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# ON SOME NEW INEQUALITIES OF HERMITE HADAMARD TYPES FOR HYPERBOLIC $p$-CONVEX FUNCTIONS

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Abstract. In this paper, we show that the power function $f^n(x)$ is hyperbolic $p$-convex function. Furthermore, we establish some new integral inequalities for higher powers of hyperbolic $p$-convex functions. Also, some applications for special means are provided as well.

## 1. Introduction

Let $f : I \to \mathbb{R}$ be a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. There are many generalizations of the notion of convex functions see [2, 3, 5, 6]. One way to generalize the notion of convex function is to replace linear functions by another family of functions in the sense of Beckenbach [2]. In this paper, we deal with a family of hyperbolic functions

$$H(x) = A \cosh px + B \sinh px,$$

where $A, B$ arbitrary constants and $p \in \mathbb{R} \setminus \{0\}$.

The Hermite-Hadamard integral inequality for convex functions $f : [a, b] \to \mathbb{R}$

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is well known in the literature and has many applications for special means, see for example [7, 8, 9]. This inequality 1 was extended for hyperbolic $p$-convex functions in [1] as

$$\frac{2}{p} f(\frac{a+b}{2}) \sinh p(\frac{b-a}{2}) \leq \int_a^b f(x)dx \leq \frac{1}{p}[f(a)+f(b)] \tanh p(\frac{b-a}{2}).$$

In current work, we proved that the higher powers of f(x) is hyperbolic $p$-convex function in addition to establish some new integral inequalities for higher powers of hyperbolic $p$-convex functions.

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## 2. Definitions and Preliminary Results

In this section, we introduce the basic definitions and results which will be used later. For more informations see [1], [4], [10].

**Definition 1.** *A function $f : I \to \mathbb{R}$ is said to be sub $H$-function on $I$ or hyperbolic $p$-convex function, if for any arbitrary closed subinterval $[u, v]$ of $I$ the graph of $f(x)$ for $x \in [u, v]$ lies nowhere above the function, determined by the equation:*
$H(x) = H(x, u, v, f) = A \cosh px + B \sinh px; \quad p \in \mathbb{R} \setminus \{0\}$
*where $A$ and $B$ are chosen such that $H(u) = f(u)$, and $H(v) = f(v)$.*
*Equivalently, for all $x \in [u, v]$*

$$f(x) \leq H(x) = \frac{f(u) \sinh p(v - x) + f(v) \sinh p(x - u)}{\sinh p(v - u)}. \tag{2}$$

**Remark 2.** *The hyperbolic $p$-convex functions possess a number of properties analogous to those of convex functions. For example: If $f : I \to \mathbb{R}$ is hyperbolic $p$-convex function, then for any $u, v \in I$, the inequality $f(x) \geq H(x)$ holds outside the interval $[u, v]$.*

**Definition 3.** *Let a function $f : I \to \mathbb{R}$ be hyperbolic $p$-convex function*
$S_u(x) = A \cosh px + B \sinh px$
*is said to be supporting function for $f(x)$ at the point $u \in (a, b)$ if*

(1) $S_u(u) = f(u)$
(2) $S_u(x) \leq f(x) \ \forall x \in I$.

*That is, if $f(x)$ and $S_u(x)$ agree at $x = u$ the graph of $f(x)$ does not lie under the support curve.*

**Proposition 4.** *If $f : I \to \mathbb{R}$ is a differentiable hyperbolic $p$-convex function, then the supporting function for $f(x)$ at the point $u \in I$ has the form*

$$S_u(x) = f(u) \cosh p(x - u) + \frac{f'(u)}{p} \sinh p(x - u). \tag{3}$$

**Theorem 5.** *Let $f : I \to \mathbb{R}$ be a two times continuously differentiable function. Then $f$ is hyperbolic $p$-convex function on $I$ if and only if $f''^2 f(x) \geq 0$ for all $x$ in $I$.*

**Example 6.** *Let $f_s : (0, \infty) \to (0, \infty)$, $f_s(x) = x^s$ with $p \in \mathbb{R} \setminus \{0\}$. If $s \in (-\infty, 0) \cup [1, \infty)$ and*

$$f_s''(x) - p^2 f_s(x) = s(s - 1)x^{s-2} - p^2 x^s = p^2 x^{s-2} (\frac{s(s - 1)}{p^2} - x^2).$$

