ideals of M and $(\Phi^*U) \cap (\Phi^*V) = \Phi^*(U \cap V)$, then $U \cap V$ is also a left (resp. right, two-sided) approximately Γ-ideal of M.*

Proof. Since U and V are both left approximately Γ-ideals of M, $M\Gamma U \subseteq \Phi^*U$ and $M\Gamma V \subseteq \Phi^*V$,

$$\begin{aligned} M\Gamma(U\cap V) &= \{a\alpha x : a \in M, \alpha \in \Gamma, x \in U \cap V\} \\ &= \{a\alpha x : a \in M, \alpha \in \Gamma, x \in U \text{ and } x \in V\} \\ &= \{a\alpha x : a \in M, \alpha \in \Gamma, x \in U\} \cap \{a\alpha x : a \in M, \alpha \in \Gamma, x \in V\} \\ &= M\Gamma U \cap M\Gamma V \\ &\subseteq (\Phi^*U) \cap (\Phi^*V) \\ &= \Phi^*(U \cap V). \end{aligned}$$

Therefore $M\Gamma(U \cap V) \subseteq \Phi^*(U \cap V)$, that is, $U \cap V$ is a left approximately Γ-ideal of M. The other cases can be seen in a similar way. □

Corollary 2. *Let (X, δ_Φ) be a descriptive proximity space, $M \subseteq X$ be an approximately Γ-ring and $U_i \subseteq M$ ($1 \leq i \leq n$, $n \geqslant 2$). If U_i are left (resp. right, two-sided) approximately Γ-ideals of M and $\bigcap\limits_{1\leq i\leq n} \Phi^*U_i = \Phi^*\left(\bigcap\limits_{1\leq i\leq n} U_i\right)$, then $\bigcap\limits_{1\leq i\leq n} U_i$ is also a left (resp. right, two-sided) approximately Γ-ideal of M.*

Let M be an approximately Γ-ring and K an approximately Γ-subring of M. The left compatible (weak equivalence) relation "ω_l" defined as

$$a\omega_l b :\Leftrightarrow (-a) + b \in K \cup \{e\}$$

for $a, b \in M$.

A weak class defined by relation "ω_l" is called approximately left coset. The approximately left coset that contains the element $a \in M$ is denoted by $\tilde{a}_l$, that is,

$$\tilde{a}_l = \{a + k \mid k \in K,\ a \in M,\ a + k \in M\} \cup \{a\}.$$

Similarly, we can define the approximately right coset that contains the element $a \in M$ is denoted by $\tilde{a}_r$, that is,

$$\tilde{a}_r = \{k + a \mid k \in K,\ a \in M,\ k + a \in M\} \cup \{a\}.$$

We can easily show that $\tilde{a}_l = a + K$ and $\tilde{a}_r = K + a$. Since $(M, +)$ is an abelian approximately group, $\tilde{a}_l = \tilde{a}_r$ and so we use only notation $\tilde{a}$. Then

$$M/_\omega = \{a + K \mid a \in M\}$$

is a set of all approximately cosets of M by K. In this case, if we consider Φ^*M instead of approximately Γ-ring M

$$(\Phi^*M)/_\omega = \{a + K \mid a \in \Phi^*M\}.$$

Definition 15. *Let (X, δ_Φ) be a descriptive proximity space , $M \subseteq X$ be an approximately Γ-ring and K an approximately Γ-subring of M. For $a, b \in M$, let $a+K$ and $b+K$ be two approximately cosets that determined the elements a and b, respectively. Then sum of two approximately cosets that determined by $a+b \in \Phi^*M$ can be defined as*

$$(a+b)+K = \{(a+b)+k \mid k \in K,\ a+b \in \Phi^*M,\ (a+b)+k \in M\} \cup \{a+b\}$$

and denoted by

$$(a+K) \oplus (b+K) = (a+b)+K.$$

Definition 16. *Let (X, δ_Φ) be a descriptive proximity space , $M \subseteq X$ be an approximately Γ-ring and K an approximately Γ-subring of M. For $a, b \in M$, let $a+K$ and $b+K$ be two approximately cosets that determined the elements a and b, respectively. Then product of two approximately cosets that determined by $a\alpha b \in \Phi^*M$ can be defined as*

