**MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES**
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# A New Continuous Lifetime Distribution and its Application to the Indemnity and Aircraft Windshield Datasets

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**Abstract**
Kharazmi and Saadatinik [21] introduced a new family of distribution called hyperbolic cosine – F (HCF) distributions. They studied some properties of this model and obtained the estimates of its parameters by different methods. In this paper, it is focused on a special case of HCF family with Weibull distribution as a baseline model. Various properties of the proposed distribution including explicit expressions for the moments, quantiles, moment generating function, failure rate function, mean residual lifetime, order statistics and expression of the entropies are derived. Superiority of this model is proved in some simulations and applications.

***Keywords:*** *Hyperbolic cosine function; Weibull distribution; Hazard function; Mean residual life time; Maximum likelihood estimates.*

***AMS Subject Classification (2010):*** *Primary: 62E15.*

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## 1. Introduction

Real world phenomena are commonly described using statistical distributions. Numerous classical distributions have been extensively used over the past decades for modeling data in several areas such as engineering, actuarial, environmental, biological studies, economics, finance and insurance. However, in many applied areas such as lifetime analysis, finance and insurance, it is needed to extend these distributions. For that reason, several methods for generating new families of distributions have been studied. The well-known generators are the following: Azzalini's skew family by Azzalini [8], Marshal-Olkin generated family (MO-G) by Marshall and Olkin [22], exponentiated family (EF) of distributions by Gupta et al. [19], beta-G by Eugene et al. [17] and Jones [20], Kumaraswamy-G (Kw-G) by Cordeiro and de Castro [12], McDonald-G (Mc-G) by Alexander et al. [1], gamma-G (type 1) by Zografos and Balakrishnan [33], gamma-G (type 2) by Ristić and Balakrishnan [27], gamma-G (type 3) by Torabi and Hedesh [31], log-gamma-G by Amini et al. [7], logistic-G by Tahir et al. [29], exponentiated generalized-G by Cordeiro et al. [15], geometric exponential-Poisson family by Nadarajah et al. [24], truncated-exponential skew-symmetric family by Nadarajah et al. [25], logistic-generated (Lo-G) family by Torabi and Montazari [32], Transformed-Transformer (T-X) by Alzaatreh et al. [5], exponentiated (T-X) by Alzaghal et al. [6], Weibull-G by Bourguignon et al. [11], Exponentiated half logistic generated family by Cordeiro et al. [14], Kumaraswamy Odd log-logistic-G by Alizadeh et al. [3], Kumaraswamy Marshall-Olkin by Alizadeh et al. [4], Beta Marshall-Olkin by Alizadeh et al. [2], Type Half-Logistic family of distributions by Cordeiro et al. [13] and Odd generalized exponential-G by Tahir et al. [30]. These families of distributions have received a great deal of attention in recent years.
Kharazmi and Saadatinik [21] introduced a family of distributions using Hyperbolic Cosine function. The hyperbolic

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cosine has similar name to the trigonometric functions, but it is defined in terms of the exponential function as follows:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \tag{1.1}$$

The function $\cosh(x)$ is odd and has a Taylor series expression with only even exponents for x.

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \tag{1.2}$$

According to Kharazmi and Saadatinik [21] a random variable X has a Hyperbolic Cosine-F (HCF) distribution if its cumulative distribution function (cdf) is given by

$$G(x) = \frac{2e^a}{e^{2a}-1} \sinh\Big(aF(x)\Big), \tag{1.3}$$

where $x > 0,\ a > 0$.
$F(x)$ can be the cdf of any random variable. Kharazmi and Saadatinik [21] assumed $F(x) = 1 - e^{-\lambda x}$ (the exponential distribution function). This distribution is called HCE by them and is studied some properties of this model.

The aim of this paper is to introduce Hyperbolic Cosine-Weibull (HCW) distribution and study some of its mathematical properties. The rest of the paper is organized as follows. In Section 2, we introduce the HCW model and discuss some general properties of this family of distributions. In Section 3, we obtain some statistical and reliability functions of HCW model. Maximum likelihood estimation of unknown parameters is investigated in section 4. Some simple simulations are designed in section 5 to compare the new model with older ones. The Application of this distribution is studied using two real data sets in Section 6.

