



Some Additive Inequalities for Heinz Operator Mean

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Abstract

In this paper we obtain some new additive inequalities for Heinz operator mean, namely the operator $H_\nu(A, B) := \frac{1}{2}(A\sharp_\nu B + A\sharp_{1-\nu}B)$ where $A\sharp_\nu B := A^{1/2}(A^{-1/2}BA^{-1/2})^\nu A^{1/2}$ is the weighted geometric mean for the positive invertible operators A and B , and $\nu \in [0, 1]$.

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1. Introduction

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators and $\nu \in [0, 1]$

$$A\nabla_\nu B := (1 - \nu)A + \nu B,$$

the weighted operator arithmetic mean, and

$$A\sharp_\nu B := A^{1/2}(A^{-1/2}BA^{-1/2})^\nu A^{1/2},$$

the weighted operator geometric mean [14]. When $\nu = \frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

Define the Heinz operator mean by

$$H_\nu(A, B) := \frac{1}{2}(A\sharp_\nu B + A\sharp_{1-\nu}B).$$

The following interpolatory inequality is obvious

$$A\sharp B \leq H_\nu(A, B) \leq A\nabla B \tag{1.1}$$

for any $\nu \in [0, 1]$.

We recall that Specht's ratio is defined by [16]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h \in (0, 1) \cup (1, \infty), \\ 1 & \text{if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h \rightarrow 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0, 1)$ and increasing on $(1, \infty)$. The following result provides an upper and lower bound for the Heinz mean in terms of the operator geometric mean $A\sharp B$:

Theorem 1.1 (Dragomir, 2015 [6]). Assume that A and B are positive invertible operators and the constants $M > m > 0$ are such that

$$mA \leq B \leq MA. \tag{1.3}$$

Then we have

$$\omega_\nu(m, M)A\sharp B \leq H_\nu(A, B) \leq \Omega_\nu(m, M)A\sharp B, \tag{1.4}$$

where

$$\Omega_v(m, M) := \begin{cases} S(m^{2v-1}) & \text{if } M < 1, \\ \max \{S(m^{2v-1}), S(M^{2v-1})\} & \text{if } m \leq 1 \leq M, \\ S(M^{2v-1}) & \text{if } 1 < m \end{cases} \quad (1.5)$$

and

$$\omega_v(m, M) := \begin{cases} S(M^{v-\frac{1}{2}}) & \text{if } M < 1, \\ 1 & \text{if } m \leq 1 \leq M, \\ S(m^{v-\frac{1}{2}}) & \text{if } 1 < m, \end{cases} \quad (1.6)$$

where $v \in [0, 1]$.

We consider the Kantorovich's constant defined by

$$K(h) := \frac{(h+1)^2}{4h}, \quad h > 0. \quad (1.7)$$

The function K is decreasing on $(0, 1)$ and increasing on $[1, \infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K(\frac{1}{h})$ for any $h > 0$.

We have:

Theorem 1.2 (Dragomir, 2015 [7]). Assume that A and B are positive invertible operators and the constants $M > m > 0$ are such that the condition (1.3) is valid. Then for any $v \in [0, 1]$ we have

$$(A \sharp B \leq) H_v(A, B) \leq \exp[\Theta_v(m, M) - 1] A \sharp B \quad (1.8)$$

where

$$\Theta_v(m, M) := \begin{cases} K(m^{2v-1}) & \text{if } M < 1, \\ \max \{K(m^{2v-1}), K(M^{2v-1})\} & \text{if } m \leq 1 \leq M, \\ K(M^{2v-1}) & \text{if } 1 < m \end{cases} \quad (1.9)$$

and

$$(0 \leq) H_v(A, B) - A \sharp B \leq \frac{1}{4m^{1-v}} \max_{x \in [m, M]} D(x^{2v-1}) A, \quad (1.10)$$

where the function $D: (0, \infty) \rightarrow [0, \infty)$ is defined by $D(x) = (x-1) \ln x$.

