Konuralp Journal of Mathematics, 7 (1) (2019) 186-191

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

Generalized Simpson Type Integral Inequalities

Mehmet Zeki Sarıkaya[1*] and Sakine Bardak[1]

[1] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey*
[*] *Corresponding author E-mail: sarikayamz@gmail.com*

Abstract

In this paper, we have established some generalized Simpson type inequalities for convex functions. Furthermore, inequalities obtained in special case present a refinement and improvement of previously known results.

Keywords: *Simpson type inequalities, convex functions, integral inequalities.*
2010 Mathematics Subject Classification: *26D07, 26D10, 26D15, 26A33.*

1. Introduction

The following inequality is well known in the literature as Simpson's inequality.

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup\left|f^{(4)}(x)\right|<\infty$.Then, the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see ([1]-[21]).
In [2], Dragomir et. al. proved the following some recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth.

Theorem 1.2. *Suppose $f:[a,b]\to\mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a,b) and $f'\in L[a,b]$. Then, the following inequality*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{b-a}{3}\left\|f'\right\|_1 \tag{1.1}$$

holds, where $\|f'\|_1=\int_a^b|f'(x)|\,dx$.

The bound of (1.1) for L-Lipschitzian mapping was given in [2] by $\frac{5}{36}L(b-a)$. Also, the following inequality was obtained in [2].

Theorem 1.3. *Suppose $f:[a,b]\to\mathbb{R}$ is an absolutely continuous mapping on $[a,b]$ whose derivative belongs to $L_p[a,b]$. Then the following inequality holds,*

$$\begin{aligned}&\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\;&\frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\|f'\right\|_p\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$

The aim of this paper is to establish new generalization of Simpson's type inequalities for the class of functions whose derivatives in absolute value are convex. Furthermore, inequalities obtained in special case present a refinement and improvement of previously known results.

Email addresses: **sarikayamz@gmail.com (Mehmet Zeki Sarikaya), sakine.bardak@gmail.com (Sakine Bardak)**

2. Main Results

Let's begin start the following Lemma which helps us to obtain the main results:

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous on I° such that $f' \in L_1[a,b]$ with $a,b \in I$, $a<b$. Then the following equality holds:*

$$\begin{aligned}
& \frac{(b-w)^2}{2(b-a)}\int_0^1\left(\frac{1}{3}-\frac{t}{2}\right)f'\left(\frac{1+t}{2}w+\frac{1-t}{2}b\right)dt \\
& +\frac{(w-a)^2}{2(b-a)}\int_0^1\left(\frac{t}{2}-\frac{1}{3}\right)f'\left(\frac{1+t}{2}w+\frac{1-t}{2}a\right)dt \\
= & \frac{1}{6}f(w)+\frac{1}{3(b-a)}\left[(b-w)f\left(\frac{w+b}{2}\right)+(w-a)f\left(\frac{a+w}{2}\right)\right] \\
& -\frac{1}{b-a}\int_{\frac{a+w}{2}}^{\frac{w+b}{2}}f(u)\,du
\end{aligned}$$

where $w = ha+(1-h)b$, $h \in [0,1]$.

Proof. If we apply integration by parts, we have

$$\begin{aligned}
I_1 &= \int_0^1\left(\frac{1}{3}-\frac{t}{2}\right)f'\left(\frac{1+t}{2}w+\frac{1-t}{2}b\right)dt \\
&= -\frac{2}{b-w}\left(\frac{1}{3}-\frac{t}{2}\right)f\left(\frac{1+t}{2}w+\frac{1-t}{2}b\right)\Bigg|_0^1-\frac{1}{b-w}\int_0^1 f\left(\frac{1+t}{2}w+\frac{1-t}{2}b\right)dt \\
&= \frac{2}{6(b-w)}f(w)+\frac{2}{3(b-w)}f\left(\frac{w+b}{2}\right)-\frac{2}{(b-w)^2}\int_w^{\frac{w+b}{2}}f(u)\,du
\end{aligned}$$

and similary methots we can show that

$$\begin{aligned}
I_2 &= \int_0^1\left(\frac{t}{2}-\frac{1}{3}\right)f'\left(\frac{1+t}{2}w+\frac{1-t}{2}a\right)dt \\
&= \frac{2}{6(w-a)}f(w)+\frac{2}{3(w-a)}f\left(\frac{a+w}{2}\right)-\frac{2}{(w-a)^2}\int_{\frac{w+a}{2}}^{w}f(u)\,du.
\end{aligned}$$