## 2. Hyperbolic cosine – Weibull (HCW) distribution

If in (1.3) we get $F(x) = 1 - e^{-\lambda x^\beta}$, Hyperbolic cosine-Weibull (HCW) is obtained. The probability density function (pdf) of this distribution is given by

$$g(x; a, \beta, \lambda) = \frac{2a\, e^a}{e^{2a}-1} \lambda\, \beta\, x^{\beta-1}\, e^{-\lambda x^\beta}\ \cosh\Big(a(1 - e^{-\lambda x^\beta})\Big) \tag{2.1}$$

where $x > 0, a > 0 \beta > 0, \lambda > 0$. If a random variable X comes from HCW distribution, we noted $X \sim HCW(a, \beta, \lambda)$. Figure 1 shows the shapes of $HCW(a, \beta, \lambda)$ for different values of parameters. Clearly, changing parameters of model cause different skewness in pdf. So this model can be applied in many applications.

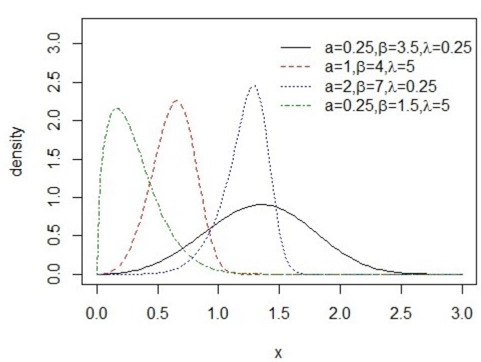


**Figure 1.** Plots of the $HCW(a, \beta, \lambda)$ for different values of parameters.

A motivation for introducing this new model is an application in reliability. Suppose that the failure of a device occurs due to the presence of an unknown number, 2N+1, of initial defects of some kind.

Let $Y_1, \cdots, Y_{2N+1}$ denote the failure times of the initial defects. Let X denote the failure time of the device. Then $X = max(Y_1, \cdots, Y_{2N+1})$. Suppose N is a discrete random variable with a new probability mass function:

$$P(N = n) = \begin{cases} \frac{2e^a}{e^{2a}-1} \frac{a^{2n+1}}{(2n+1)!} & n = 0, 1, 2, \dots \\ 0 & o.w. \end{cases}$$

Where $0 < a < \infty$. Suppose also that $Y_1, \cdots, Y_{2N+1}$ is a random sample from the Weibull distribution with pdf $f(x) = \lambda \, \beta \, x^{\beta-1} \, e^{-\lambda x^\beta}$ and cdf $F(x) = 1 - e^{-\lambda x^\beta}$, then

$$f_{X|N=n}(x) = (2n + 1) f(x) \, F^{2n}(x)$$

So the marginal probability density function of X is given by (2.1).
On the other hand, the new model can be investigated as an infinite mixtures of generalized-F $(F^\alpha(x))$ distributions. Using the series expansion

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \tag{2.2}$$

The HCW distribution can be state as follows:

$$g(x) = \frac{2a \, e^a}{e^{2a} - 1} f(x) \cosh\Big(a(F(x))\Big) = \sum_{n=0}^{\infty} w(a, n) f_U(x)$$

where $U \sim$ generalized-F with only even exponents.

$$f_U(x) = (2n + 1) f(x) \, F^{2n}(x)$$

and $w(a, n) = \frac{2a \, e^a}{e^{2a}-1} \frac{a^{2n}}{(2n+1)!}$.

## 3. Statistical and reliability properties

In this section, we study the several statistical and reliability properties of the HCW distribution, such as the survival function (SF), conditional survival function (CSF), failure rate (or hazard) function (FR), moment generating function (MGF), mean residual life (MRL) time and kth moment.

### 3.1 Survival, quantile, conditional reliability and failure rate function

The cdf of HCW using (1.3) can be written as

$$G(x; a, \beta, \lambda) = \frac{2e^a}{e^{2a} - 1} \sinh\left(a(1 - e^{-\lambda x^\beta})\right)$$