*Then,*

$$f_s''(x) - p^2 f_s(x) \geq 0 \ for \ x \in (0, \frac{\sqrt{s(s - 1)}}{|p|})$$

*Hence, the power function $f_s$ for $s \in (-\infty, 0) \cup [1, \infty)$ is hyperbolic $p$-convex function on $(0, \frac{\sqrt{s(s-1)}}{|p|})$.*

**Theorem 7.** *A function $f : I \to \mathbb{R}$ is hyperbolic $p$-convex function on $I$ if and only if there exist a supporting function for $f(x)$ at each point $x \in I$.*

## 3. Main Results

**Theorem 8.** *Let $f : I \to \mathbb{R}$ be non-negative, two times continuously differentiable and hyperbolic $p$-convex functions then the higher powers of $f(x)$ is hyperbolic $p$-convex function.*

*Proof.* Since, $f(x)$ be non-negative and hyperbolic $p$-convex function, then using Theorem 5, we get

$$f(x) \geq 0 \text{ and } f''^2 f(x) \geq 0 \;\; \forall x \in I. \tag{4}$$

Hence,

$$f''^2 f(x) \geq \frac{p^2}{n} f(x) \;\; \forall n \in \mathbb{N}. \tag{5}$$

$$(f^n(x))'^{n-1}(x) f'(x)$$

$$(f^n(x))''^{n-2}(x)(f'^2 + n f^{n-1}(x) f''(x)$$

$$\begin{aligned} (f^n(x))''^2 f^n(x) &= n(n-1) f^{n-2}(x)(f'^2 + n f^{n-1}(x) f''^2 f^n(x) \\ &= n(n-1) f^{n-2}(x)(f'^2 + n f^{n-1}(x)(f''(x) - \frac{p^2}{n} f(x)). \end{aligned}$$

Now using (4), (5) we conclude that

$$(f^n(x))''^2 f^n(x) \geq 0.$$

Hence, $f^n(x)$ is hyperbolic $p$-convex function $\forall n \in \mathbb{N}$. □

**Theorem 9.** *Let $f : I \to \mathbb{R}$ be a non-negative hyperbolic $p$-convex function, $n \in \mathbb{N}$, and $a, b \in I$ with $a < b$, Then*

$$\int_a^b f^n(x) dx \leq \sinh^{-n} p(b-a) \sum_{r=0}^{n} \frac{1}{\mu} \binom{n}{r} [f(a)]^{n-r} [f(b)]^r [e^{\mu b + \lambda} - e^{\mu a + \lambda}], \tag{6}$$

*where $\lambda = pb(n-r) - arp$, and $\mu = (2r-n)p$.*

*Proof.* Since, $f(x)$ is hyperbolic $p$-convex function, then from Definition 1 we have

$$f(x) \leq H(x) \;\; \forall x \in [a, b].$$

As f(x) is non-negative, we get:

$$f^n(x) \leq H^n(x) \;\; \forall n \in \mathbb{N}$$

Thus, using (2), one obtains

$$\begin{aligned}
&\int_a^b f^n(x)dx \le \int_a^b H^n(x)dx \\
&= \frac{1}{\sinh^n p(b-a)} \int_a^b [f(a)\sinh p(b-x) + f(b)\sinh p(x-a)]^n dx \\
&= \sinh^{-n} p(b-a) \sum_{r=0}^{n} \binom{n}{r} [f(a)]^{n-r}[f(b)]^r \int_a^b \sinh^r p(x-a)\sinh^{n-r} p(b-x)dx \\
&= \sinh^{-n} p(b-a) \sum_{r=0}^{n} \binom{n}{r} [f(a)]^{n-r}[f(b)]^r \\
&\times \int_a^b [\frac{e^{p(x-a)} - e^{-p(x-a)}}{2}]^r [\frac{e^{p(b-x)} - e^{-p(b-x)}}{2}]^{n-r} dx \\
&\le \sinh^{-n} p(b-a) \sum_{r=0}^{n} \binom{n}{r} [f(a)]^{n-r}[f(b)]^r \int_a^b e^{rp(x-a)} e^{p(n-r)(b-x)} dx \\
&= \sinh^{-n} p(b-a) \sum_{r=0}^{n} \binom{n}{r} [f(a)]^{n-r}[f(b)]^r \int_a^b e^{rp(x-a)+p(n-r)(b-x)} dx \\
&= \sinh^{-n} p(b-a) \sum_{r=0}^{n} \binom{n}{r} [f(a)]^{n-r}[f(b)]^r \int_a^b e^{p(2r-n)x+pb(n-r)-arp} dx \\
&= \sinh^{-n} p(b-a) \sum_{r=0}^{n} \frac{1}{\mu}\binom{n}{r} [f(a)]^{n-r}[f(b)]^r [e^{\mu b+\lambda} - e^{\mu a+\lambda}],
\end{aligned}$$

