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Common fixed and coincidence point theorems for maps in Menger space with Hadzic type t - norm

K.P.R.Rao* , K.Bhanu Lakshmi† and N.Srinivasa Rao‡

Abstract

In this paper, we obtain a unique common fixed point theorem for two weakly compatible mappings in a Menger space and also obtain a common coincidence point theorem for two hybrid pairs of mappings.

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1. Introduction and preliminaries

In 1942, Menger [6] introduced the notion of a statistical metric space as a generalization of a metric space (M, d) in which the distance $d(x, y)$, $(x, y \in M)$ between x and y is replaced by a distribution function $F_{x,y}$. Schweizer and Sklar [9] studied this concept and established some fundamental results on this space . First , we give some known preliminaries.

1.1. Definition. . A mapping $F : R \to [0, 1]$ is said to be a distribution function if

(i) F is non-decreasing,
(ii) F is left continuous,
(iii) $\inf_{x\in R} F(x) = 0$ and $\sup_{x\in R} F(x) = 1$.

We denote the set of all distribution functions by $\mathbb{D}$.

*Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar -521 510, A.P., INDIA.
Email: kprrao2004@yahoo.com
†Department of Science and Humanities, Lakireddy Balireddy College of Engineering, Mylavaram -522 230, A.P., INDIA.
Email:bhanulaks@gmail.com
‡Department of Science and Humanities, Vignan University,Vadlamudi-522 213, Guntur Dt., A.P., INDIA.
Email:srinivasunimmala@yahoo.co.in

1.2. Definition. ([9]). A probabilistic metric space is an ordered pair (M, F), where M is a non empty set and F is a function defined on $M \times M$ to $\mathbb{D}$ which satisfies the following conditions: For $x, y, z \in M$,

(i) $F_{x,\ y}(0) = 0$,
(ii) $F_{x,\ y}(s) = 1$ for all $s > 0$ if and only if $x = y$,
(iii) $F_{x,\ y}(s) = F_{y,\ x}(s)$ for all $s \in R$ and
(iv) $F_{x,\ y}(s_1) = 1$ and $F_{y,\ z}(s_2) = 1$ for all $s_1, s_2 > 0$ imply
$F_{x,\ z}(s_1 + s_2) = 1$.

1.3. Definition. ([9]). A function $t : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a triangular norm or t - norm if it satisfies the following conditions: For $a, b, c, d \in [0, 1]$,

(i) $t(a, 1) = a$,
(ii) $t(a, b) = t(b, a)$,
(iii) $t(c, d) \geq t(a, b)$ if $c \geq a$ and $d \geq b$,
(iv) $t(t(a, b), c) = t(a, t(b, c))$.

1.4. Definition. ([9]). Let M be a nonempty set, ' t ' is a t - norm and $F : M \times M \to \mathbb{D}$ satisfy:

(i) $F_{x,y}(0) = 0$ for all $x, y \in M$,
(ii) $F_{x,y}(s) = 1$ for all $s > 0$ if and only if $x = y$,
(iii) $F_{x,y}(s) = F_{y,x}(s)$ for all $s \in R$ and
(iv) $F_{x,y}(u + v) \geq t(F_{x,z}(u), F_{z,y}(v))$ for all $u, v \geq 0$ and $x, y, z \in M$.

Then the triplet (M, F, t) is called a Menger space.

1.5. Remark. If (M, d) is a metric space then ' d ' induces a mapping $F : M \times M \to \mathbb{D}$, where F is defined by $F_{p,q}(x) = H(x - d(p, q))$, where $H(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0 \end{cases}$ is the Heaviside function.

Further, if $t : [0, 1] \times [0, 1] \to [0, 1]$ is defined by $t(a, b) = \min\{a, b\}$, then (M, F, t) is a Menger space. It is complete if the metric space (M, d) is complete.

1.6. Definition. ([9]). Let (M, F, t) be a Menger space. Let $x \in M$. For $\epsilon > 0$ and $0 < \lambda < 1$, the (ϵ, λ) - neighbourhood of x is defined as
$N_x(\epsilon, \lambda) = \{y \in M : F_{x,y}(\epsilon) > 1 - \lambda\}$.

The topology induced by the family $\{N_p(\epsilon, \lambda) : p \in M, \epsilon > 0, 0 < \lambda < 1\}$ is known as the (ϵ, λ)- topology.