So survival and quantile functions are simply given by

$$\bar{G}(x; a, \beta, \lambda) = 1 - G(x; a, \beta, \lambda) = 1 - \frac{2e^a}{e^{2a} - 1} \sinh\left(a(1 - e^{-\lambda x^\beta})\right), \tag{3.1}$$

$$\begin{aligned} x_p &= \Big[ -\frac{1}{\lambda}\Big( \log\Big(1 - \frac{arcsinh(\frac{e^{2a}-1}{2e^a} p)}{a}\Big)\Big)\Big]^{1/\beta} \\ &= \Big[ -\frac{1}{\lambda}\Big( \log\Big(1 - \frac{\log\Big(\frac{e^{2a}-1}{2e^a} p + \sqrt{(\frac{e^{2a}-1}{2e^a} p)^2 + 1}\Big)}{a}\Big)\Big)\Big]^{1/\beta}, \quad 0 \le p \le 1, \end{aligned} \tag{3.2}$$

The last equation comes from this fact that $arcsinh(x) = ln(x + \sqrt{x^2 + 1})$. Also conditional reliability function is given by

$$\bar{G}(x; a, \beta, \lambda | t) = \frac{\bar{G}(x + t; a, \beta, \lambda)}{\bar{G}(x; a, \beta, \lambda)} = \frac{1 - \frac{2e^a}{e^{2a}-1} \sinh\left(a(1 - e^{-\lambda(x+t)^\beta})\right)}{1 - \frac{2e^a}{e^{2a}-1} \sinh\left(a(1 - e^{-\lambda t^\beta})\right)}; \quad x > 0, \, t > 0. \tag{3.3}$$

From (2.1) and (3.1) it is easy to verify that the failure rate function is given by

$$h(x;a,\beta,\lambda) = \frac{2e^{a}\,\lambda\beta\,x^{\beta-1}\,e^{-\lambda x^{\beta}}\cosh\left(a(1-e^{-\lambda x^{\beta}})\right)}{e^{2a}-1-2e^{a}\sinh\left(a(1-e^{-\lambda x^{\beta}})\right)}. \tag{3.4}$$

Figure 2 illustrate some samples of possible shapes of the failure rate function for certain values of the parameters. It is clear that this model has both increasing and decreasing failure rates with different values of parameters.

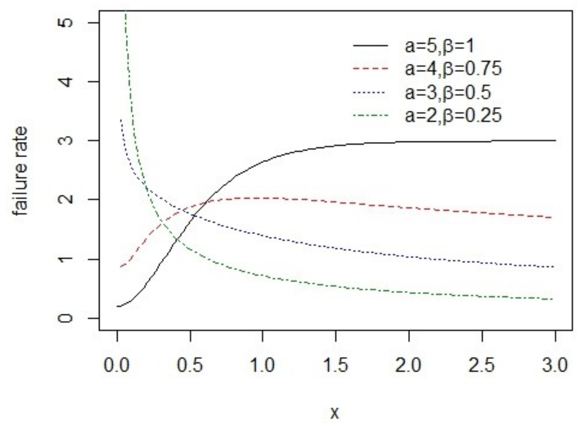


**Figure 2.** failure rate function shapes for selected values of the parameters when $\lambda = 3$.

The failure rate is a key notion in reliability and survival analysis for measuring the ageing process. Understanding the shape of the failure rate is important in reliability theory, risk analysis and other disciplines. The concepts of increasing and decreasing, bathtub shaped (first decreasing and then increasing) and upside-down bathtub shaped (first increasing and then decreasing) failure rates for univariate distributions have been found very useful in reliability theory.

### 3.2 Moment generating function and mean residual life time

Now let us consider different moments of the $HCW(a,\beta,\lambda)$ distribution. Some of the most important features and characteristics of a distribution can be studied through its moments, such as moment generating function, the kth moment and interested reliability properties such as mean residual life time. The moment generating function of $HCW(a,\beta,\lambda)$ using (2.1) and (2.2) is immediately written as

$$M_X(t) = E(e^{tX}) = \frac{2ae^{a}}{e^{2a}-1}\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\sum_{j=0}^{\infty}\frac{t^j}{j!}\,\frac{a^{2n}(-1)^k\binom{2n}{k}}{(k+1)(2n)!}\,\frac{\Gamma\left(1+\frac{j}{\beta}\right)}{\left(\lambda(k+1)\right)^{\frac{j}{\beta}}}$$

The jth moment and jth central moment of the HCW distribution can be derived as

$$\mu_j = E(X^j) = \frac{2ae^{a}}{e^{2a}-1}\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\frac{a^{2n}(-1)^k\binom{2n}{k}}{(k+1)(2n)!}\,\frac{\Gamma\left(1+\frac{j}{\beta}\right)}{\left(\lambda(k+1)\right)^{\frac{j}{\beta}}}$$

