https://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser.A1 Math.Stat.
Volume 69, Number 2, Pages 1320–1328 (2020)
DOI: 10.31801/cfsuasmas.559608
ISSN 1303–5991 E-ISSN 2618-6470

Received by the editors: May 01, 2019; Accepted: August 26, 2020

ON A JANOWSKI FORMULA BASED ON A GENERALIZED DIFFERENTIAL OPERATOR

Rabha W. IBRAHIM

Informetrics Research Group, Ton Duc Thang University, Ho Chi Minh City, VIETNAM
Faculty of Mathematics & Statistics, Ton Duc Thang University, Ho Chi Minh City, VIETNAM

Abstract. The central purpose of the current paper is to consider a set of beneficial possessions including inequalities for a generalized subclass of Janowski functions (analytic functions) which are formulated here by revenues of a generalized Sàlàgean's differential operator. Numerous recognized consequences of the outcomes are also indicated. We present some results involving the subordination and superordination inequalities. Moreover, growth inequalities are indicated in the sequel. Real and special cases are suggested containing the differential operator.

1. Introduction

The differential operators regularly characterize physical capacities, the derivatives signify their proportions of modification, and the operator expresses a relationship between the two. Because such relatives are exceptionally common, differential operators play a flat role in many categories involving physics, economics, engineering and biology. In this direction, Ibrahim and Darus introduced the following mixed operator [1]: let Λ be the class of normalized function formulated by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U = \{z : |z| < 1\}. \tag{1}$$

2020 *Mathematics Subject Classification.* 30C45.
Keywords and phrases. Differential operator, unit disk, analytic function, subordination and superordination, univalent function.
rabhaibrahim@tdtu.edu.vn, rabhaibrahim@yahoo.com
0000-0001-9341-025X.

then, we have

$$\begin{aligned} D_\kappa^0 f(z) &= f(z) \\ D_\kappa^1 f(z) &= zf'(z) + \frac{\kappa}{2}\left(f(z) - f(-z) - 2z\right), \quad \kappa \in \mathbb{R} \\ &\vdots \\ D_\kappa^m f(z) &= D_\kappa(D_\kappa^{m-1} f(z)) \\ &= z + \sum_{n=2}^{\infty} [n + \frac{\kappa}{2}(1 + (-1)^{n+1})]^m \, a_n z^n. \end{aligned} \tag{2}$$

Obviously, if we let $\kappa = 0$, we get the Sàlàgean's differential operator [2]. We title D_κ^m the Sàlàgean-difference operator. In addition, D_κ^m is a modified Dunkl operator of complex variables [3] and for recent work [4]. Dunkl operator characterizes a major generalization of partial derivatives and attains the commutative law in $\mathbb{R}^n$. In geometry, it acquires the reflexive relation, which is plotting the space into itself as a set of fixed points.

The Hadamard product or convolution of two power series is denoted by $(*)$ achieving

$$\begin{aligned} f(z) * h(z) &= \Big(z + \sum_{n=2}^{\infty} a_n z^n\Big) * \Big(z + \sum_{n=2}^{\infty} \eta_n z^n\Big) \\ &= z + \sum_{n=2}^{\infty} a_n \eta_n z^n. \end{aligned} \tag{3}$$

Thus, we have

$$D_\kappa^m f(z) = \mathfrak{D}(z) * f(z),$$

where

$$\begin{aligned} \mathfrak{D}(z) :&= z + \sum_{n=2}^{\infty} [n + \frac{\kappa}{2}(1 + (-1)^{n+1})]^m \, z^n \\ &:= z + \sum_{n=2}^{\infty} \mathfrak{N}^m \, z^n \end{aligned}$$

Recall that $f \prec g$ then there exists a Schwarz function $\omega \in U$ such that $\omega(0) = 0, |\omega(z)| < 1$, $z \in U$ satisfying $f(z) = g(\omega(z))$ for all $z \in U$ (see [5]). The inequality $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Furthermore, let $\mathbb{J}(A, B)$ denote the family of all functions φ that are analytic in the open unit disk U with $\varphi(0) = 1$ and achieve

$$\varphi(z) \prec \frac{1 + Az}{1 + Bz}, \; -1 \leq B < A \leq 1.$$

Note that the function $\dfrac{1+Az}{1+Bz}$ is univalent in the open unit disk U.