The following bounds for the Heinz mean $H_v(A, B)$ in terms of $A \nabla B$ are also valid:

Theorem 1.3 (Dragomir, 2015 [7]). With the assumptions of Theorem 1.2 we have

$$(0 \leq) A \nabla B - H_v(A, B) \leq v(1-v) \Upsilon(m, M) A, \quad (1.11)$$

where

$$\Upsilon(m, M) := \begin{cases} (m-1) \ln m & \text{if } M < 1, \\ \max \{(m-1) \ln m, (M-1) \ln M\} & \text{if } m \leq 1 \leq M, \\ (M-1) \ln M & \text{if } 1 < m \end{cases} \quad (1.12)$$

and

$$A \nabla B \exp[-4v(1-v)(F(m, M) - 1)] \leq H_v(A, B) (\leq A \nabla B) \quad (1.13)$$

where

$$F(m, M) := \begin{cases} K(m) & \text{if } M < 1, \\ \max \{K(m), K(M)\} & \text{if } m \leq 1 \leq M, \\ K(M) & \text{if } 1 < m. \end{cases} \quad (1.14)$$

For other recent results on operator geometric mean inequalities, see [1]-[13], [15] and [17]-[18].

Motivated by the above results, we establish in this paper some inequalities for the quantities

$$H_v(A, B) - A \sharp B \quad \text{and} \quad A \nabla B - H_v(A, B)$$

under various assumptions for positive invertible operators A and B .

2. Bounds for $H_\nu(A, B) - A\sharp B$

We first notice the following simple result:

Theorem 2.1. Assume that A and B are positive invertible operators and the constants $M > m > 0$ are such that the condition (1.3) holds. If we consider the function $f_\nu : [0, \infty) \rightarrow \mathbb{R}$ for $\nu \in [0, 1]$ defined by

$$f_\nu(x) = \frac{1}{2} (x^\nu + x^{1-\nu}),$$

then we have

$$f_\nu(m)A \leq H_\nu(A, B) \leq f_\nu(M)A. \quad (2.1)$$

Proof. We observe that

$$f'_\nu(x) = \frac{1}{2} (\nu x^{\nu-1} + (1-\nu)x^{-\nu}),$$

which is positive for $x \in (0, \infty)$.

Therefore f_ν is increasing on $(0, \infty)$ and

$$f_\nu(m) = \min_{x \in [m, M]} f_\nu(x) \leq f_\nu(x) \leq \max_{x \in [m, M]} f_\nu(x) = f_\nu(M)$$

for any $x \in [m, M]$.

Using the continuous functional calculus, we have for any operator X with $mI \leq X \leq MI$ that

$$f_\nu(m)I \leq \frac{1}{2} (X^\nu + X^{1-\nu}) \leq f_\nu(M)I. \quad (2.2)$$

From (1.3) we have, by multiplying both sides with $A^{-1/2}$ that

$$mI \leq A^{-1/2}BA^{-1/2} \leq MI.$$

Now, writing the inequality (2.2) for $X = A^{-1/2}BA^{-1/2}$, we get

$$f_\nu(m)I \leq \frac{1}{2} \left[(A^{-1/2}BA^{-1/2})^\nu + (A^{-1/2}BA^{-1/2})^{1-\nu} \right] \leq f_\nu(M)I. \quad (2.3)$$

Finally, if we multiply both sides of (2.3) by $A^{1/2}$ we get the desired result (2.1). \square

Corollary 2.2. Let A and B be two positive operators. For positive real numbers m, m', M, M' , put $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ and let $\nu \in [0, 1]$.

(i) If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, then

$$f_\nu(h')A \leq H_\nu(A, B) \leq f_\nu(h)A. \quad (2.4)$$

(ii) If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then

$$\frac{f_\nu(h)}{h}A \leq H_\nu(A, B) \leq \frac{f_\nu(h')}{h'}A. \quad (2.5)$$

Proof. If the condition (i) is valid, then we have for $X = A^{-1/2}BA^{-1/2}$

$$I < \frac{M'}{m'}I = h'I \leq X \leq hI = \frac{M}{m}I,$$

which, by (2.2) gives the desired result (2.4).

If the condition (ii) is valid, then we have

$$0 < \frac{1}{h}I \leq X \leq \frac{1}{h'}I < I,$$

which, by (2.2) gives

$$f_\nu\left(\frac{1}{h}\right)A \leq H_\nu(A, B) \leq f_\nu\left(\frac{1}{h'}\right)A$$

that is equivalent to (2.5), since

$$f_\nu\left(\frac{1}{h}\right) = \frac{f_\nu(h)}{h}.$$

\square

We need the following lemma in order to prove our first main result:

Lemma 2.3. Consider the function $g_v : [0, \infty) \rightarrow \mathbb{R}$ for $v \in (0, 1)$ defined by

$$g_v(x) = \frac{1}{2} (x^v + x^{1-v}) - \sqrt{x} \geq 0. \tag{2.6}$$

Then $g_v(0) = g_v(1) = 0$, g_v is increasing on $(0, x_v)$ with a local maximum in

$$x_v := \left(\frac{v}{1-v} \right)^{\frac{2}{1-2v}} \in (0, 1), \tag{2.7}$$

is decreasing on $(x_v, 1)$ with a local minimum in $x = 1$ and increasing on $(1, \infty)$ with $\lim_{x \rightarrow \infty} g_v(x) = \infty$.