In particular, its mean and variance are given by

$$E(X) = \frac{2ae^{a}}{e^{2a}-1}\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\frac{a^{2n}(-1)^k\binom{2n}{k}}{(k+1)(2n)!}\,\frac{\Gamma\left(1+\frac{1}{\beta}\right)}{\left(\lambda(k+1)\right)^{\frac{1}{\beta}}}$$

and

$$Var(X) = E(X-\mu)^2.$$

One of the well-known properties of the life time distribution is mean residual life time. For the HCW distribution it can be written as

$$m(t) = E(X-t|X>t) = \frac{2a\,e^{a}}{e^{2a}-1-2e^{a}\sinh\left(a(1-e^{-\lambda t^{\beta}})\right)}\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\frac{a^{2n}(-1)^{k}\binom{2n}{k}}{(k+1)(2n)!}\int_{0}^{\infty}e^{-\lambda(k+1)(u+t)^{\beta}}\mathrm{d}u$$

### 3.3 Order statistics, stress-strength parameter and Shannon entropy measure

Here, we provide an order statistics result. Let $X_1,\cdots,X_n$ be a random sample from a $HCW(a,\beta,\lambda)$, and let $X_{i:n}$ denote the $i^{th}$ order statistic. The pdf of $X_{i:n}$ is given by

$$f_{x(i)}(x) = \frac{n!}{(i-1)!(n-i)!}\frac{2ae^{a}\,\lambda\beta\,x^{\beta-1}\,e^{-\lambda x^{\beta}}\cosh\left(a(1-e^{-\lambda x^{\beta}})\right)}{e^{2a}-1}$$
$$\times\Big(\frac{2e^{a}\,\sinh\left(a(1-e^{-\lambda x^{\beta}})\right)}{e^{2a}-1}\Big)^{i-1}\Big(1-\frac{2e^{a}\,\sinh\left(a(1-e^{-\lambda x^{\beta}})\right)}{e^{2a}-1}\Big)^{n-i}.$$

Now we discuss about the stress-strength parameter. Suppose $X_1\sim HCW(a_1,\beta_1,\lambda_1)$ and $X_2\sim HCW(a_2,\beta_2,\lambda_2)$ are independently distributed, then

$$P(X_1<X_2) = \frac{2a_1\,e^{a_1}}{e^{2a_1}-1}\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\frac{a_1^{2n}(-1)^{k}\binom{2n}{k}}{(k+1)(2n)!}$$
$$-\frac{2a_1\,e^{a_1}}{e^{2a_1}-1}\frac{2a_2\,e^{a_2}}{e^{2a_2}-1}\sum_{n=0}^{\infty}\sum_{t=0}^{\infty}\sum_{k=0}^{2n}\frac{(-1)^{t}(\lambda_1(k+1))^{t}}{t!}\frac{a_1^{2n}a_2^{2n}(-1)^{2k}\binom{2n}{k}^{2}}{(k+1)^{2}((2n)!)^{2}}\frac{\Gamma\Big(1+\frac{t\beta_1}{\beta_2}\Big)}{\Big(\lambda_2(k+1)\Big)^{\frac{t\beta_1}{\beta_2}}}$$

The entropy of a random variable measures the variation of the uncertainty. A large value of entropy indicates the greater uncertainty in the data. Shannon entropy (Shannon [28]), of $HCW(a,\beta,\lambda)$ can

$$H(X) = -\log(\frac{2ae^{a}}{e^{2a}-1})-\log(\lambda\beta)+\frac{2ae^{a}}{e^{2a}-1}(\beta-1)\sum_{n=0}^{\infty}\sum_{k=0}^{2n+1}\frac{a^{2n+1}(-1)^{k}\binom{2n+1}{k}}{(2n+1)!}\int_{0}^{\infty}\frac{e^{-\lambda kx^{\beta}}}{x}\mathrm{d}x$$
$$+\frac{2ae^{a}}{e^{2a}-1}\sum_{k=0}^{2n}\frac{a^{2n+1}(-1)^{k}\binom{2n}{k}}{(k+1)^{2}(2n)!}-\log(\frac{e^{2a}-1}{2e^{a}})-\frac{2e^{a}}{e^{2a}-1}arctanh(\frac{e^{2a}-1}{2e^{a}})+1.$$