Recently, Arif et all. [6] introduced anew class of analytic functions along with the concepts of Janowski functions as follows:

Definition 1.1. *If $f \in \Lambda$, then $f \in \mathbb{J}^b(A,B,j)$ if and only if*

$$1+\frac{1}{b}\Big(\frac{2D_0^{j+1}f(z)}{D_0^j f(z)-D_0^j f(-z)}\Big) \prec \frac{1+Az}{1+Bz},$$

$$\Big(z\in U,\ -1\le B<A\le 1,\ j=1,2,...,\ b\in\mathbb{C}\setminus\{0\}\Big),$$

where $D_0^{j+1}f(z)$ is the Sàlàgean's differential operator.

In our study, we shall extend the above class as follows:
If $f\in\Lambda$, then $f\in\mathbb{J}^b_\kappa(A,B,j)$ if and only if

$$1+\frac{1}{b}\Big(\frac{2D_\kappa^{j+1}f(z)}{D_\kappa^j f(z)-D_\kappa^j f(-z)}\Big) \prec \frac{1+Az}{1+Bz},$$

$$\Big(z\in U,\ -1\le B<A\le 1,\ j=1,2,...,\ b\in\mathbb{C}\setminus\{0\},\ \kappa\ge 0\Big),$$

1.1. **Special cases.**

- $\kappa=0 \Longrightarrow$ [6];
- $\kappa=0, B=0 \Longrightarrow$ [7];
- $\kappa=0, A=1, B=-1, b=2 \Longrightarrow$ [8].

Lemma 1.2. *If $P\in\mathbb{J}(A,B)$ then its coefficients satisfy*

$$|p_n|\le (A-B), \quad \forall n\ge 1,$$

where

$$P(z)=1+p_1z+p_2z^2+p_3z^3+...., \quad z\in U.$$

2. Results

Our results based on two expressions involving the differential operator D_κ^m.

2.1. **Special class of the expression $D_\kappa^{j+1}/D_\kappa^j$.** We have our first result as follows:

Theorem 2.1. *If $f\in\mathbb{J}^b_\kappa(A,B,j)$ then the odd function*

$$\mathfrak{O}(z)=\frac{1}{2}[f(z)-f(-z)], \quad z\in U$$

achieves the following inequality

$$1+\frac{1}{b}\Big(\frac{D_\kappa^{j+1}\mathfrak{O}(z)}{D_\kappa^j\mathfrak{O}(z)}-1\Big) \prec \frac{1+Az}{1+Bz},$$

$$\Big(z \in U,\ -1 \le B < A \le 1,\ j = 1, 2, ...,\ b \in \mathbb{C} \setminus \{0\},\ \kappa \ge 0\Big),$$

Proof. Since $f \in \mathbb{J}^b_\kappa(A, B, j)$ then there is a function $P \in \mathbb{J}(A, B)$ such that

$$b(P(z) - 1) = \Big(\frac{2D^{j+1}_\kappa f(z)}{D^j_\kappa f(z) - D^j_\kappa f(-z)}\Big)$$

and

$$b(P(-z) - 1) = \Big(\frac{-2D^{j+1}_\kappa f(-z)}{D^j_\kappa f(z) - D^j_\kappa f(-z)}\Big).$$

This implies that

$$1 + \frac{1}{b}\Big(\frac{D^{j+1}_\kappa \mathfrak{O}(z)}{D^j_\kappa \mathfrak{O}(z)} - 1\Big) = \frac{P(z) + P(-z)}{2}.$$

Also, since

$$P(z) \prec \frac{1 + Az}{1 + Bz}$$

where $\frac{1 + Az}{1 + Bz}$ is univalent then by the definition of the subordination, we obtain

$$1 + \frac{1}{b}\Big(\frac{D^{j+1}_\kappa \mathfrak{O}(z)}{D^j_\kappa \mathfrak{O}(z)} - 1\Big) \prec \frac{1 + Az}{1 + Bz}.$$

□

Note that

- When $\kappa = 0$, we get the following result, which can be found in [6]

$$1 + \frac{1}{b}\Big(\frac{D^{j+1}_0 \mathfrak{O}(z)}{D^j_0 \mathfrak{O}(z)} - 1\Big) \prec \frac{1 + Az}{1 + Bz}.$$

