Hacettepe Journal of Mathematics and Statistics
Volume 43 (6) (2014), 915 – 922

FG-morphisms and FG-extensions

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Abstract

We investigate the relations between Fan-Gottesman compactification and categories. We deal with maps having an extension to a homeomorphism between the Fan-Gottesman compactification of their domains and ranges.

2000 AMS Classification: 54D35, 18A40.

Keywords: Fan-Gottesman compactification, categories.

Received 31 : 05 : 2012 : Accepted 25 : 10 : 2013 Doi : 10.15672/HJMS.2014437526

The first section of this paper contains some preliminaries about categories. Category theory provides the language and mathematical foundations for discussing properties of large classes of mathematical objects such as the class of all sets or all groups while circumventing problems such as Russell's paradox. In fact S.Eilenberg and S. MacLane [10,11] give a lot of informations about categories and functors. Category theory has also played a foundational role for formalizing new concepts such as schemes which are fundamental to major areas of contemporary research. Pioneering work of this nature was done by A.Grothendieck [7], K. Morita [12,13,14,15] and others.

The second section of this paper contains some preliminaries about the Fan-Gottesman compactification. In 1952, Ky Fan and Noel Gottesman defined a compactification that is similar to the Wallman compactification, introduced by Henry Wallman in 1938 [17], and afterwards called Fan-Gottesman compactification of regular spaces with a normal base [5]. We investigated the relations between the Fan-Gottesman and Wallman compactification and showed that Fan-Gottesman compactification of some specific and interesting spaces such as normal A_2 and T_4 is Wallman-type compactification [4]. In this section we show that Fan-Gottesman compactification can be obtained via base consisting of open ultrafilters.

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In [9], Herrlich has stated that it is of interest to determine if the Wallman compactification may be regarded as a functor, especially as an epireflection functor, on a suitable category of spaces. This problem was solved affirmatively by Harris in [8].

In [3], Belaid and Echi characterize when Wallman extensions of maps are homeomorphisms.

The third section of this paper, we define $FG-$morphism and $FG-$extension. Let X, Y be two T_3 spaces and $q : X \to Y$ a continuous map. An $FG-$extension of q is a continuous map $F(q) : FX \to FY$ such that $F(q) \circ f_X = f_Y \circ q$, where FX is the Fan-Gottesman compactification of X and $f_X : X \to FX$ is the canonical embedding of X into its Fan-Gottesman compactification FX. We will characterize when Fan-Gottesman extensions of maps are homeomorphisms.

1. Categories

A category C consist of a certain collection of object $Ob(C)$ and for any two object $b, c \in Ob(C)$, there is a set $morph(b, c)$ of morphism (function) between b and c. This collection may be empty, but an identity morphism 1_b must be contained in $morph(b, b)$. Furthermore if there are morphism $morph(b, c)$ and morphism $morph(a, b)$, then their composition must be in $morph(a, c)$. Given two categories C and D, then a map can be defined between these, the so called functor, $F : C \to D$. A functor send object of C to object of D and morphism in C to morphism in D subject to certain condition. Furthermore, it is possible to define maps between functors, the so called natural transformation [11].

One usually requires the morphisms to preserve the mathematical structure of the objects. So if the objects are all groups, a good choice for a morphism would be a group homomorphism. Similarly, for vector spaces, one would choose linear maps, and for differentiable manifolds, one would choose differentiable maps.

In the category of topological spaces, morphisms are usually continuous maps between topological spaces. However, there are also other category structures having topological spaces as objects, but they are not nearly as important as the "standard" category of topological spaces and continuous maps.

We denote by Top the category of topological spaces with continuous maps as morphisms, and by Top_i the full subcategory of Top whose objects are the T_i spaces. There are several ways to generalize the usual separation properties T_0, T_1, T_2, T_3 and T_4 of topology to topological categories [1,2]. All the above categories are full reflective subcategories of Top. There is a universal T_i -space for every topological space X, we denote it by $\mathbf{T}_i(X)$. The assignment $X \to \mathbf{T}_i(X)$ defines a functor $\mathbf{T}_i$ from Top onto Top_i, which is a left adjoint functor of the inclusion functor $Top_i \to Top$.