1.7. Proposition. ([9]). If t is continuous then (ϵ, λ)- topology is a Hausdorff topology on M.

1.8. Definition. ([9]). Let (M, F, t) be a Menger space. A sequence $\{x_n\}$ in M converges to $x \in M$, if for any $\epsilon > 0$ and $0 < \lambda < 1$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \geq N$. A sequence $\{x_n\}$ in (M, F, t) is said to be Cauchy sequence in M if for $\epsilon > 0$ and $0 < \lambda < 1$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n, m \geq N$. A Menger space (M, F, t), where t is continuous, is said to be complete if every Cauchy sequence in M is convergent in (ϵ, λ) - topology.

In 1972, Sehgal and Reid [10] introduced the notion of contraction mapping on a probabilistic metric space and proved fixed point theorems for such mappings.

1.9. Definition. ([10]). Let (M, F, t) be a Menger space. A map $T : M \to M$ is said to be a contraction mapping if there exists a constant $0 < p < 1$ such that $F_{Tx,Ty}(s) \geq F_{x,y}(\frac{s}{p})$ for each $x, y \in M$ and for all $s > 0$.

1.10. Theorem. ([10]). Let (M, F, t) be a complete Menger space, where 't' is a continuous function satisfying $t(x, x) \geq x$ for each $x \in [0, 1]$. If $T : M \to M$ is a contraction mapping then there is a unique $p \in M$ such that $Tp = p$. Moreover $T^n q \to p$ for each $q \in M$.

In 1978, Hadzic [4] introduced a class $\mathcal{F}$ of t- norms $t \neq t_{\min}$, for which every contraction in a complete Menger space (M, F, t) has a fixed point.

1.11. Definition. ([4]). We say that the t - norm t is of Hadzic - type and we write $t \in \mathcal{F}$ if the family $\{t^n\}_{n \in N}$ of it's iterates defined, for each $x \in [0, 1]$ by $t^0(x) = 1$ and $t^{n+1}(x) = t(t^n(x), x)$ for all $n \geq 0$ is equicontinuous at $x = 1$.

i.e., for each $\epsilon \in (0, 1)$, there exists $\delta \in (0, 1)$ such that $x > 1 - \delta$ implies $t^n x > 1 - \epsilon$ for all $n \geq 1$.

1.12. Theorem. ([4]). Let (M, F, t) be a complete Menger space, where 't' is a continuous t - norm of Hadzic type. If $T : M \to M$ is a contraction mapping then there is a unique $p \in M$ such that $Tp = p$. Moreover $T^n q \to p$ for each $q \in M$.

Recently Choudhury and Das [1], proved the following

1.13. Theorem. ([1]). Let (M, F, t_M) be a complete Menger space with continuous t-norm t_M given by $t_M(a, b) = \min\{a, b\}$ and $f : M \to M$ be satisfying $F_{fx,fy}(\varphi(s)) \geq F_{x,y}(\varphi(\frac{s}{c}))$ for all $x, y \in M$ and for $s \geq 0$, where $0 < c < 1$ and $\varphi : R \to R^+$ satisfies

(i) $\varphi(t) = 0$ iff $t = 0$,
(ii) $\varphi(t)$ is increasing and $\varphi(t) \to \infty$ as $t \to \infty$,
(iii) φ is left continuous on $(0, \infty)$,
(iv) φ is continuous at 0 .

Then f has a unique fixed point in M.

Later several authors obtained fixed point theorems in Menger spaces using an altering distance function,for example refer [2],[3],[7]etc.
Sastry et.al. [8] , defined altering function of type (S) as follows :

1.14. Definition. ([8]) A function $\varphi : R^+ \to R^+$ is said to be an altering distance function of type (S) if it satisfies

(i) $\varphi(t) = 0$ iff $t = 0$,
(ii) $\varphi(t) \to \infty$ as $t \to \infty$,
(iii) φ is continuous at 0 .

1.15. Lemma. ([8]) Let (M, F, t) be a Menger space with a continuous Hadzic type t - norm , $0 < c < 1$ and φ be an altering distance function of type (S). Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence in M such that for any $r > 0$, $F_{x_n,x_{n+1}}(\varphi(r)) \geq F_{x_0,x_1}(\varphi(\frac{r}{c^n}))$. Then $\{x_n\}$ is a Cauchy sequence.

1.16. Theorem. ([8]) Let (M, F, t) be a complete Menger space with a continuous Hadzic type t - norm 't' and φ be an altering distance function of type (S), $P : M \to M$ be satisfying $F_{Px,Py}(\varphi(s)) \geq F_{x,y}(\varphi(\frac{s}{c}))$ for all $x, y \in M$ and for $s > 0$ and $0 < c < 1$. Then P has a unique fixed point $z \in M$. Moreover, $P^n x \to z$ for each $x \in M$.