- When $\kappa = 0, b = 1$, we have a result given in [10]

$$\Big(\frac{D^{j+1}_0 f(z)}{D^j_0 f(z)}\Big) \prec \frac{1 + Az}{1 + Bz}.$$

- When $\kappa = 0, b = 1, A = 1 - 2\alpha, -1$ we attain a result given in [10]

$$\Big(\frac{D^{j+1}_0 f(z)}{D^j_0 f(z)}\Big) \prec \frac{1 + (1 - 2\alpha)z}{1 - z}.$$

Corollary 2.2. *If $f \in \mathbb{J}^b_\kappa(A, B, j)$ then the odd function*

$$\mathfrak{O}(z) = \frac{1}{2}[f(z) - f(-z)], \quad z \in U$$

achieves

$$\Re\Big(\frac{z\mathfrak{O}(z)'}{\mathfrak{O}(z)}\Big) \ge \frac{1 - r^2}{1 + r^2}, \quad |z| = r < 1.$$

Proof. In view of Theorem 2.1, the function $\mathfrak{O}(z)$ is starlike in the open unit disk. The subordination concept implies that

$$\frac{z\mathfrak{O}(z)'}{\mathfrak{O}(z)} \prec \frac{1-z^2}{1+z^2}$$

that is, there exists a Schwarz function $\wp \in U, |\wp(z)| \le |z| < 1, \wp(0) = 0$ such that

$$\Phi(z) := \frac{z\mathfrak{O}(z)'}{\mathfrak{O}(z)} \prec \frac{1-\wp(z)^2}{1+\wp(z)^2}$$

which yields that there is $\xi, |\xi| = r < 1$ such that

$$\wp^2(\xi) = \frac{1-\Phi(\xi)}{1+\Phi(\xi)}, \quad \xi \in U.$$

A calculation gives that

$$\Big|\frac{1-\Phi(\xi)}{1+\Phi(\xi)}\Big| = |\wp(\xi)|^2 \le |\xi|^2.$$

Hence, we have the following conclusion

$$\Big|\Phi(\xi) - \frac{1+|\xi|^4}{1-|\xi|^4}\Big|^2 \le \frac{4|\xi|^4}{(1-|\xi|^4)^2}$$

or

$$\Big|\Phi(z) - \frac{1+|\xi|^4}{1-|\xi|^4}\Big| \le \frac{2|\xi|^2}{(1-|\xi|^4)}.$$

This implies that

$$\Re(\Phi(z)) \ge \frac{1-r^2}{1+r^2}, \quad |\xi| = r < 1,$$

which completes the proof. □

Theorem 2.3. *If $f \in \mathbb{J}^b_\kappa(A, B, j)$ then*

$$\Re\Big(D^{j+1}_\kappa f(z)\Big) \ge \Re\Big(\frac{z}{2}(1 + b(P(z) - 1))e^{\psi(z)}\Big),$$

where $P(z) \in \mathbb{J}(A, B)$ and

$$\psi(z) := \frac{b}{2}\int_0^z \frac{(P(\tau) + P(-\tau) - 2)}{\tau} d\tau.$$

Proof. Let $f \in \mathbb{J}^b_\kappa(A, B, j)$. It has been shown in [6], Theorem 5 that

$$D^{j+1}_0 f(z) = \frac{z}{2}(1 + b(P(z) - 1))e^{\psi(z)}.$$

But

$$\Re\Big(D^{j+1}_\kappa f(z)\Big) \ge \Re\Big(D^{j+1}_0 f(z)\Big), \quad \kappa \ge 0,$$

it follows that

$$\Re\Big(D^{j+1}_\kappa f(z)\Big) \ge \Re\Big(\frac{z}{2}(1 + b(P(z) - 1))e^{\psi(z)}\Big).$$

This completes the proof. $\square$

Theorem 2.4. *If $f \in \mathbb{J}_\kappa^b(A,B,j)$ then*

$$\frac{(1-|b|Ar)r+(1-|b|Br^2)}{(1+r^2)(1-Br)} \leq |D_\kappa^{j+1}f(z)| \leq \frac{\big((1+|b|Ar)r+(1-|b|Br^2)\big)}{(1-r^2)(1+Br)}.$$