$$(a\alpha b)+K = \{(a\alpha b)+k \mid k \in K,\ a\alpha b \in \Phi^*M,\ (a\alpha b)+k \in M\} \cup \{a\alpha b\}$$

and denoted by

$$(a+K)\,\alpha\,(b+K) = (a\alpha b)+K.$$

Theorem 9. *Let $M \subseteq X$ be an approximately Γ-ring, K an approximately Γ-subring of M and $M/_\omega$ be a set of all approximately cosets of M by K. If $(\Phi^*M)/_\omega \subseteq \Phi^*(M/_\omega)$, then $M/_\omega$ is an approximately Γ-ring under the operations given by $(a+K) \oplus (b+K) = (a+b)+K$ and $(a+K)\,\alpha\,(b+K) = (a\alpha b)+K$ for all $a, b \in M$ and $\alpha \in \Gamma$.*

Proof. Let $(\Phi^*M)/_\omega \subseteq \Phi^*(M/_\omega)$. Since M is an approximately Γ-ring and Theorem 4, $(M/_\omega, \oplus)$ is a abelian approximately group of all approximately cosets of M by K. Furthermore,

$(\mathcal{A}\Gamma_1)$ Since M is an approximately Γ-ring, $a\alpha b \in \Phi^*M$ and then $(a+K)\,\alpha\,(b+K) = (a\alpha b)+K \in (\Phi^*M)/_\omega$. From the hypothesis, $(a+K)\,\alpha\,(b+K) \in \Phi^*(M/_\omega)$.

$(\mathcal{A}\Gamma_2)$ For all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, distributive properties holds in Φ^*M. From the Definitions 15 and 16 for all $(a+K), (b+K), (c+K) \in M/_\omega$,

$$\begin{aligned} & ((a+K) \oplus (b+K))\,\alpha\,(c+K) \\ = \ & ((a+b)+K)\,\alpha\,(c+K) \\ = \ & ((a+b)\,\alpha c)+K \end{aligned}$$

and

$$\begin{aligned} & ((a+K)\,\alpha\,(c+K)) \oplus ((b+K)\,\alpha\,(c+K)) \\ = \ & ((a\alpha c)+K) \oplus ((b\alpha c)+K) \\ = \ & ((a\alpha c)+(b\alpha c))+K \\ = \ & ((a+b)\,\alpha c)+K, \end{aligned}$$

where $((a+b)\,\alpha c)+K \in (\Phi^* M)\,/_\omega$. Thus $((a+K) \oplus (b+K))\,\alpha\,(c+K) = ((a+K)\,\alpha\,(c+K)) \oplus ((b+K)\,\alpha\,(c+K))$ holds in $(\Phi^* M)\,/_\omega$. From the hypothesis, right distributive law holds in $\Phi^*\,(M/_\omega)$. Similarly, we can show that

$$\begin{aligned} & (a+K)\,(\alpha+\beta)\,(b+K) \\ = \; & ((a+K)\,\alpha\,(b+K)) \oplus ((a+K)\,\beta\,(b+K)) \end{aligned}$$

and

$$\begin{aligned} & (a+K)\,\alpha((b+K) \oplus (c+K)) \\ = \; & ((a+K)\,\alpha\,(b+K)) \oplus ((a+K)\,\alpha\,(c+K)) \end{aligned}$$

properties hold in $\Phi^*\,(M/_\omega)$ for all $(a+K)\,,(b+K)\,,(c+K) \in M/_\omega$.