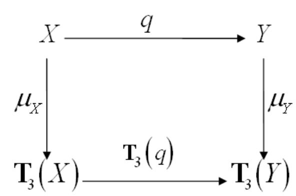
It is recalled that a continuous map $q : Y \to Z$ is said to be a *quasihomeomorphism*, if $U \to q^{-1}(U)$ defines a bijection $O(Z) \to O(Y)$ [7], where $O(Y)$ is the set of all open subsets of the space Y. If Z is T_2 space and, q is not onto, thus q is not a quasihomeomorphism. As showed by the open sets Z, $Z \backslash \{z\}$ for some $z \in Z$. On the other hand, if Z is $\mathbb{R}$, with open sets $\{(-\infty, c) : c \in (-\infty, \infty]\}$ and Y is its subspace $\mathbb{Q}$, then the embedding is a quasihomeomorphism. A subset S of a topological space X is said to be strongly dense in X, if S meets every nonempty locally closed subset of X [9]. In here, locally closed means that every point x of S has a neighbourhood such that $V_x \cap S$ is a closed subset of V_x. In other words, S is locally closed if and only if $S = O \cap F$ for some open subset O of X and some closed subset F of X. In addition, one most evident definition is equivalent to closedness. Thus, a subset S of X is strongly dense if and only if the canonical injection $S \to X$ is a quasihomeomorphism. Besides, a continuous map

$q : X \rightarrow Y$ is a quasihomeomorphism if and only if the topology of X is the inverse image of Y by q and the subset $q(X)$ is strongly dense in Y [7].

It is known that T_0 -identification of a topological space is done by Stone [17].

Now, we will construct T_3 reflection for X in Top. Firstly, we construct regular reflection by taking the supremum of all regular topologies which are coarser than the topology of X. This is a bireflection in Top, in other words, the underlying set stays the same. Then, apply it to the T_0- reflection. We get a space which is regular and T_0, hence regular and T_1. The composite of the two reflection is T_3-reflection.

Let X be a topological space and define $\sim$ on X by $x \sim y$ if and only if $cl_X \{x\} = cl_X \{y\}$. Then, $\sim$ is an equivalence relation on X and the resulting quotient space $X/\sim$ is T_0-space. This procedure and the space it produces are referred to as the T_0-identification of X. Clearly $\mathbf{T}_0(X) = X/\sim$. $\mathbf{T}_0(X)$ is called T_0- reflection. The canonical onto map from X onto its T_0- identification $\mathbf{T}_0(X)$ will be denoted by μ_X. It is clear that μ_X is an onto quasihomeomorphism. If $q : X \rightarrow Y$ is a continuous map,



then the diagram is commutative. $\mathbf{T}_0$ defines a (covariant) functor from Top to itself. Thus, we get a space which is regular and T_0, hence regular and T_1. The composite of the two reflections is T_3-reflection.

2. Fan-Gottesman Compactification

A compactification of a topological space X is a compact Hausdorff space Y containing X as a subspace such that $cl_Y X = Y$. In addition there are a lot of compactification methods applying different topological space such as Aleksandrov (one-point), Wallman, Stone-Cech. But, we study with Fan-Gottesman compactification.

Let β be a class of open sets in X. If it satisfies the following three conditions, it is called a *normal base*.

(1) β is closed under finite intersections
(2) If $B \in \beta$, then $X - cl_X B \in \beta$, where $cl_X B$ denotes the closure of B in X.
(3) For every open set U in X and every $B \in \beta$ such that $cl_X B \subset U$, there exists a set $D \in \beta$ such that $cl_X B \subset D \subset cl_X D \subset U$.

We consider a regular space having a normal base for open sets i.e., which satisfies the above three properties of normal base. A *chain family on* β is a non-empty family of sets of β such that

$$cl_X B_1 \cap cl_X B_2 \cap ... \cap cl_X B_n \neq \varnothing$$

for any finite number of sets B_i of the family. Every chain family on β is contained in at least one maximal chain family on β by Zorn's lemma. Maximal chain families on β will be denoted by letters as $a^*, b^*, ...$, and also the set of all maximal chain families on β will be denoted by $(X, \beta)^*$. Whose topology is defined as follow. For each $B \in \beta$, let

$$\tau(B) = \{b^* \in (X, \beta)^* : \text{there exists a } A \in b^* \text{ with } cl_X B \subset A\}$$

Then, the topology of $(X,\beta)^*$ is defined by taking

$$\beta^* = \{\tau(B) : B \in \beta\}$$

as a base of open sets. $(X,\beta)^*$ is a compact Hausdorff space and is a compactification of our regular space. Afterwards this compactification is called Fan-Gottesman compactification [6].

Now, we determine the Fan-Gottesman compactification via open ultrafilters.