Proof. Let $f \in \mathbb{J}_\kappa^b(A,B,j)$. This implies the following equality

$$|D_\kappa^{j+1}f(z)| = |\eta(z)|\Big|1+\frac{b(A-B)\omega(z)}{1+B\omega(z)}\Big|, \quad |\omega(z)|<|z|=r<1, \tag{4}$$

where

$$\eta(z) = \frac{1}{2}[D_\kappa^j f(z) - D_\kappa^j f(-z)].$$

The function η is univalent in the open unit disk U thus in view of the Growth Theorem, we have

$$\frac{r}{(1+r)^2} \leq |\eta(z)| \leq \frac{r}{(1-r)^2}, \quad |z|=r<1.$$

Moreover, by taking $|\omega(z)|<|z|=r$, a computation gives the inequality

$$\begin{aligned}\Big|1+\frac{b(A-B)\omega(z)}{1+B\omega(z)}\Big| &= \Big|\frac{(1+B\omega(z))+b(A-B)\omega(z)}{1+B\omega(z)}\Big| \\ &\leq \frac{(1-|b|Ar)+(1-|b|)Br}{1-Br)} \\ &\leq \frac{(1+|b|Ar)+(1-|b|)Br}{1+Br)}.\end{aligned}$$

By employing the last two inequalities in (4), we have the desire result. $\square$

Theorem 2.5. *For $f \in \Lambda$, define the functional*

$$\Psi(z) = \frac{D_\kappa^j f(z)}{z}, \quad z \in U \setminus \{0\}$$

such that $\Re(\Psi(z))>0$. Then

$$\Re\Big(\frac{z\Psi(z)'}{\Psi(z)}\Big) \leq \frac{2r}{1-r^2}, \quad |z|=r<1.$$

Proof. It is clear that $\Psi(0)=1$. According to the condition of the theorem, there exists a Schwarz function $\wp \in U, |\wp(z)| \leq |z| < 1, \wp(0)=0$ such that

$$\Psi(z) = \frac{1+\wp(z)}{1-\wp(z)}.$$

This gives the equality

$$\frac{z\Psi(z)'}{\Psi(z)} = \frac{2z\wp'(z)}{1-\wp(z)^2}.$$

Hence, by utilizing the Schwarz-Pick Theorem, we attain

$$|\frac{z\Psi(z)'}{\Psi(z)}| \le \frac{2|z|\,|\wp'(z)|}{1-|\wp(z)|^2} \le \frac{2r}{1-|\wp(z)|^2}.\frac{1-|\wp(z)|^2}{1-r^2} = \frac{2r}{1-r^2}.$$

□

2.2. **Special class of the expression** $z(D^m_\kappa f(z))'^m_\kappa f(z)$**.** A function $f \in \Lambda$ is said to be in the class $\mathbb{S}^m_\kappa(h)$ if and only if the expression $z(D^m_\kappa f(z))'^m_\kappa f(z)$ takes all values in the conic domain $\Omega := h(U)$, where $h(z)$ is convex univalent then, we can describe the class as follows:

$$\Big(\frac{z(D^m_\kappa f(z))'}{D^m_\kappa f(z)}\Big) \prec h(z), \quad z \in U. \tag{5}$$

Next result shows the upper bound of the operator $D^m_\kappa f(z)$, when $f \in \mathbb{S}^m_\kappa(h)$ and the upper and lower bound of the expression $D^m_\kappa f(z)/z$.

Theorem 2.6. *Let $f \in \mathbb{S}^m_\kappa(h)$, where $h(z)$ is convex univalent function in U. Then*

$$D^m_\kappa f(z) \prec z\exp\Big(\int_0^z \frac{h(\omega(\xi))-1}{\xi}d\xi\Big),$$

where $\omega(z)$ is analytic in $\mathbb{U}$, with $\omega(0)=0$ and $|\omega(z)|<1$. Furthermore, for $|z|=\eta$, $D^m_\kappa f(z)$ achieves the inequality

$$\exp\Big(\int_0^1 \frac{h(\omega(-\eta))-1}{\eta}\Big)d\eta \le \Big|\frac{D^m_\kappa f(z)}{z}\Big| \le \exp\Big(\int_0^1 \frac{h(\omega(\eta))-1}{\eta}\Big)d\eta.$$

Proof. Since $f \in \mathbb{S}^m_\kappa(h)$, we have

$$\Big(\frac{z(D^m_\kappa f(z))'}{D^m_\kappa f(z)}\Big) \prec h(z), \quad z \in \mathbb{U},$$

which means that there exists a Schwarz function with $\omega(0)=0$ and $|\omega(z)|<1$ such that

$$\Big(\frac{z(D^m_\kappa f(z))'}{D^m_\kappa f(z)}\Big) = h(\omega(z)), \quad z \in \mathbb{U},$$

which implies that

$$\Big(\frac{(D^m_\kappa f(z))'}{D^m_\kappa f(z)}\Big) - \frac{1}{z} = \frac{h(\omega(z))-1}{z}.$$