## 4. Maximum likelihood estimation

Let $X_1,\cdots,X_n$ be a random sample from the distribution with density $f(x;\theta)$. The likelihood function based on observed values $x_1,\cdots,x_n$ is given by

$$L(\theta,\underline{x}) = \prod_{i=1}^{n}f(x_i,\theta) \tag{4.1}$$

By maximizing (4.1) the Maximum likelihood estimate of $\theta$ (MLE) is obtained. In case of the HCW distribution, the log-likelihood function of the parameter is given as

$$l(a,\beta,\lambda,\underline{x}) = \log\Big(L(a,\beta,\lambda,\underline{x})\Big) = n\log(\frac{2ae^{a}}{e^{2a}-1})+n\log(\lambda\beta)+(\beta-1)\sum_{i=1}^{n}\log(x_i)-\lambda\sum_{i=1}^{n}x_i^{\beta}$$
$$+\sum_{i=1}^{n}\log\Big(\cosh\Big(a(1-e^{-\lambda x_i^{\beta}})\Big)\Big)$$

So, the MLEs of $a,\beta$ and $\lambda$, say $\hat{a},\hat{\beta}$ and $\hat{\lambda}$, respectively, can be obtained as the solutions of

$$\frac{\partial l}{\partial a} = n\frac{2e^{3a}(1-a)-2e^{a}(1+a)}{2a\,e^{a}(e^{2a}-1)}+\sum_{i=1}^{n}(1-e^{-\lambda x_i^{\beta}})\tanh\Big(a(1-e^{-\lambda x_i^{\beta}})\Big) = 0,$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log(x_i) - \lambda \sum_{i=1}^{n} x_i^{\beta} \, \log(x_i) + a\lambda \sum_{i=1}^{n} x_i^{\beta} \, \log(x_i) \, e^{-\lambda x_i^{\beta}} \tanh\left(a(1 - e^{-\lambda x_i^{\beta}})\right) = 0,$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i^{\beta} + a \sum_{i=1}^{n} x_i^{\beta} \, e^{-\lambda x_i^{\beta}} \tanh\left(a(1 - e^{-\lambda x_i^{\beta}})\right) = 0.$$

Due to the non-linearity of these equations the MLEs of parameters can be obtained numerically. We use the optim function from the statistical software R (R Development Core Team, [26]) to solve these equations.

## 5. Simulations

In this section, we perform a small simulation study to investigate the finite sample properties of ML estimators described in Section 4. To conduct the experimental study, we generate 5000 synthetic samples of size n = 10, 30, 50 and 100 from HCW and HCE with true parameters $a = 1$, $\beta = 2$ and $\lambda = 3$. To examine the estimation accuracies, the absolute bias and the mean squared error (MSE) are computed. Figures 3 and 4 show a graphical representation of the absolute bias and the MSE of the parameter estimates as a function of sample size n.