Therefore, we obtain the following result

$$\begin{aligned}
& \frac{(b-w)^2}{2(b-a)}I_1+\frac{(w-a)^2}{2(b-a)}I_2 \\
= & \frac{b-w}{6(b-a)}f(w)+\frac{b-w}{3(b-a)}f\left(\frac{b+w}{2}\right)-\frac{1}{b-a}\int_w^{\frac{w+b}{2}}f(u)\,du \\
& +\frac{w-a}{6(b-a)}f(w)+\frac{w-a}{3(b-a)}f\left(\frac{a+w}{2}\right)-\frac{1}{b-a}\int_{\frac{a+w}{2}}^{w}f(u)\,du \\
= & \frac{1}{6}f(w)+\frac{1}{3(b-a)}\left[(b-w)f\left(\frac{w+b}{2}\right)+(w-a)f\left(\frac{a+w}{2}\right)\right] \\
& -\frac{1}{b-a}\int_{\frac{a+w}{2}}^{\frac{w+b}{2}}f(u)\,du
\end{aligned}$$

which completes the proof. □

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a,b]$ with $a,b \in I$, $a<b$. If $|f'|$ is a convex function on $[a,b]$ then the following inequality holds:*

$$\begin{aligned}
& \left|\frac{1}{6}f(w)+\frac{1}{3(b-a)}\left[(b-w)f\left(\frac{w+b}{2}\right)+(w-a)f\left(\frac{a+w}{2}\right)\right]\right. \\
& \left.-\frac{1}{b-a}\int_{\frac{a+w}{2}}^{\frac{w+b}{2}}f(u)\,du\right| \\
\leq & \frac{61\left[(b-w)^2+(w-a)^2\right]|f'(w)|+29\left[(b-w)^2|f'(b)|+(w-a)^2|f'(a)|\right]}{6^4(b-a)}
\end{aligned}$$

where $w = ha+(1-h)b$, $h \in [0,1]$.

Proof. From Lemma 2.1 and by using the convexity of $|f'|$, we get

$$\begin{aligned}
&\left|\frac{1}{6}f(w)+\frac{1}{3(b-a)}\left[(b-w)f\left(\frac{w+b}{2}\right)+(w-a)f\left(\frac{a+w}{2}\right)\right]\right.\\
&\left.-\frac{1}{b-a}\int_{\frac{a+w}{2}}^{\frac{w+b}{2}}f(u)\,du\right|\\
\leq\;& \frac{(b-w)^2}{2(b-a)}\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\left[\frac{1+t}{2}\left|f'(w)\right|+\frac{1-t}{2}\left|f'(b)\right|\right]dt\\
&+\frac{(w-a)^2}{2(b-a)}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[\frac{1+t}{2}\left|f'(w)\right|+\frac{1-t}{2}\left|f'(a)\right|\right]dt\\
=\;& \frac{(b-w)^2}{4(b-a)}\left|f'(w)\right|\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|(1+t)dt+\frac{(b-w)^2}{4(b-a)}\left|f'(b)\right|\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|(1-t)dt\\
&+\frac{(w-a)^2}{4(b-a)}\left|f'(w)\right|\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|(1+t)dt+\frac{(w-a)^2}{4(b-a)}\left|f'(a)\right|\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|(1-t)dt.
\end{aligned}$$

By simple computation, we have

$$\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|(1+t)dt=\frac{61}{3^4 2^2} \tag{2.1}$$

and

$$\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|(1-t)dt=\frac{29}{3^4 2^2}. \tag{2.2}$$

This completes the proof. □

Remark 2.3. *If we choose $h=0$ in Theorem 2.2, then we have $w=b$ and*

$$\begin{aligned}
&\left|\frac{1}{6}f(b)+\frac{1}{3}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}f(x)\,dx\right| \\
\leq\;&(b-a)\frac{61\left|f'(b)\right|+29\left|f'(a)\right|}{6^4}
\end{aligned}\tag{2.3}$$

Similarly, if we choose $h=1$ in Theorem 2.2, then we have $w=a$ and

$$\begin{aligned}
&\left|\frac{1}{6}f(a)+\frac{1}{3}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{\frac{a+b}{2}}f(x)\,dx\right| \\
\leq\;&(b-a)\frac{61\left|f'(a)\right|+29\left|f'(b)\right|}{6^4}.
\end{aligned}\tag{2.4}$$

Combining the inequalities (2.3) and (2.4), we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\
\leq\;&\frac{5}{72}(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]
\end{aligned}$$

which is given by Sarikaya et al. in [13, (for $s=1$)].