$(\mathcal{A}\Gamma_3)$ For all $a,b,c \in M$ and $\alpha,\beta \in \Gamma$, associative property holds in $\Phi^* M$. Then

$$\begin{aligned} & ((a+K)\,\alpha\,(b+K))\,\beta\,(c+K) \\ = \; & ((a\alpha b)+K)\,\beta\,(c+K) \\ = \; & ((a\alpha b)\,\beta c)+K \end{aligned}$$

and

$$\begin{aligned} & (a+K)\,\alpha\,((b+K)\,\beta\,(c+K)) \\ = \; & (a+K)\,\alpha\,((b\beta c)+K) \\ = \; & (a\alpha\,(b\beta c))+K \\ = \; & ((a\alpha b)\,\beta c)+K \end{aligned}$$

where $((a\alpha b)\,\beta c)+K \in (\Phi^* M)\,/_\omega$. Thus

$((a+K)\,\alpha\,(b+K))\,\beta\,(c+K) = (a+K)\,\alpha\,((b+K)\,\beta\,(c+K))$ holds in $(\Phi^* M)\,/_\omega$. From the hypothesis, associative property holds in $\Phi^*\,(M/_\omega)$.

Consequently, $M/_\omega$ is an approximately Γ-ring. □

Definition 17. *Let $M \subseteq X$ be an approximately Γ-ring and K an approximately Γ-subring of M. The approximately ring $M/_\omega$ is called an approximately Γ-ring of all approximately cosets of M by K or shortly approximately quotient Γ-ring and denoted by $M/_\omega K$.*

Definition 18. *Let (X,δ_Φ) be a descriptive proximity space, $M,N \subseteq X$ be approximately Γ-rings and Θ be a mapping from $\Phi^* M$ into $\Phi^* N$ such that $\Phi^* M, \Phi^* N$ be additive groupoids and Γ-groupoids. If $\Theta\,(a+b) = \Theta\,(a) + \Theta\,(b)$ and $\Theta\,(a\alpha b) = \Theta\,(a)\,\alpha\Theta\,(b)$ for all $a,b \in M$ and $\alpha \in \Gamma$, then Θ is called an approximately Γ-homomorphism.*

An approximately Γ-homomorphism Θ from $\Phi^* M$ into $\Phi^* N$ is called

(i) an approximately Γ-monomorphism if Θ is injective,
(ii) an approximately Γ-epimorphism if Θ is surjective,
(iii) an approximately Γ-isomorphism if Θ is bijective.

Also, M is called approximately Γ-homomorphic to N, denoted by $M \simeq_\Gamma N$, if Θ is an approximately Γ-epimorphism.

Theorem 10. *Let (X, δ_Φ) be a descriptive proximity space, $M, N \subseteq X$ be approximately Γ-rings and Θ an approximately Γ-homomorphism from Φ^*M into Φ^*N. Then the following properties hold:*

(1) $\Theta(0_M) = 0_N$, *where $0_N \in \Phi^*N$ is the additive approximately identity element of N.*
(2) $\Theta(-a) = -\Theta(a)$ *for all $a \in M$.*

Proof. (1) Since Θ is an approximately Γ-homomorphism, $\Theta(0_M) + \Theta(0_M) = \Theta(0_M + 0_M) = \Theta(0_M) = \Theta(0_M) + 0_N$. Thus we have that $\Theta(0_M) = 0_N$ by the Theorem 3.

(2) Let $a \in M$. Then $\Theta(a) + \Theta(-a) = \Theta(a + (-a)) = \Theta(0_M) = 0_N$ by (1). Similarly, we can obtain that $\Theta(-a) + \Theta(a) = 0_N$ for all $a \in M$. Since $\Theta(a)$ has a unique inverse from Theorem 2 (2), $\Theta(-a) = -\Theta(a)$ for all $a \in M$. $\square$

Theorem 11. *Let (X, δ_Φ) be a descriptive proximity space, $M, N \subseteq X$ be approximately Γ-rings and Θ an approximately Γ-homomorphism from Φ^*M into Φ^*N, $K \subseteq M$ and Φ^*K be an additive groupoid. If K is an (commutative) approximately Γ-subring of M and $\Theta(\Phi^*K) = \Phi^*\Theta(K)$, then $\Theta(K) = \{\Theta(k) : k \in K\}$ is an (commutative) approximately Γ-subring of N.*