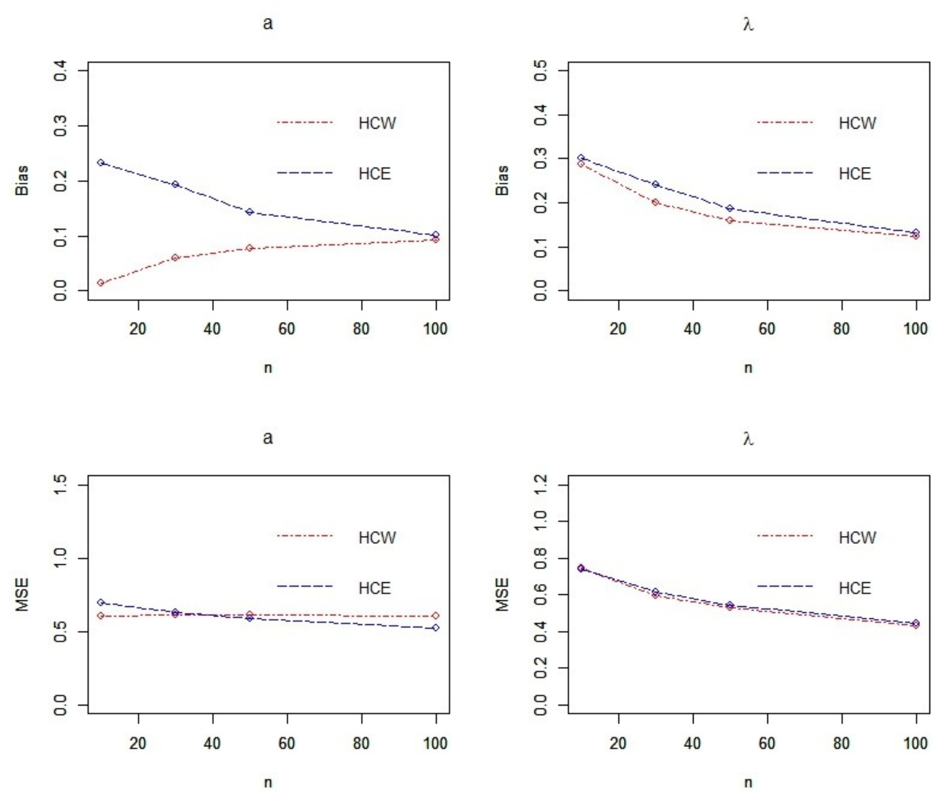


**Figure 3.** Absolute bias (above) and MSE's (bottom) of parameters a and $\lambda$ for HCE and HCW models.

Clearly, in small sample sizes, the absolute bias and MSE values of HCW estimates are smaller than HCE. But for large n, the bias and MSE of estimator of parameter a in HCW become larger than HCE model. For the parameter $\beta$ these two indices converge reasonably well to zero when n increases.

## 6. Applications

In this section, two real datasets are used to establish the usefulness of HCW distribution. Since the pdf of HCW has different shapes, this model can be applied in many applications. Here, we present two applications of this model. The first dataset is a familiar test in reliability analysis that is cited in several papers. The second application comes from insurance studies.

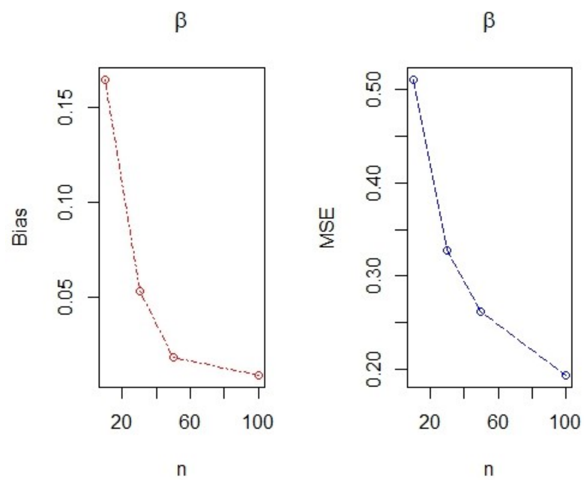


**Figure 4.** Absolute bias (left) and MSE (right) of parameter $\beta$ for HCW model.

### 6.1 Aircraft Windshield

The first dataset (Table 16.11 of Murthy et al. [23]) represent the service times for a particular model Aircraft windshield. The sample size is 63 and we fit three models to these data: Weibull-Geometric(WG), HCE and HCW. The pdf of WG is shown in (6.1). For further details about WG see Barreto-Souza et al. [10].

$$f_{wg}(x) = (1-a)\beta\,\lambda^{\beta}\,x^{\beta-1}e^{-(\lambda x)^{\beta}}\,(1-a\,e^{-(\lambda x)^{\beta}})^{-2}, \quad x \geq 0. \tag{6.1}$$

Table 1 shows the MLEs of parameters and two information criteria: Akaike information criterion (AIC) and Bayesian information criterion (BIC). These criteria are given by

$$AIC = -2\hat{l}(\theta) + 2m; \qquad BIC = -2\hat{l}(\theta) + m\log(n)$$

where m and n are the number of parameters and sample size, respectively and $\hat{l}$ is the maximized log-likelihood. As a rule of thumb, the model with smaller AIC or BIC value is considered to provide a better fit. According to Table 1, HCW provides the best model for the service times of Aircraft Windshields among three models.