Theorem 2.4. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L_1[a,b]$ where $a,b\in I$, $a<b$. If $|f'|^q$ is a convex function on $[a,b]$, $q>1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(w)}{6}+\frac{1}{3(b-a)}\left[(b-w)f\left(\frac{w+b}{2}\right)+(w-a)f\left(\frac{a+w}{2}\right)\right]\right.\\
&\left.-\frac{1}{b-a}\int_{\frac{a+w}{2}}^{\frac{b+w}{2}}f(x)\,dx\right|\\
\leq\;&\frac{1}{12(b-a)}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\\
&\times\left\{(b-w)^2\left[\frac{3\left|f'(w)\right|^q+\left|f'(b)\right|^q}{4}\right]^{\frac{1}{q}}+(w-a)^2\left[\frac{3\left|f'(w)\right|^q+\left|f'(a)\right|^q}{4}\right]^{\frac{1}{q}}\right\}
\end{aligned}\tag{2.5}$$

where $w=ha+(1-h)b$, $h\in[0,1]$.

Proof. From Lemma 2.1 and by Hölder's inequality we get

$$\begin{aligned}
& \left|\frac{f(w)}{6}+\frac{1}{3(b-a)}\left[(b-w) f\left(\frac{w+b}{2}\right)+(w-a) f\left(\frac{a+w}{2}\right)\right]\right. \\
& \left.-\frac{1}{b-a} \int_{\frac{a+w}{2}}^{\frac{b+w}{2}} f(x) d x\right| \\
\leq & \frac{(b-w)^2}{2(b-a)} \int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'\left(\frac{1+t}{2} w+\frac{1-t}{2} b\right)\right| d t \\
& +\frac{(w-a)^2}{2(b-a)} \int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2} w+\frac{1-t}{2} a\right)\right| d t \\
\leq & \frac{(b-w)^2}{2(b-a)}\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|^p d t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2} w+\frac{1-t}{2} b\right)\right|^q d t\right)^{\frac{1}{q}} \\
& +\frac{(w-a)^2}{2(b-a)}\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^p d t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2} w+\frac{1-t}{2} a\right)\right|^q d t\right)^{\frac{1}{q}} .
\end{aligned} \tag{2.6}$$

By simple computation, we have

$$\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|^p d t=\frac{2\left(1+2^{p+1}\right)}{6^{p+1}(p+1)}. \tag{2.7}$$

Since $|f'|^q$ is a convex on $[a,b]$, we know that for $t \in [0,1]$

$$\begin{aligned}
& \int_0^1\left|f'\left(\frac{1+t}{2} w+\frac{1-t}{2} b\right)\right|^q d t \\
\leq & \int_0^1\left[\frac{1+t}{2}\left|f'(w)\right|^q+\frac{1-t}{2}\left|f'(b)\right|^q\right] d t \\
= & \frac{3\left|f'(w)\right|^q+\left|f'(b)\right|^q}{4}
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
& \int_0^1\left|f'\left(\frac{1+t}{2} w+\frac{1-t}{2} a\right)\right|^q d t \\
\leq & \int_0^1\left[\frac{1+t}{2}\left|f'(w)\right|^q+\frac{1-t}{2}\left|f'(a)\right|^q\right] d t \\
= & \frac{3\left|f'(w)\right|^q+\left|f'(a)\right|^q}{4}
\end{aligned} \tag{2.9}$$

Using the equations (2.7)-(2.9) in (2.6), then we obtain the (2.5). □

Remark 2.5. *If we choose $h=0$ in Theorem 2.4, then we have $w=b$ and*

$$\left|\frac{f(b)}{6}+\frac{1}{3} f\left(\frac{a+b}{2}\right)-\frac{1}{b-a} \int_{\frac{a+b}{2}}^b f(x) d x\right| \leq \frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left[\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right]^{\frac{1}{q}}. \tag{2.10}$$

Similarly, if we choose $h=1$ in Theorem 2.4, then we have $w=a$ and

$$\left|\frac{f(a)}{6}+\frac{1}{3} f\left(\frac{a+b}{2}\right)-\frac{1}{b-a} \int_a^{\frac{a+b}{2}} f(x) d x\right| \leq \frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left[\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right]^{\frac{1}{q}}. \tag{2.11}$$

Combining the inequalities (2.10) and (2.11), we have

$$\begin{aligned}
& \left|\frac{1}{6}\left[f(a)+4 f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a} \int_a^b f(x) d x\right| \\
\leq & \frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}} \\
& \times\left[\left[\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right]^{\frac{1}{q}}\right]
\end{aligned}$$

which is proved by Sarikaya et al. in [13, (for $s=1$)].