Proof. Let K be an approximately Γ-subring of M. Then $0_K \in \Phi^*K$ and by Theorem 10 (1), $\Theta(0_K) = 0_N$, where $0_N \in \Phi^*N$. Thus $0_N = \Theta(0_K) \in \Theta(\Phi^*K) = \Phi^*\Theta(K)$. This means that $\Theta(K) \neq \emptyset$. Let $\Theta(k) \in \Theta(K)$, where $k \in K$. Since K is an approximately Γ-subring of M, $-k \in K$ for all $k \in K$. Thus from Theorem 10 (2) $-\Theta(k) = \Theta(-k) \in \Theta(K)$ for all $\Theta(k) \in \Theta(K)$. Hence by Theorem 5, $\Theta(K)$ is an approximately Γ-subring of N.

Let K be a commutative approximately Γ-subring. Thus $\Theta(k)\alpha\Theta(l) = \Theta(k\alpha l) = \Theta(l\alpha k) = \Theta(l)\alpha\Theta(k)$ for all $\Theta(k), \Theta(l) \in \Theta(K)$ and $\alpha \in \Gamma$. Hence $\Theta(K)$ is a commutative approximately Γ-subring of N. $\square$

Definition 19. *Let $M, N \subseteq X$ be approximately Γ-rings in (X, δ_Φ) and Θ be an approximately Γ-homomorphism from Φ^*M into Φ^*N. The kernel of Θ, denoted by $Ker\Theta$, is defined to be the set*

$$Ker\Theta = \{x \in M : \Theta(x) = 0_N\}.$$

Theorem 12. *Let $M, N \subseteq X$ be approximately Γ-rings in (X, δ_Φ), Θ be an approximately homomorphism from Φ^*M into Φ^*N and $\Phi^*Ker\Theta$ additive groupoid and Γ-groupoid. Then $Ker\Theta$ is an approximately Γ-ideal of M.*

Proof. Let $x \in Ker\Theta$. Since $\Theta(-x) = -\Theta(x) = -0_N = 0_N$, $-x \in Ker\Theta$ Hence by Theorem 1, $Ker\Theta$ is an additive approximately subgroup of M.

Let $z \in M\Gamma(Ker\Theta)$. Then $z = a\alpha x$ where $a \in M, \alpha \in \Gamma, x \in Ker\Theta$. $\Theta(z) = \Theta(a\alpha x) = \Theta(a)\alpha\Theta(x) = \Theta(a)\alpha 0_N = 0_N$ by Lemma 3. Hence $z \in Ker\Theta$ and since $Ker\Theta \subseteq \Phi^*(Ker\Theta)$, $z \in \Phi^*(Ker\Theta)$. Therefore $M\Gamma(Ker\Theta) \subseteq \Phi^*(Ker\Theta)$ and so $Ker\Theta$ is a left approximately Γ-ideal of M. Similarly, we can show that

$(Ker\Theta)\,\Gamma M \subseteq \Phi^*(Ker\Theta)$. Hence $Ker\Theta$ is a right approximately Γideal of M. Consequently, $Ker\Theta$ is an approximately Γideal of M. □

Corollary 3. *Let $M, N \subseteq X$ be approximately Γ-rings in (X, δ_Φ), Θ be an approximately homomorphism from Φ^*M into Φ^*N and $\Phi^*Ker\Theta$ additive groupoid and Γ-groupoid. Then $Ker\Theta$ is an approximately Γ-subring of M.*

Proof. It is obvious from Remark 1. □

Theorem 13. *Let M be an approximately Γ-ring in (X, δ_Φ), K an approximately Γ-subring of M and $\Phi^*M, \Phi^*\left(M/_\omega K\right)$ additive groupoids and Γ-groupoids. Then the mapping*

$$\Pi : \Phi^*M \to \Phi^*\left(M/_\omega K\right)$$

defined by $\Pi\left(a\right) = a + K$ for all $a \in \Phi^*M$, is an approximately Γ-homomorphism.