where $\lambda = pb(n-r) - arp$, and $\mu = (2r-n)p$.
Hence, the theorem follows. □

**Theorem 10.** *Let $f : I \to \mathbb{R}$ be a differentiable hyperbolic $p$-convex function, $n \in \mathbb{N}$, and $a, b \in I$ with $a < b$, Then*

$$\int_a^b f^{2n-1}(x)dx \ge \sum_{r=0}^{2n-1} \binom{2n-1}{r} f^{2n-r-1}(a)(\frac{f'(a)}{p})^r \sum_{k=0}^{r} \frac{(-1)^k}{2^r \alpha}\binom{r}{k}[e^{\alpha b+\beta} - e^{\alpha a+\beta}], \tag{7}$$

*where $\alpha = p(r-2k)$, and $\beta = ap(2k-r)$.*

*Proof.* Since, $f(x)$ is hyperbolic $p$-convex function, then from Definition 3, we have

$$f(x) \ge S_a(x) \quad \forall x \in I$$

and consequently,

$$f^{2n-1}(x) \ge S_a^{2n-1}(x) \quad \forall n \in \mathbb{N}$$

Thus, using (3) and $\cosh p(x-a) \geq 1$, one has

$$\int_a^b f^{2n-1}(x)dx \geq \int_a^b S_a^{2n-1}(x)dx$$
$$= \int_a^b [f(a)\cosh p(x-a) + \frac{f'(a)}{p}\sinh p(x-a)]^{2n-1}dx$$
$$= \sum_{r=0}^{2n-1} \binom{2n-1}{r} f^{2n-r-1}(a)(\frac{f'(a)}{p})^r \int_a^b \cosh^{2n-r-1} p(x-a)\sinh^r p(x-a)dx.$$
$$\geq \sum_{r=0}^{2n-1} \binom{2n-1}{r} f^{2n-r-1}(a)(\frac{f'(a)}{p})^r \int_a^b \sinh^r p(x-a)dx$$
$$= \sum_{r=0}^{2n-1} \binom{2n-1}{r} f^{2n-r-1}(a)(\frac{f'(a)}{p})^r \int_a^b [\frac{e^{p(x-a)} - e^{-p(x-a)}}{2}]^r dx$$
$$= \sum_{r=0}^{2n-1} \binom{2n-1}{r} f^{2n-r-1}(a)(\frac{f'(a)}{p})^r \int_a^b \sum_{k=0}^{r} \frac{(-1)^k}{2^r}\binom{r}{k} e^{p(r-k)(x-a)}e^{-pk(x-a)}dx$$
$$= \sum_{r=0}^{2n-1} \binom{2n-1}{r} f^{2n-r-1}(a)(\frac{f'(a)}{p})^r \sum_{k=0}^{r} \frac{(-1)^k}{2^r}\binom{r}{k} \int_a^b e^{p(r-2k)x+ap(2k-r)}$$
$$= \sum_{r=0}^{2n-1} \binom{2n-1}{r} f^{2n-r-1}(a)(\frac{f'(a)}{p})^r \sum_{k=0}^{r} \frac{(-1)^k}{2^r\alpha}\binom{r}{k} [e^{\alpha b+\beta} - e^{\alpha a+\beta}],$$

where $\alpha = p(r-2k)$, and $\beta = ap(2k-r)$.
Hence, the theorem follows. □

**Theorem 11.** *Let $f : [0,\infty) \to \mathbb{R}$ be a differentiable hyperbolic $p$-convex function, $n \in \mathbb{N}$, and $a, b \in [0,\infty]$ with $a < b$. Such that $f(0) > 0, f'(0) > 0$. Then*

$$\int_a^b f^{2n}(x)dx \geq \sum_{r=0}^{2n} \binom{2n}{r} f^{2n-r}(0)(\frac{f'(0)}{p})^r \sum_{k=0}^{r} \frac{(-1)^k}{2^r\gamma}\binom{r}{k}[e^{\gamma b} - e^{\gamma a}],$$