Integrating both sides, we have

$$\log D^m_\kappa f(z) - \log z = \int_0^z \frac{h(\omega(\xi))-1}{\xi}d\xi.$$

Consequently, this yields

$$\log\frac{D^m_\kappa f(z)}{z} = \int_0^z \frac{h(\omega(\xi))-1}{\xi}d\xi. \tag{6}$$

By using the definition of subordination, we get

$$D_\kappa^m f(z) \prec z \exp\Big(\int_0^z \frac{h(\omega(\xi))-1}{\xi} d\xi\Big).$$

In addition, we note that the function $h(z)$ maps the disk $0<|z|<\eta<1$ onto a region which is convex and symmetric with respect to the real axis, that is

$$h(-\eta|z|) \le \Re(h(\omega(\eta z))) \le h(\eta|z|), \quad \eta \in (0,1),$$

which yields the following inequalities:

$$h(-\eta) \le h(-\eta|z|), \quad h(\eta|z|) \le h(\eta)$$

and

$$\int_0^1 \frac{h(\omega(-\eta|z|))-1}{\eta} d\eta \le \Re\Big(\int_0^1 \frac{h(\omega(\eta))-1}{\eta} d\eta\Big) \le \int_0^1 \frac{h(\omega(\eta|z|))-1}{\eta} d\eta.$$

By using the above relations and Eq. (6), we conclude that

$$\int_0^1 \frac{h(\omega(-\eta|z|))-1}{\eta} d\eta \le \log\Big|\frac{D_\kappa^m f(z)}{z}\Big| \le \int_0^1 \frac{h(\omega(\eta|z|))-1}{\eta} d\eta.$$

This equivalence to the inequality

$$\exp\Big(\int_0^1 \frac{h(\omega(-\eta|z|))-1}{\eta} d\eta\Big) \le \Big|\frac{D_\kappa^m f(z)}{z}\Big| \le \exp\Big(\int_0^1 \frac{h(\omega(\eta|z|))-1}{\eta} d\eta\Big).$$

Thus, we obtain

$$\exp\Big(\int_0^1 \frac{h(\omega(-\eta))-1}{\eta}\Big) d\eta \le \Big|\frac{D_\kappa^m f(z)}{z}\Big| \le \exp\Big(\int_0^1 \frac{h(\omega(\eta))-1}{\eta}\Big) d\eta.$$

This completes the proof. □

3. Conclusion

From above, we conclude that the generalized differential operator is used to generate a set of new classes of analytic functions in terms of the Janowski formula. Different cases are recognized for recent efforts. One can develop the above work using another classes of univalent functions such as harmonic, multivalent and meromorphic.

References

[1] Ibrahim, R.W., Darus, M., Subordination inequalities of a new Salagean-difference operator, *International Journal of Mathematics and Computer Science,* 14 (3) (2019), 573–582.

[2] Sàlàgean, G.S., Subclasses of univalent functions, Complex Analysis-Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), *Lecture Notes in Math.,* vol. 1013, Springer, Berlin, (1983), 362–372.

[3] Dunkl, C.F., Differential-difference operators associated with reflections groups, *Trans. Am. Math. Soc.,* 311 (1989), 167–183.

[4] Ibrahim, R. W., New classes of analytic functions determined by a modified differential-difference operator in a complex domain, *Karbala International Journal of Modern Science,* 3 (1) (2017), 53–58.

[5] Miller, S.S., Mocanu, P. T., Differential subordinations: theory and applications, CRC Press, 2000.

[6] Arif, M. et al., A New Class of Analytic Functions Associated with Sàlàgean Operator, *Journal of Function Spaces,* 2019 (2019).

[7] Sakaguchi, K., On a certain univalent mapping, J*ournal of the Mathematical Society of Japan,* 11 (1959), 72–75.

[8] Das, R. N., Singh, P., On subclasses of schlicht mapping, *Indian Journal of Pure and Applied Mathematics,* 8 (1977), 864–872.

[9] Kuroki, K., Owa, S., Some subordination criteria concerning the Salagean operator, *Journal of Inequalities in Pure and Applied Mathematics,* 10 (2) (2009), 1–11.

[10] Noor, K.I. et al., Some applications of certain integral operators involving functions, *Tamkang Journal of Mathematics,* 49 (2018), 25–34.