Proof. (i). If $v \in (0, \frac{1}{2})$, then

$$\begin{aligned} g'_v(x) &= \frac{1}{2} \left(\frac{v}{x^{1-v}} + \frac{1-v}{x^v} - \frac{1}{x^{1/2}} \right) \\ &= \frac{1}{2} \frac{v + (1-v)x^{1-2v} - x^{\frac{1-2v}{2}}}{x^{1-v}}. \end{aligned}$$

If we denote $u = x^{\frac{1-2v}{2}}$, then we have

$$\begin{aligned} v + (1-v)x^{1-2v} - x^{\frac{1-2v}{2}} &= (1-v)u^2 - u + v \\ &= (1-v) \left(u - \frac{v}{1-v} \right) (u-1) \\ &= (1-v) \left(x^{\frac{1-2v}{2}} - \frac{v}{1-v} \right) \left(x^{\frac{1-2v}{2}} - 1 \right). \end{aligned}$$

We observe that $g'_v(x) = 0$ only for $x = 1$ and $x_v = \left(\frac{v}{1-v} \right)^{\frac{2}{1-2v}} \in (0, 1)$. Also $g'_v(x) > 0$ for $x \in (0, x_v) \cup (1, \infty)$ and $g'_v(x) < 0$ for $x \in (x_v, 1)$. These imply the desired conclusion.

(ii) If $v \in (\frac{1}{2}, 1)$, then

$$g'_v(x) = \frac{1}{2} \frac{1-v + vx^{2v-1} - x^{\frac{2v-1}{2}}}{x^v}.$$

If we denote $z = x^{\frac{2v-1}{2}}$, then we have

$$\begin{aligned} 1-v + vx^{2v-1} - x^{\frac{2v-1}{2}} &= vz^2 - z + 1-v \\ &= v \left(z - \frac{1-v}{v} \right) (z-1) \\ &= v \left(x^{\frac{2v-1}{2}} - \frac{1-v}{v} \right) \left(x^{\frac{2v-1}{2}} - 1 \right). \end{aligned}$$

We observe that $g'_v(x) = 0$ only for $x = 1$ and $x_v = \left(\frac{1-v}{v} \right)^{\frac{2}{2v-1}} = \left(\frac{v}{1-v} \right)^{\frac{2}{1-2v}} \in (0, 1)$. Also $g'_v(x) > 0$ for $x \in (0, x_v) \cup (1, \infty)$ and $g'_v(x) < 0$ for $x \in (x_v, 1)$. These imply the desired conclusion. \square

The above lemma allows us to obtain various bounds for the nonnegative quantity

$$H_v(A, B) - A\sharp B$$

when some conditions for the involved operators A and B are known.

Theorem 2.4. Assume that A and B are positive invertible operators with $B \leq A$. Then for $v \in (0, 1)$ we have

$$(0 \leq) H_v(A, B) - A\sharp B \leq g_v(x_v)A, \tag{2.8}$$

where g_v is defined by (2.6) and x_v by (2.7).

Proof. From Lemma 2.3 we have for $v \in (0, 1)$ that

$$0 \leq \frac{1}{2} (x^v + x^{1-v}) - \sqrt{x} \leq g_v(x_v)$$

for any $x \in [0, 1]$.

Using the continuous functional calculus, we have for any operator X with $0 \leq X \leq I$ that

$$0 \leq \frac{1}{2} (X^v + X^{1-v}) - X^{1/2} \leq g_v(x_v) \tag{2.9}$$

for $v \in (0, 1)$.

By multiplying both sides of the inequality $0 \leq B \leq A$ with $A^{-1/2}$ we get

$$0 \leq A^{-1/2}BA^{-1/2} \leq I.$$

If we use the inequality (2.9) for $X = A^{-1/2}BA^{-1/2}$, then we get

$$0 \leq \frac{1}{2} \left[\left(A^{-1/2}BA^{-1/2} \right)^v + \left(A^{-1/2}BA^{-1/2} \right)^{1-v} \right] - \left(A^{-1/2}BA^{-1/2} \right)^{1/2} \leq g_v(x_v)I \tag{2.10}$$

for $v \in (0, 1)$.