Theorem 2.6. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a,b]$ where $a,b \in I$, $a<b$. If $|f'|^q$ is a convex function on $[a,b]$, $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(w)}{6}+\frac{1}{3(b-a)}\left[(b-w)f\left(\frac{w+b}{2}\right)+(w-a)f\left(\frac{a+w}{2}\right)\right]\right.\\
&\left.-\frac{1}{b-a}\int_{\frac{a+w}{2}}^{\frac{b+w}{2}}f(x)\,dx\right|\\
\leq\;& \frac{1}{2(b-a)}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left[(b-w)^2\left(\frac{61|f'(w)|^q+29|f'(b)|^q}{3^42^3}\right)^{\frac{1}{q}}\right.\\
&\left.+(w-a)^2\left(\frac{61|f'(w)|^q+29|f'(a)|^q}{3^4 2^3}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where $w=ha+(1-h)b$, $h\in[0,1]$.

Proof. From Lemma 2.1 and by using power main inequality, we get

$$\begin{aligned}
&\left|\frac{f(w)}{6}+\frac{1}{3(b-a)}\left[(b-w)f\left(\frac{w+b}{2}\right)+(w-a)f\left(\frac{a+w}{2}\right)\right]\right.\\
&\left.-\frac{1}{b-a}\int_{\frac{a+w}{2}}^{\frac{b+w}{2}}f(x)\,dx\right|\\
=\;& \frac{(b-w)^2}{2(b-a)}\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'\left(\frac{1+t}{2}w+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{(w-a)^2}{2(b-a)}\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}w+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is a convex function on $[a,b]$, by using the equalities (2.1) and (2.2), we obtain

$$\begin{aligned}
&\left|\frac{f(w)}{6}+\frac{1}{3(b-a)}\left[(b-w)f\left(\frac{w+b}{2}\right)+(w-a)f\left(\frac{a+w}{2}\right)\right]\right.\\
&\left.-\frac{1}{b-a}\int_{\frac{a+w}{2}}^{\frac{b+w}{2}}f(x)\,dx\right|\\
\leq\;& \frac{(b-w)^2}{2(b-a)}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{|f'(w)|^q}{2}\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|(1+t)\,dt+\frac{|f'(b)|^q}{2}\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|(1-t)\,dt\right)^{\frac{1}{q}}\\
&+\frac{(w-a)^2}{2(b-a)}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{|f'(w)|^q}{2}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|(1+t)\,dt+\frac{|f'(a)|^q}{2}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|(1-t)\,dt\right)^{\frac{1}{q}}\\
=\;& \frac{1}{2(b-a)}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left[(b-w)^2\left(\frac{61}{3^42^3}|f'(w)|^q+\frac{29}{3^42^3}|f'(b)|^q\right)^{\frac{1}{q}}\right.\\
&\left.+(w-a)^2\left(\frac{61}{3^42^3}|f'(w)|^q+\frac{29}{3^42^3}|f'(a)|^q\right)^{\frac{1}{q}}\right]
\end{aligned}$$

which completes the proof. □

Remark 2.7. *If we choose $h=0$ in Theorem 2.6, then we have $w=b$ and*

$$\begin{aligned}
&\left|\frac{f(b)}{6}+\frac{1}{3}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}f(x)\,dx\right| \qquad (2.12)\\
\leq\;& \frac{b-a}{2}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left(\frac{29|f'(a)|^q+61|f'(b)|^q}{3^42^3}\right)^{\frac{1}{q}}.
\end{aligned}$$

If we choose $h=1$ *in Theorem 2.6, then we have* $w=a$ *and*

$$\left|\frac{f(a)}{6}+\frac{1}{3}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^{\frac{a+b}{2}}f(x)\,dx\right| \tag{2.13}$$
$$\leq \frac{b-a}{2}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left(\frac{61\,|f'(a)|^q+29\,|f'(b)|^q}{3^4 2^3}\right)^{\frac{1}{q}}.$$