**Table 1.** MLE's with standard errors in parenthesis and information criteria for service times data

| **Model** | **ML estimates** | | | **AIC** | **BIC** |
|---|---|---|---|---|---|
| | $\hat{a}$ | $\hat{\beta}$ | $\hat{\lambda}$ | | |
| **WG** | 0.001 (0.572) | 1.629 (0.241) | 0.423 (0.094) | 206.63 | 213.06 |
| **HCE** | 3.694 (0.678) | | 0.895 (0.097) | 203.63 | 209.79 |
| **HCW** | 2.592 (0.905) | 1.303 (0.219) | 0.565 (0.208) | 203.36 | 207.92 |

Figure 5 depicts density plots of three competitive fitted models for the dataset. It can be observed that the HCW model provides a more adequate fit to histogram of the data than the other competitors. Also the P-P plot of the data for HCW model is shown in this Figure, that is indicated that the data comes from this model.

### 6.2 US indemnity data

For the second application, we consider a dataset widely used in the actuarial literature. This dataset is comprised of US indemnity losses used in Frees and Valdez [18]. The data consist of 1500 general liability claims,

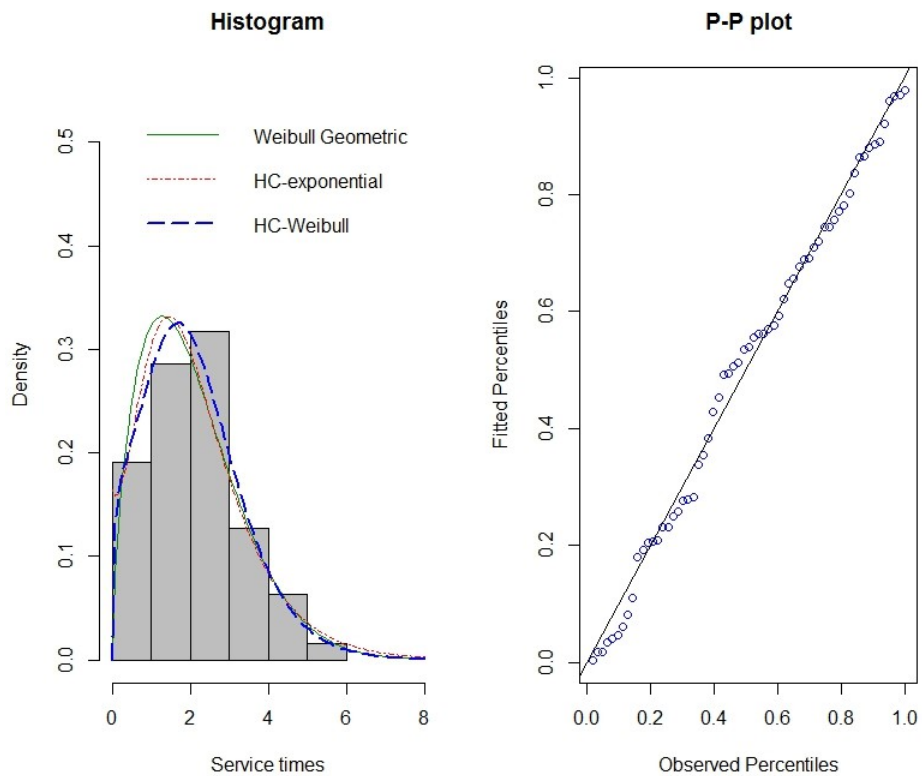


**Figure 5.** The fitted pdf's and the relative histogram (left) and P-P plots of fitted HCW (right).

giving for each the indemnity payment and the allocated loss adjustment expense both in USD. The dataset can be found in the R packages copula. We focus here on the first column in dataset ( pure loss data). Eling [16] fitted several distributions to the real data and logarithm of them. In both cases, he suggested two distributions that are fitted better than another models to data: Skew normal (SN) and Skew-t (ST). Eling [16] showed that ST is the best model for the logarithm of losses among all studied models. Here, we fit HCW distribution to this dataset and compare it with SN and ST models. The pdf of these distributions are as follow. For further information about skew models and their applications see Azzalini [9].

$$f_{SN}(x) = \frac{2}{\sigma}\phi(\frac{x-\mu}{\sigma})\,\Phi(\lambda\frac{x-\mu}{\sigma}); \qquad x,\mu,\lambda \in (-\infty,\infty), \sigma > 0.$$

$$f_{ST}(x) = \frac{2}{\sigma}t(\frac{x-\mu}{\sigma};v)\,T(\lambda\frac{x-\mu}{\sigma};v); \qquad x,\mu,\lambda \in (-\infty,\infty), \sigma, v > 0.$$

where $\phi(.)$ and $\Phi(.)$ are pdf and cdf of standard normal variable, respectively. Also $t(.;v)$ and $T(.;v)$ are pdf and cdf of a t-student variable with degree of freedom, respectively.