Proof. From the definition of Π, Π is a mapping from Φ^*M into $\Phi^*\left(M/_\omega K\right)$. By using the Definitions 15 and 16,

$$\Pi\left(a+b\right) = \left(a+b\right) + K = \left(a+K\right) \oplus \left(b+K\right) = \Pi\left(a\right) \oplus \Pi\left(b\right),$$

$$\Pi\left(a\alpha b\right) = \left(a\alpha b\right) + K = \left(a+K\right)\alpha\left(b+K\right) = \Pi\left(a\right)\alpha\Pi\left(b\right)$$

for all $a, b \in M$ and $\alpha \in \Gamma$. Thus Π is an approximately Γ-homomorphism from Definition 18. □

Definition 20. *In the above theorem, the approximately Γ-homomorphism Π is called an approximately natural Γ-homomorphism from Φ^*M into $\Phi^*\left(M/_\omega K\right)$.*

Definition 21. *Let $M, N \subseteq X$ be approximately Γ-rings in (X, δ_Φ), K be a non-empty subset of M and $\Phi^*M, \Phi^*K, \Phi^*N$ additive groupoids and Γ-groupoids. Let*

$$\tau : \Phi^*M \longrightarrow \Phi^*N$$

be a mapping and

$$\tau_K = \tau|_K : K \longrightarrow \Phi^*N$$

a restricted mapping. If $\tau\left(a+b\right) = \tau_{_K}\left(a+b\right) = \tau_{_K}\left(a\right) + \tau_{_K}\left(b\right) = \tau\left(a\right) + \tau\left(b\right)$ and $\tau\left(a\alpha b\right) = \tau_{_K}\left(a\alpha b\right) = \tau_{_K}\left(a\right)\alpha\tau_{_K}\left(b\right) = \tau\left(a\right)\alpha\tau\left(b\right)$ for all $a, b \in K$ and $\alpha \in \Gamma$, then τ is called a restricted approximately Γ-homomorphism and also if τ is surjective, then M is called restricted approximately Γ-homomorphic to N, denoted by $M \simeq_r N$.

Theorem 14. *Let $M, N \subseteq X$ be approximately Γ-rings in (X, δ_Φ) and τ be an approximately Γ-homomorphism from Φ^*M into Φ^*N . Let $\Phi^*Ker\tau$ be additive groupoid and Γ-groupoid, and $\left(\Phi^*M\right)/_\omega$ be a set of all approximately cosets of Φ^*M by $Ker\tau$. If $\left(\Phi^*M\right)/_\omega \subseteq \Phi^*\left(M/_\omega Ker\tau\right)$ and $\Phi^*\tau\left(M\right) = \tau\left(\Phi^*M\right)$, then*

$$M/_\omega Ker\tau \simeq_r \tau\left(M\right).$$

Proof. Since $\Phi^* Ker\tau$ be additive groupoid and Γ-groupoid, from Corollary 3 $Ker\tau$ is an approximately Γ-subring of M. Since $Ker\tau$ is an approximately Γ-subring of M and $(\Phi^* M)/_\omega \subseteq \Phi^*(M/_\omega Ker\tau)$, then $M/_\omega Ker\tau$ is an approximately Γ-ring of all approximately cosets of M by $Ker\tau$ from Theorem 9. Since $\Phi^*\tau(M) = \tau(\Phi^* M)$, $\tau(M)$ is an approximately Γ-subring of N by Theorem 11. Define

$$\begin{array}{rcl} \mu : \Phi^*(M/_\omega Ker\tau) & \longrightarrow & \Phi^*\tau(M) \\ K & \longmapsto & \mu(K) = \begin{cases} \mu_{M/_\omega Ker\tau}(K) & , K \in (\Phi^* M)/_\omega \\ e_{\tau(M)} & , K \notin (\Phi^* M)/_\omega \end{cases} \end{array}$$

where

$$\mu_{M/_\omega Ker\tau} = \mu|_{M/_\omega Ker\tau} : M/_\omega Ker\tau \longmapsto \Phi^*\tau(M)\, a+Ker\tau \longmapsto \mu|_{M/_\omega Ker\tau}(a+Ker\tau) = \tau(a)$$

for all $a + Ker\tau \in M/_\omega Ker\tau$.