1.17. Definition. ([5]) A pair of self mappings is called weakly compatible if they commute at their coincidence points.

In this paper, we extend Theorem 1.16 for two pairs of weakly compatible mappings.

2. Main results

2.1. Theorem. Let (M, F, t) be a Menger space with continuous Hadzic type t-norm 't' and φ be an altering distance function of type (S). Let $P, Q, f, g : M \to M$ be maps such that

(2.1.1) $F_{Px,Qy}(\varphi(s)) \geq F_{fx,gy}(\varphi(\frac{s}{c}))$ for all $x, y \in M$ and for $s > 0$ and $0 < c < 1$.
(2.1.2) $P(M) \subseteq g(M)$, $Q(M) \subseteq f(M)$,
(2.1.3) either $f(M)$ or $g(M)$ is complete,
(2.1.4) the pairs (f, P) and (g, Q) are weakly compatible .

Then f, g, P and Q have a unique common fixed point in M .

Proof. Let $x_0 \in M$.
Since $P(M) \subseteq g(M)$, there exists $x_1 \in M$ such that $y_1 = gx_1 = Px_0$.
Since $Q(M) \subseteq f(M)$, there exists $x_2 \in M$ such that $y_2 = fx_2 = Qx_1$.
Continuing in this way, we get sequences $\{x_n\}$ and $\{y_n\}$ in M such that $y_{2n+1} = gx_{2n+1} = Px_{2n}$ and $y_{2n+2} = fx_{2n+2} = Qx_{2n+1}$, $n = 0, 1, 2 \cdots$
Since φ is continuous at 0 and vanishes only at 0, it follows that for given $s > 0$ there exists $r > 0$ such that $\frac{s}{2} > \varphi(r)$. Now

$$\begin{aligned} F_{y_{2n+1},\ y_{2n+2}}(s) &\geq F_{y_{2n+1},\ y_{2n+2}}(\varphi(r)), \\ &= F_{Px_{2n},\ Qx_{2n+1}}(\varphi(r)), \\ &\geq F_{fx_{2n},\ gx_{2n+1}}(\varphi(\tfrac{r}{c})), \\ &= F_{y_{2n},\ y_{2n+1}}(\varphi(\tfrac{r}{c})) \end{aligned}$$

Similarly,

$$F_{y_{2n+1},\ y_{2n}}(s) \geq F_{y_{2n},\ y_{2n-1}}(\varphi(\frac{r}{c})).$$

Thus

$$F_{y_{n+1},\ y_n}(s) \geq F_{y_n,\ y_{n-1}}(\varphi(\frac{r}{c})) \geq \cdots \geq F_{y_1,\ y_0}(\varphi(\frac{r}{c^n})).$$

Hence from Lemma 1.15 , $\{y_n\}$ is Cauchy.
Suppose $g(M)$ is complete.
Then there exist $z, v \in M$ such that $y_{2n+1} = gx_{2n+1} \to z = gv$.
Since $\{y_n\}$ is Cauchy , we have $y_n \to z$.
Again for given $s > 0$ there exists $r > 0$ such that $\frac{s}{2} > \varphi(r)$. Now,

$$\begin{aligned} F_{z,\ Qv}(s) &\geq t(F_{z,\ y_{2n+1}}(\varphi(r)), F_{y_{2n+1},\ Qv}(s - \varphi(r))), \\ &\geq t(F_{z,\ y_{2n+1}}(\varphi(r)), F_{Px_{2n},\ Qv}(\varphi(r))), \\ &\geq t(F_{z,\ y_{2n+1}}(\varphi(r)), F_{fx_{2n},\ gv}(\varphi(\tfrac{r}{c}))), \quad \text{from} \quad (2.1.1) \\ &= t(F_{z,\ y_{2n+1}}(\varphi(r)), F_{y_{2n},\ gv}(\varphi(\tfrac{r}{c}))), \\ &\to 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

since $y_n \to z$ and t is a continuous Hadzic type t-norm .
Thus $z = Qv$. Hence

(2.1) $\quad gv = z = Qv.$

Since $z = Qv \in Q(M) \subseteq f(M)$, there exists $u \in M$ such that

(2.2) $\quad z = fu.$

Now

$$F_{Pu,\ z}(\varphi(s)) = F_{Pu,\ Qv}(\varphi(s)) \geq F_{fu,\ gv}(\varphi(\frac{s}{c})) = F_{z,\ z}(\varphi(\frac{s}{c})) = 1.$$