Finally, if we multiply both sides of (2.10) with $A^{1/2}$, then we get the desired result (2.8). □

Theorem 2.5. Assume that A and B are positive invertible operators and the constants $M > m \geq 0$ are such that the condition (1.3) holds. Let $v \in (0, 1)$.

(i) If $0 \leq m < M \leq 1$, then

$$\gamma_v(m, M)A \leq H_v(A, B) - A\sharp B \leq \Gamma_v(m, M)A, \tag{2.11}$$

where

$$\gamma_v(m, M) := \begin{cases} g_v(m) & \text{if } 0 \leq m < M \leq x_v, \\ \min \{g_v(m), g_v(M)\} & \text{if } 0 \leq m \leq x_v \leq M \leq 1, \\ g_v(M) & \text{if } x_v \leq m < M \end{cases} \tag{2.12}$$

and

$$\Gamma_v(m, M) := \begin{cases} g_v(M) & \text{if } 0 \leq m < M \leq x_v, \\ g_v(x_v) & \text{if } 0 \leq m \leq x_v \leq M \leq 1, \\ g_v(m) & \text{if } x_v \leq m \leq M \leq 1, \end{cases} \tag{2.13}$$

where g_v is defined by (2.6) and x_v by (2.7).

(ii) If $1 \leq m < M < \infty$, then

$$g_v(m)A \leq H_v(A, B) - A\sharp B \leq g_v(M)A. \tag{2.14}$$

Proof. (i) If $0 \leq m < M \leq 1$ then by Lemma 2.3 we have for $v \in (0, 1)$ that

$$\begin{aligned} & \begin{cases} g_v(m) & \text{if } 0 \leq m < M \leq x_v \\ \min \{g_v(m), g_v(M)\} & \text{if } 0 \leq m \leq x_v \leq M \leq 1 \\ g_v(M) & \text{if } x_v \leq m < M \end{cases} \\ & \leq g_v(x) \\ & \leq \begin{cases} g_v(M) & \text{if } 0 \leq m < M \leq x_v \\ g_v(x_v) & \text{if } 0 \leq m \leq x_v \leq M \leq 1 \\ g_v(m) & \text{if } x_v \leq m < M \leq 1 \end{cases} \end{aligned}$$

for any $x \in [m, M]$.

Now, on making use of a similar argument to the one in the proof of Theorem 2.4, we obtain the desired result (2.13). □

(ii) Obvious by the properties of function g_v .

The interested reader may obtain similar bounds for other locations of $0 \leq m < M < \infty$. The details are omitted.

The following particular case holds:

Corollary 2.6. Let A and B be two positive operators. For positive real numbers m, m', M, M' , put $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ and let $v \in (0, 1)$.

(i) If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, then

$$g_v(h^I)A \leq H_v(A, B) - A\sharp B \leq g_v(h)A. \tag{2.15}$$

(ii) If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then

$$\tilde{\gamma}_v(h, h^I)A \leq H_v(A, B) - A\sharp B \leq \tilde{\Gamma}_v(h, h^I)A, \tag{2.16}$$

where

$$\tilde{\gamma}_v(h, h^I) := \begin{cases} \frac{g_v(h)}{h} & \text{if } 0 \leq \frac{1}{h} < \frac{1}{h'} \leq x_v, \\ \min \left\{ \frac{g_v(h)}{h}, \frac{g_v(h^I)}{h^I} \right\} & \text{if } 0 \leq \frac{1}{h} \leq x_v \leq \frac{1}{h'} \leq 1, \\ \frac{g_v(h^I)}{h^I} & \text{if } x_v \leq \frac{1}{h} < \frac{1}{h'} \end{cases} \tag{2.17}$$

and

$$\bar{\Gamma}_v(h, h') := \begin{cases} \frac{g_v(h')}{h'} & \text{if } 0 \leq \frac{1}{h} < \frac{1}{h'} \leq x_v, \\ g_v(x_v) & \text{if } 0 \leq \frac{1}{h} \leq x_v \leq \frac{1}{h'} \leq 1, \\ \frac{g_v(h)}{h} & \text{if } x_v \leq \frac{1}{h} < \frac{1}{h'} \leq 1. \end{cases} \tag{2.18}$$

3. Bounds for $A \nabla B - H_v(A, B)$

In order to provide some upper and lower bounds for the quantity

$$A \nabla B - H_v(A, B)$$

where A and B are positive invertible operators, we need the following lemma.