*where, $\gamma = p(r-2k)$.*

*Proof.* Since, $f(x)$ is hyperbolic $p$-convex function, then from Definition 3, we have

$$f(x) \geq S_0(x) \;\; \forall x \in [0,\infty)$$

As $f(0) > 0$ and $f'(0) > 0$,
using Proposition 4, we conclude that $S_0(x) > 0$, $\forall x \in [0,\infty)$ and consequently,

$$f^{2n}(x) \geq S_0^{2n}(x) \;\; \forall n \in \mathbb{N}$$

Thus, using (3) and $\cosh px \geq 1$, one has

$$\int_a^b f^{2n}(x)dx \quad \geq \quad \int_a^b S_0^{2n}(x)dx$$

$$
\begin{aligned}
&= \int_a^b [f(0)\cosh px + \frac{f'(0)}{p}\sinh px]^{2n} dx \\
&= \sum_{r=0}^{2n} \binom{2n}{r} f^{2n-r}(0)(\frac{f'(0)}{p})^r \int_a^b \cosh^{2n-r} px \sinh^r px \, dx \\
&\geq \sum_{r=0}^{2n} \binom{2n}{r} f^{2n-r}(0)(\frac{f'(0)}{p})^r \int_a^b \sinh^r px \, dx \\
&= \sum_{r=0}^{2n} \binom{2n}{r} f^{2n-r}(0)(\frac{f'(0)}{p})^r \int_a^b [\frac{e^{px}-e^{-px}}{2}]^r dx \\
&= \sum_{r=0}^{2n} \binom{2n}{r} f^{2n-r}(0)(\frac{f'(0)}{p})^r \sum_{k=0}^{r} \frac{(-1)^k}{2^r}\binom{r}{k} \int_a^b e^{p(r-k)x} e^{-pkx} dx \\
&= \sum_{r=0}^{2n} \binom{2n}{r} f^{2n-r}(0)(\frac{f'(0)}{p})^r \sum_{k=0}^{r} \frac{(-1)^k}{2^r}\binom{r}{k} \int_a^b e^{p(r-2k)x} dx \\
&= \sum_{r=0}^{2n} \binom{2n}{r} f^{2n-r}(0)(\frac{f'(0)}{p})^r \sum_{k=0}^{r} \frac{(-1)^k}{2^r \gamma}\binom{r}{k} [e^{\gamma b} - e^{\gamma a}],
\end{aligned}
$$

where, $\gamma = p(r-2k)$.
Hence, the theorem follows. □

**Theorem 12.** *Let $f : [0,\infty) \to \mathbb{R}$ be a non-negative differentiable hyperbolic $p$-convex function, $n \in \mathbb{N}$, and $a, b \in [0,\infty)$ with $a < b$. Such that $f'(0) = 0$, then has the following inequalities*

$$
\begin{aligned}
\int_a^b f^{2n}(x)dx &\geq (\frac{f(0)}{2})^{2n}[\binom{2n}{n}(b-a) \\
&+ 2\sum_{r=0}^{n-1} \frac{1}{p(n-r)}\binom{2n}{r} \cosh p(n-r)(b+a) \sinh p(n-r)(b-a)],
\end{aligned}
$$

$$
\begin{aligned}
\int_a^b f^{2n-1}(x)dx &\geq 4(\frac{f(0)}{2})^{2n-1} \sum_{r=0}^{n-1} \frac{2n-1}{pr(2n-2r-1)} \cosh p(n-r-\frac{1}{2})(b+a) \\
&\times \sinh p(n-r-\frac{1}{2})(b-a).
\end{aligned}
$$