Thus $Pu = z$. Hence

(2.3) $\quad Pu = z = fu.$

Since (f, P) is weakly compatible and from (2.3), we have $Pz = fz$.
Now from (2.1.1), we have
$F_{Pz,\ z}(\varphi(s)) = F_{Pz,\ Qv}(\varphi(s)) \geq F_{fz,\ gv}(\varphi(\frac{s}{c})) = F_{Pz,\ z}(\varphi(\frac{s}{c}))$

$$\geq F_{Pz,\ z}(\varphi(\frac{s}{c^2})) \cdots \geq F_{Pz,\ z}(\varphi(\frac{s}{c^n})) \to 1 \quad \text{as} \quad n \to \infty.$$

Thus $Pz = z$. Hence

(2.4) $\quad z = Pz = fz.$

Since (g, Q) is weakly compatible, from(2.1), we have $gz = Qz$.
From (2.1.1), we have
$F_{z,\ Qz}(\varphi(s)) = F_{Pu,\ Qz}(\varphi(s)) \geq F_{fu,\ gz}(\varphi(\frac{s}{c})) = F_{z,\ Qz}(\varphi(\frac{s}{c}))$

$$\geq F_{z,\ Qz}(\varphi(\frac{s}{c^2})) \cdots \geq F_{z,\ Qz}(\varphi(\frac{s}{c^n})) \to 1 \quad \text{as} \quad n \to \infty.$$

Thus $Qz = z$. Hence

(2.5) $\quad z = Qz = gz.$

From (2.4) and (2.5), z is a common fixed point of P, Q, f and g.
Suppose z' is another common fixed point of P, Q, f and g. Then
From (2.1.1), we have
$F_{z,\ z'}(\varphi(s)) = F_{Pz,\ Qz'}(\varphi(s)) \geq F_{fz,\ gz'}(\varphi(\frac{s}{c})) = F_{z,\ z'}(\varphi(\frac{s}{c}))$

$$\geq F_{z,\ z'}(\varphi(\frac{s}{c^2})) \cdots \geq F_{z,\ z'}(\varphi(\frac{s}{c^n})) \to 1 \quad \text{as} \quad n \to \infty.$$

Thus $z' = z$. Hence z is the unique common fixed point of P, Q, f and g.
Similarly the theorem holds when $f(M)$ is complete. □

Recently, Sastry et.al. [8] proved the following theorem for a multivalued map in a complete Menger space with Hadzic type t-norm.

2.2. Theorem. ([8]). Let (M, F, t) be a complete Menger space with a continuous Hadzic type t - norm 't' , φ be an altering distance function of type (S) and P be a multivalued map of M into the class of nonempty subsets of M. Suppose that there exists $0 < c < 1$ such that for any $x, y \in M$, $F_{u,\ v}(\varphi(s)) \geq F_{x,\ y}(\varphi(\frac{s}{c}))$ for all $s > 0$, whenever $u \in Px$, $v \in Py$.
Then P has a unique fixed point $z \in M$ and $Pz = \{z\}$.

Now we extend this theorem for two pairs of hybrid mappings.

2.3. Definition. Let (M, F, t) be a Menger space and $f : M \to M$, P be a multi valued map of M into the class of nonempty subsets of M. Then f is said to be P- weakly commuting at $x \in M$ if $f^2x \in Pfx$.

2.4. Theorem. Let (M, F, t) be a Menger space with a continuous Hadzic type t - norm 't' and φ be an altering distance function of type (S). Let P and Q be multivalued maps of M into the class of nonempty subsets of M and f and g be self maps on M. Suppose that there exists $0 < c < 1$ such that for any $x, y \in M$,

(2.4.1) $F_{u,\ v}(\varphi(s)) \geq F_{fx,\ gy}(\varphi(\frac{s}{c}))$ for all $s > 0$, whenever $u \in Px$, $v \in Qy$.
(2.4.2) $P(M) \subseteq g(M), \quad Q(M) \subseteq f(M)$,
(2.4.3) either $f(M)$ or $g(M)$ is complete,
(2.4.4) f is P-weakly commuting and g is Q-weakly commuting at their coincidence points.

Then the pairs (f, P) and (g, Q) have a common coincidence point in M.