Table 2 shows the results. According to this Table the HCW model provides the better fit than SN and ST models. Figure 6 indicate that the new model has better fit than two old models. Also by P-P plot, it is clear that HCW model is accurately fit to these data.

## 7. Conclusions

We define a new model, so-called the HCW distribution, that is a special cases of HCF family of distributions proposed by Kharazmi and Saadatinik [21]. In simulations and applications, it is showed that this model is better than HCE model that is introduced by Kharazmi and Saadatinik [21]. Plots of the density and failure rate functions showed that this model can be applied in several fields from reliability to finance and environmental sciences. Two applications of the HCW distribution to real data sets are provided to illustrate that this distribution provides a better fit than Weibull-Geometric, Hyperbolic cosine exponential, Skew normal and Skew-t distributions. Moreover,

**Table 2.** MLE's with standard errors in parenthesis and information criteria for US indemnity data

| Model | ML estimates | | | | | | AIC | BIC |
|---|---|---|---|---|---|---|---|---|
| | $\hat{a}$ | $\hat{\beta}$ | $\hat{\lambda}$ | $\hat{\mu}$ | $\hat{\sigma}$ | $\hat{v}$ | | |
| **SN** | | | 0.001 | 9.371 | 1.637 | | 5742.43 | 5758.37 |
| | | | (2.07) | (2.71) | (0.031) | | | |
| **ST** | | | -0.67 | 10.184 | 1.771 | 33.78 | 5738.06 | 5759.31 |
| | | | (0.32) | (0.33) | (0.17) | (26.09) | | |
| **HCW** | 6.786 | 3.044 | 0.002 | | | | 5733.61 | 5749.55 |
| | (0.61) | (0.04) | (0.001) | | | | | |

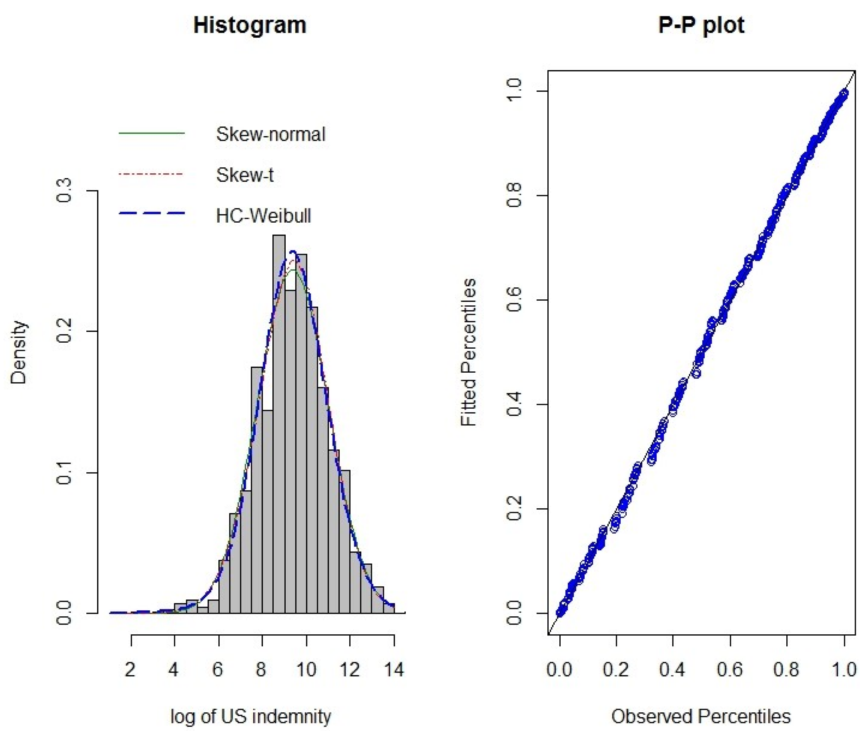


**Figure 6.** The fitted pdf's and the relative histogram (left) and P-P plots of fitted HCW (right).

in a simple simulation design, we showed the proficiency of this new model. A multivariate version of this model is a good extension for future works.

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