*Proof.* Since, $f(x)$ is hyperbolic $p$-convex function, then from Definition 3, we have

$$f(x) \geq S_0(x) \ \ \forall x \in [0,\infty) \tag{8}$$

Since, $f(x)$ is differentiable and $f'(0) = 0$, then from Proposition 4, the supporting function $S_0(x)$ for $f(x)$ at the point $0 \in [0,\infty)$ can be written in the form

$$S_0(x) = f(0)\cosh px. \tag{9}$$

Hence, $S_0(x) \geq 0 \ \forall x \in [0, \infty)$ Thus, using (8), one obtains

$$f^n(x) \geq S_0^n(x) \ \ \forall n \in \mathbb{N} \tag{10}$$

Therefore, from (9) and (10), the following two cases arise,
Case 1.

$$\begin{aligned}
\int_a^b f^{2n}(x)dx &\geq \int_a^b S_0^{2n}(x)dx \\
&= f^{2n}(0)\int_a^b \cosh^{2n} px\ dx \\
&= (\frac{f(0)}{2})^{2n}\int_a^b [\binom{2n}{n} + \sum_{r=0}^{n-1} 2\binom{2n}{r}\cosh 2p(n-r)x]dx \\
&= (\frac{f(0)}{2})^{2n}[\binom{2n}{n}(b-a) \\
&+ 2\sum_{r=0}^{n-1}\frac{1}{p(n-r)}\binom{2n}{r}\cosh p(n-r)(b+a)\sinh p(n-r)(b-a)].
\end{aligned}$$

Case 2.

$$\begin{aligned}
\int_a^b f^{2n-1}(x)dx &\geq \int_a^b S_0^{2n-1}(x)dx \\
&= f^{2n-1}(0)\int_a^b \cosh^{2n-1} px\ dx \\
&= 2(\frac{f(0)}{2})^{2n-1}\int_a^b \sum_{r=0}^{n-1}\binom{2n-1}{r}\cosh p(2n-2r-1)xdx \\
&= 4(\frac{f(0)}{2})^{2n-1}\sum_{r=0}^{n-1}\frac{2n-1}{pr(2n-2r-1)}\cosh p(n-r-\frac{1}{2})(b+a) \\
&\times \sinh p(n-r-\frac{1}{2})(b-a).
\end{aligned}$$

□

**Theorem 13.** *Let $f : [0, \infty) \to \mathbb{R}$ be an increasing differentiable hyperbolic p-convex function, $n \in \mathbb{N}$, and $a, b \in [0, \infty)$ with $a < b$. Such that $f(0) = 0$, then has the following inequalities*

$$\begin{aligned}
\int_a^b f^{2n}(x)dx &\geq (\frac{f(0)}{2})^{2n}[\binom{2n}{n}(b-a) \\
&+ 2\sum_{r=0}^{n-1}\frac{1}{p(n-r)}\binom{2n}{r}\cosh p(n-r)(b+a)\sinh p(n-r)(b-a)],
\end{aligned}$$

$$\int_a^b f^{2n-1}(x)dx \geq 4(\frac{f(0)}{2})^{2n-1}\sum_{r=0}^{n-1}\frac{1}{p(2n-2r-1)}\frac{2n-1}{r}\cosh p(n-r-\frac{1}{2}) \times (b+a)\sinh p(n-r-\frac{1}{2})(b-a).$$

*Proof.* Since, $f(x)$ is hyperbolic $p$-convex function, then from Definition 3, we have

$$f(x) \geq S_0(x) \ \ \forall x \in [0,\infty). \tag{11}$$

Since, $f(x)$ is increasing, then $f'(0) \geq 0$. Since, $f(x)$ is differentiable and $f(0)=0$, then from Proposition 4, the supporting function $S_0(x)$ for $f(x)$ at the point $0 \in [0,\infty)$ can be written in the form

$$S_0(x) = \frac{f'(0)}{p}\sinh px. \tag{12}$$

Hence, $S_0(x) \geq 0 \ \forall x \in [0,\infty)$ Thus, using (11), one obtains

$$f^n(x) \geq S_0^n(x) \ \ \forall n \in \mathbb{N} \tag{13}$$

Therefore, from (12) and (13), the following two cases arise,
Case 1.