Lemma 3.1. Consider the function $h_v : [0, \infty) \rightarrow \mathbb{R}$ for $v \in (0, 1)$ defined by

$$h_v(x) = \frac{x+1}{2} - \frac{1}{2}(x^v + x^{1-v}) \geq 0. \tag{3.1}$$

Then h_v is decreasing on $[0, 1)$ and increasing on $(1, \infty)$ with $x = 1$ its global minimum. We have $h_v(0) = \frac{1}{2}$, $\lim_{x \rightarrow \infty} h_v(x) = \infty$ and h_v is convex on $(0, \infty)$.

Proof. We have

$$h'_v(x) = \frac{1}{2} \left(1 - \frac{v}{x^{1-v}} - \frac{1-v}{x^v} \right)$$

and

$$h''_v(x) = \frac{1}{2} v(1-v) (x^{v-2} + x^{-v-1})$$

for any $x \in (0, \infty)$ and $v \in (0, 1)$.

We observe that $h'_v(1) = 0$ and $h''_v(x) > 0$ for any $x \in (0, \infty)$ and $v \in (0, 1)$. These imply that the equation $h'_v(x) = 0$ has only one solution on $(0, \infty)$, namely $x = 1$. Since $h'_v(x) < 0$ for $x \in (0, 1)$ and $h'_v(x) > 0$ for $x \in (1, \infty)$, then we deduce the desired conclusion. \square

Theorem 3.2. Assume that A and B are positive invertible operators, the constants $M > m \geq 0$ are such that the condition (1.3) holds and $v \in (0, 1)$. Then we have

$$\delta_v(m, M)A \leq A \nabla B - H_v(A, B) \leq \Delta_v(m, M)A, \tag{3.2}$$

where

$$\delta_v(m, M) := \begin{cases} h_v(M) & \text{if } M < 1, \\ 0 & \text{if } m \leq 1 \leq M, \\ h_v(m) & \text{if } 1 < m \end{cases} \tag{3.3}$$

and

$$\Delta_v(m, M) := \begin{cases} h_v(m) & \text{if } M < 1, \\ \max\{h_v(m), h_v(M)\} & \text{if } m \leq 1 \leq M, \\ h_v(M) & \text{if } 1 < m, \end{cases} \tag{3.4}$$

where h_v is defined by (3.1).

Proof. Using Lemma 3.1 we have

$$\begin{cases} h_v(M) & \text{if } M < 1, \\ 0 & \text{if } m \leq 1 \leq M, \\ h_v(m) & \text{if } 1 < m, \end{cases} \leq h_v(x) \leq \begin{cases} h_v(m) & \text{if } M < 1, \\ \max\{h_v(m), h_v(M)\} & \text{if } m \leq 1 \leq M, \\ h_v(M) & \text{if } 1 < m \end{cases}$$

for any $x \in [m, M]$ and $v \in (0, 1)$.

Using the continuous functional calculus, we have for any operator X with $mI \leq X \leq MI$ that

$$\delta_v(m, M)I \leq \frac{X+I}{2} - \frac{1}{2} \left(X^v + X^{1-v} \right) \leq \Delta_v(m, M)I. \quad (3.5)$$

From (1.3) we have, by multiplying both sides with $A^{-1/2}$ that

$$mI \leq A^{-1/2}BA^{-1/2} \leq MI.$$

Now, writing the inequality (3.5) for $X = A^{-1/2}BA^{-1/2}$, we get

$$\begin{aligned} \delta_v(m, M)I & \\ & \leq \frac{A^{-1/2}BA^{-1/2} + I}{2} - \frac{1}{2} \left(\left(A^{-1/2}BA^{-1/2} \right)^v + \left(A^{-1/2}BA^{-1/2} \right)^{1-v} \right) \\ & \leq \Delta_v(m, M)I. \end{aligned} \quad (3.6)$$

Finally, if we multiply both sides of (3.6) by $A^{1/2}$ we get the desired result (3.2). \square

Corollary 3.3. Let A and B be two positive operators. For positive real numbers m, m', M, M' , put $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ and let $v \in (0, 1)$.
(i) If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, then

$$h_v(h')A \leq A \nabla B - H_v(A, B) \leq h_v(h)A. \quad (3.7)$$

(ii) If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then

$$\frac{h_v(h')}{h'}A \leq A \nabla B - H_v(A, B) \leq \frac{h_v(h)}{h}A. \quad (3.8)$$

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