Proof. Let $x_0 \in M$.
Since $P(x_0) \subseteq g(M)$, there exists $x_1 \in M$ such that $y_1 = gx_1 \in Px_0$.
Since $Q(x_1) \subseteq f(M)$, there exists $x_2 \in M$ such that $y_2 = fx_2 \in Qx_1$.
Continuing in this way, we get sequences $\{x_n\}$ and $\{y_n\}$ in M such that $y_{2n+1} = gx_{2n+1} \in Px_{2n}$ and $y_{2n+2} = fx_{2n+2} \in Qx_{2n+1}$, $n = 0, 1, 2 \cdots$

$$\begin{aligned} F_{y_{2n+1},\ y_{2n+2}}(\varphi(s)) &\geq F_{fx_{2n},\ gx_{2n+1}}(\varphi(\tfrac{s}{c})), \\ &= F_{y_{2n},\ y_{2n+1}}(\varphi(\tfrac{s}{c})) \end{aligned}$$

Similarly,

$$F_{y_{2n},\ y_{2n+1}}(\varphi(s)) \geq F_{y_{2n-1},\ y_{2n}}(\varphi(\frac{s}{c})).$$

Thus

$$F_{y_n,\ y_{n+1}}(\varphi(s)) \geq F_{y_{n-1},\ y_n}(\varphi(\frac{s}{c}))$$

Since φ is continuous at 0 and vanishes only at 0, it follows that for given $s > 0$ there exists $r > 0$ such that $\frac{s}{2} > \varphi(r)$. Now

$$F_{y_n,\ y_{n+1}}(s) \geq F_{y_n,\ y_{n+1}}(\varphi(r)) \geq F_{y_{n-1},\ y_n}(\varphi(\frac{r}{c})) \geq \cdots \geq F_{y_0,\ y_1}(\varphi(\frac{r}{c^n})).$$

Hence from Lemma 1.15 , $\{y_n\}$ is Cauchy sequence in M.
Suppose $f(M)$ is complete.
Then there exist $z, p \in M$ such that $y_n \to z = fp$.
Let $z_1 \in Pp$. Since $y_{2n+2} = fx_{2n+2} \in Qx_{2n+1}$, from (2.4.1),we have

$$\begin{aligned} F_{fp,\ z_1}(s) &\geq t(F_{fp,\ fx_{2n+2}}(\varphi(r)), F_{fx_{2n+2},\ z_1}(s - \varphi(r))), \\ &\geq t(F_{z,\ y_{2n+2}}(\varphi(r)), F_{fx_{2n+2},\ z_1}(\varphi(r))), \\ &\geq t(F_{z,\ y_{2n+2}}(\varphi(r)), F_{fp,\ gx_{2n+1}}(\varphi(\tfrac{r}{c}))), \\ &= t(F_{z,\ y_{2n+2}}(\varphi(r)), F_{z,\ y_{2n+1}}(\varphi(\tfrac{r}{c}))), \\ &\to 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

since $y_n \to z$ and t is a continuous Hadzic type t-norm .
Thus $F_{fp,\ z_1}(s) = 1$ for $s > 0$ so that $fp = z_1$. Thus

(2.6) $\quad fp \in Pp.$

Since $z = fp \in Pp \subseteq g(M)$, there exists $q \in M$ such that $z = fp = gq$.
Let $z_2 \in Qq$. Since $y_{2n+1} = gx_{2n+1} \in Px_{2n}$, from (2.4.1), we have

$$\begin{aligned} F_{gq,\ z_2}(s) &\geq t(F_{gq,\ gx_{2n+1}}(\varphi(r)), F_{gx_{2n+1},\ z_2}(s - \varphi(r))), \\ &\geq t(F_{z,\ y_{2n+1}}(\varphi(r)), F_{gx_{2n+1},\ z_2}(\varphi(r))), \\ &\geq t(F_{z,\ y_{2n+1}}(\varphi(r)), F_{fx_{2n},\ gq}(\varphi(\tfrac{r}{c}))), \\ &= t(F_{z,\ y_{2n+1}}(\varphi(r)), F_{y_{2n},\ z}(\varphi(\tfrac{r}{c}))), \\ &\to 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

since $y_n \to z$ and t is a continuous Hadzic type t-norm .
Thus $F_{gq,\ z_2}(s) = 1$ for $s > 0$ so that $gq = z_2$. Thus

(2.7) $\quad gq \in Qq.$

From (2.6) and (2.7), p is a coincidence point of f and P ; q is a concidence point of g and Q.
From (2.4.4), $fz \in Pz$ and $gz \in Qz$. Thus z is a common coincidence point of the hybrid pairs (f, P) and (g, Q).

□

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