Since

$$\begin{aligned} a + Ker\tau &= \{a + x \mid x \in Ker\tau, a + x \in M\} \cup \{a\}, \\ b + Ker\tau &= \{b + y \mid y \in Ker\tau, b + y \in M\} \cup \{b\} \end{aligned}$$

and the mapping τ is an approximately Γ-homomorphism,

$$\begin{array}{rl} & a + Ker\tau = b + Ker\tau \\ \Rightarrow & a \in b + Ker\tau \\ \Rightarrow & a \in \{b + y \mid y \in Ker\tau, b + y \in M\} \text{ or } a \in \{b\} \\ \Rightarrow & a = b + y \ (y \in Ker\tau, b + y \in M) \text{ or } a = b \\ \Rightarrow & (-b) + a = ((-b) + b) + y \ (y \in Ker\tau) \text{ or } \tau(a) = \tau(b) \\ \Rightarrow & (-b) + a = y \ (y \in Ker\tau) \\ \Rightarrow & (-b) + a \in Ker\tau \\ \Rightarrow & \tau((-b) + a) = e_{\tau(M)} \\ \Rightarrow & \tau(-b) + \tau(a) = e_{\tau(M)} \\ \Rightarrow & -\tau(b) + \tau(a) = e_{\tau(M)} \\ \Rightarrow & \tau(a) = \tau(b). \end{array}$$

Therefore $\mu_{M/_\omega Ker\tau}$ is well defined.

For $K, L \in \Phi^*(M/_\omega Ker\tau)$, lets assume that $K = L$. Since the mapping $\mu_{M/_\omega Ker\tau}$ is well defined,

$$\begin{aligned} \mu(K) &= \begin{cases} \mu_{M/_\omega Ker\tau}(K) & , K \in (\Phi^* M)/_\omega \\ e_{\tau(M)} & , K \notin (\Phi^* M)/_\omega \end{cases} \\ &= \begin{cases} \mu_{M/_\omega Ker\tau}(L) & , L \in (\Phi^* M)/_\omega \\ e_{\tau(M)} & , L \notin (\Phi^* M)/_\omega \end{cases} \\ &= \mu(L). \end{aligned}$$

Consequently, μ is well defined.

For all $a + Ker\tau, b + Ker\tau \in M/_{\omega}Ker\tau \subseteq \Phi^* (M/_{\omega}Ker\tau)$,

$$\begin{aligned}
& \mu ((a + Ker\tau) \oplus (b + Ker\tau)) \\
= \ & \mu_{M/_{\omega}Ker\tau} ((a + Ker\tau) \oplus (b + Ker\tau)) \\
= \ & \mu_{M/_{\omega}Ker\tau} ((a + b) + Ker\tau) \\
= \ & \tau (a + b) \\
= \ & \tau (a) + \tau (b) \\
= \ & \mu_{M/_{\omega}Ker\tau} (a + Ker\tau) + \mu_{M/_{\omega}Ker\tau} (b + Ker\tau) \\
= \ & \mu (a + Ker\tau) + \mu (b + Ker\tau)
\end{aligned}$$

and

$$\begin{aligned}
& \mu ((a + Ker\tau) \alpha (b + Ker\tau)) \\
= \ & \mu_{M/_{\omega}Ker\tau} ((a + Ker\tau) \alpha (b + Ker\tau)) \\
= \ & \mu_{M/_{\omega}Ker\tau} ((a\alpha b) + Ker\tau) \\
= \ & \tau (a\alpha b) \\
= \ & \tau (a) \alpha\tau (b) \\
= \ & \mu_{M/_{\omega}Ker\tau} (a + Ker\tau) \alpha\mu_{M/_{\omega}Ker\tau} (b + Ker\tau) \\
= \ & \mu (a + Ker\tau) \alpha\mu (b + Ker\tau).
\end{aligned}$$

Therefore μ is a restricted approximately Γ-homomorphism by Definition 18. Hence $M/_{\omega}Ker\tau \simeq_r \tau (M)$. □

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