$$\int_a^b f^{2n}(x)dx \geq \int_a^b S_0^{2n}(x)dx$$
$$= (\frac{f'(0)}{p})^{2n}\int_a^b \sinh^{2n} px dx$$
$$= (\frac{f'(0)}{2p})^{2n}(-1)^n\int_a^b [\binom{2n}{n} + \sum_{r=0}^{n-1} 2(-1)^{n-r}\binom{2n}{r}\cosh 2(n-r)px]dx$$
$$= (\frac{f'(0)}{2p})^{2n}(-1)^n[\binom{2n}{n}(b-a) + 4\sum_{r=0}^{n-1}\frac{(-1)^{n-r}}{2p(n-r)}\binom{2n}{r}\cosh p(n-r)(b+a)$$
$$\times \sinh p(n-r)(b-a)].$$

Case 2.

$$\int_a^b f^{2n-1}(x)dx \geq \int_a^b S_0^{2n-1}(x)dx$$
$$= (\frac{f'(0)}{p})^{2n-1}\int_a^b \sinh^{2n-1} px \ dx$$
$$= 2(\frac{f'(0)}{2P})^{2n-1}(-1)^{n-1}\int_a^b\sum_{r=0}^{n-1}(-1)^{n+r-1}\binom{2n-1}{r}\sinh p(2n-2r-1)x \ dx$$
$$= 4(\frac{f(0)}{2})^{2n-1}\sum_{r=0}^{n-1}\frac{1}{p(2n-2r-1)}\frac{2n-1}{r}\cosh p(n-r-\frac{1}{2})$$

$$\times(b+a)\sinh p(n-r-\frac{1}{2})(b-a).$$

□

**Remark 14.** *For the hyperbolic expansions in Theorems 12, 13 one can refer to* [11].

## 4. Some applications for special means

Recall the following special means

(1) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}, \quad a,b \geq 0;$$

(2) The geometric mean

$$G = G(a,b) := \sqrt{ab}, \quad a,b \geq 0;$$

(3) The harmonic mean

$$H = H(a,b) := \frac{2ab}{a+b}, \quad a,b \geq 0;$$

(4) The Logarithmic mean

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}, \quad a,b \geq 0, a \neq b;$$

(5) The Identic mean

$$I = I(a,b) = \frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}}, \quad a,b \geq 0, a \neq b;$$

(6) The m-Logarithmic mean

$$L_m = L_m(a,b) := (\frac{b^{m+1}-a^{m+1}}{(m+1)(b-a)})^{\frac{1}{m}}, \; a,b \geq 0, a \neq b;$$

where, $m \in \mathbb{R} \setminus \{-1,0\}$
it is well known that $L_m$ is monotonic nondecreasing over $m \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$.

**Proposition 15.** *Let $0 < a < b$ and $m \in \mathbb{R} \setminus \{-1,0\}$. Then, we have the following inequality*

$$(b-a)L_m^m(a,b) \leq \sinh^{-n} p(b-a) \sum_{r=0}^{n} \frac{1}{\mu} \binom{n}{r} a^{s(n-r)} b^{sr} [e^{\mu b+\lambda} - e^{\mu a+\lambda}],$$

*Proof.* The assertion follows from inequality (6) in Theorem 9, for $f_s : (0,\infty) \to (0,\infty)$, $f_s(x) = x^s$ in Example 6 provided $[a,b] \subseteq (0, \frac{\sqrt{s(s-1)}}{|p|})$, $p \neq 0$ and $m = sn$. □

**Proposition 16.** *Let $0 < a < b$ and $w \in \mathbb{R} \setminus \{-1, 0\}$. Then we have the following inequality*

$$(b-a)L_w^w(a,b) \geq \sum_{r=0}^{2n-1} \binom{2n-1}{r} a^{s(2n-r-1)} (\frac{sa^{s-1}}{p})^r \sum_{k=0}^{r} \frac{(-1)^k}{2^r \alpha} \binom{r}{k} [e^{\alpha b+\beta} - e^{\alpha a+\beta}],$$

*Proof.* The assertion follows from inequality (7) in Theorem 10, for $f_s : (0, \infty) \to (0, \infty)$, $f_s(x) = x^s$ in Example 6 provided $[a, b] \subseteq (0, \frac{\sqrt{s(s-1)}}{|p|})$, $p \neq 0$ and $w = s(2n-1)$. □

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