Combining the inequalities (2.12) and (2.13), we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{2}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left[\left(\frac{29\,|f'(a)|^q+61\,|f'(b)|^q}{3^4 2^3}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{61\,|f'(a)|^q+29\,|f'(b)|^q}{3^4 2^3}\right)^{\frac{1}{q}}\right]$$

which is given by Sarikaya et al. in [13, (for $s=1$*)].*

References

[1] M. Alomari, M. Darus and S.S. Dragomir, *New inequalities of Simpsonís type for sconvex functions with applications*, RGMIA Res. Rep. Coll., 12 (4) (2009), Article 9.
[2] S.S. Dragomir, R.P. Agarwal and P. Cerone, *On Simpsonís inequality and applications*, J. of Inequal. Appl., 5(2000), 533-579.
[3] S.S. Dragomir. *On Simpson's quadrature formula for differentiable mappings whose derivatives belong to* l_p *spaces and applications*. J. KSIAM, 2 (1998), 57–65.
[4] S.S. Dragomir, *On Simpson's quadrature formula for Lipschitzian mappings and applications* Soochow J. Mathematics, 25 (1999), 175–180.
[5] T. Du, Y. Li and Z. Yang, *A generalization of Simpson's inequality via differentiable mapping using extended* (s,m)*-convex functions,* Applied Mathematics and Computation 293 (2017) 358–369
[6] S. Hussain andS. Qaisar, *More results on Simpson's type inequality through convexity for twice differentiable continuous mappings*. Springer Plus (2016), 5:77.
[7] B.Z. Liu, *An inequality of Simpson type*, Proc. R. Soc. A, 461 (2005), 2155-2158.
[8] J. Pecaric, F. Proschan and Y.L. Tong, *Convex functions, partial ordering and statistical applications*, Academic Press, New York, 1991.
[9] J. Pecaric., and S. Varosanec, *A note on Simpson's inequality for functions of bounded variation*, Tamkang Journal of Mathematics, Volume 31, Number 3, Autumn (2000), 239–242.
[10] S. Qaisar, C.J. He, S. Hussain, *A generalizations of Simpson's type inequality for differentiable functions using* (α,m)*-convex functions and applications*, J. Inequal. Appl. 2013 (2013) 13. Article 158.
[11] H. Kavurmaci, A. O. Akdemir, E. Set and M. Z. Sarikaya, *Simpson's type inequalities for* $m-$ *and* (α,m)*-geometrically convex functions*, Konuralp Journal of Mathematics, 2(1), pp:90-101, 2014.
[12] M. E. Ozdemir, A. O. Akdemir and H. Kavurmacı, *On the Simpson's inequality for convex functions on the co-Ordinates*, Turkish Journal of Analysis and Number Theory. 2014, 2(5), 165-169.
[13] M. Z. Sarikaya, E. Set and M. E. Ozdemir, *On new inequalities of Simpson's type for s-convex functions*, Computers and Mathematics with Applications 60 (2010) 2191–2199.
[14] M.Z. Sarikaya, E. Set, M.E. Özdemir, *On new inequalities of Simpson's type for convex functions*, RGMIA Res. Rep. Coll. 13 (2) (2010) Article2.
[15] M. Z.Sarikaya, E. Set and M. E. Ozdemir, *On new inequalities of Simpson's type for functions whose second derivatives absolute values are convex*, Journal of Applied Mathematics, Statistics and Informatics , 9 (2013), No. 1.
[16] M.Z. Sarıkaya, T. Tunç and H. Budak, *Simpson's type inequality for* F*-convex function*, Facta Universitatis Ser. Math. Inform., Vol. 32, No 5 (2017), 747–753.
[17] E. Set, M. E. Ozdemir and M. Z. Sarikaya, *On new inequalities of Simpson's type for quasi-convex functions with applications*, Tamkang Journal of Mathematics, 43 (2012), no. 3, 357–364.
[18] E. Set, M. Z. Sarikaya and N. Uygun, *On new inequalities of Simpson's type for generalized quasi-convex functions*, Advances in Inequalities and Applications, 2017, 2017:3, pp:1-11.
[19] K. L. Tseng, G. S. Yang and S.S. Dragomir, *On weighted Simpson type inequalities and applications* Journal of mathematical inequalities, Vol. 1, number 1 (2007), 13–22.
[20] N. Ujevic, *Double integral inequalities of Simpson type and applications*, J. Appl. Math. Comput., 14 (2004), no:1-2, p. 213-223.
[21] Z.Q. Yang, Y.J. Li andT. Du, *A generalization of Simpson type inequality via differentiable functions using* (s,m)*-convex functions*, Ital. J. Pure Appl. Math. 35 (2015) 327–338.