2.1. Definition. Let X be a T_3 space and FX the subcollection of all maximal ultrafilter of closed subsets on X. For each open set $O \subset X$, define $O^* \subset FX$ to be the set

$$O^* = \left\{\hat{G} \in FX : \hat{G} \text{ consists of } cl_X O\right\}$$

Let Φ be the family of O^*. It is clear that Φ is the base for open sets of topology on FX. FX is a compact space and it is called the Fan-Gottesman compactifications of X.

In order to avoid the confusion between FX and $(X,\beta)^*$, we will use FX when it regarded as Fan-Gottesman compactification of X.

On the other hand, for each closed set $D \subset X$, we define $D^* \subset FX$ by $D^* = \left\{\hat{G} \in FX : \hat{G} \text{ consists of } G \subseteq D \text{ for some } G\right\}$. The following properties of FX are useful;

(i) If $U \subset X$ is open, then $FX - U^* = (FX - U)^*$

(ii) If $D \subset X$ is closed, then $FX - D^* = (FX - D)^*$

(iii) If U_1 and U_2 are open in X, then $(U_1 \cap U_2)^* = U_1^* \cap U_2^*$ and $(U_1 \cup U_2)^* = U_1^* \cup U_2^*$

Properties We consider the map $f_X : X \to FX$ defined by $f_X(x) = \hat{G}_x$, the closed ultrafilter converging to x in X. In order to avoid the confusing between $\hat{G}_x$ and $\hat{G}$, we will use $\hat{G}_x$ when it regarded as the maximal filter of closures of open sets containing x. Then the following properties hold.

1. If U is open in X, then $\overline{f_X(U)} = U^*$. In particular $f_X(X)$ is dense in FX.
2. f_X is continuous and it is an embedding of X in FX if and only if X is a T_3-space.
3. If $U_1$and U_2 are open subsets of X, then $\overline{f_X(U_1 \cap U_2)} = \overline{f_X(U_1)} \cap \overline{f_X(U_2)}$.
4. FX is a compact T_2-space .

For a T_3 space, we define $FGX = F(X)$ and we call it the Fan-Gottesman compactification of X. The notation FX is reserved only for T_3 spaces so that it is better to use some other notation for topological spaces. The same for $f_X : f_X$ is reserved for topological space; for $\mathbf{T}_3$ space, we define $F_X = f_X \circ \mu_x$ where μ_x is the canonical onto map from X onto its $T_3 - reflection$, $\mathbf{T}_3(X)$.

Since μ_x is an onto quasihomeomorphism, one obtains immediately that FGX can be described exactly as FX for T_3 space. The above properties are also true for a $\mathbf{T}_3$ space.

2.2. Remark. Let X be a T_3 space. Then, the following properties hold:

1. For each open subset U of X, we have $F_X(U) \subseteq U^*$
2. For each closed subset C of X, we have $F_X(C) \subseteq C^*$
3. Let U be open and C closed in a T_3 space. Then, $U \cap C \neq \varnothing$ if and only if $U^* \cap C^* \neq \varnothing$

2.3. Proposition. *Let X be a T_3 space and*

1. U be an open subset of X. If U is compact, then $U^* = F_X(U)$.
2. V be a closed subset of X. If V is compact, then $V^* = F_X(V)$.

Proof. Suppose that V is closed in X. We have $F_X(V) \subseteq V^*$ from Remark 1. If $\hat{G} \in V^*$, then there exists $G \in \hat{G}$ such that $G \subseteq V$. Then $V - G$ is compact by compactness of V. Thus $\cap \left\{ H \cap (V - G) : H \in \hat{G} \right\} \neq \varnothing$. If $x \in \cap \left\{ H \cap (V - G) : H \in \hat{G} \right\}$, then $\hat{G} = F_X(x)$. Hence, $\hat{G} \in F_X(V)$. Thus, $V^* \subseteq F_X(V)$. Therefore, $V^* = F_X(V)$.

Now, suppose that U is open in X. Let $\hat{G} \in U^*$. Thus, $U \in \hat{G}$. Since, $\cap \left\{ H : H \in \hat{G} \right\} \neq \varnothing$, we take an $x \in \cap \left\{ H : H \in \hat{G} \right\}$. It is seen that $\hat{G} = F_X(x)$. Therefore, according to Remark 1, $U^* = F_X(U)$. □

3. FG-morphisms and FG -extensions

Recall from [3] that a subset S of a topological space X is said to be *sufficiently dense* if S meets each nonempty closed subset and each nonempty open subset of X. By an *almost -homeomorphism* (*α-homeomorphism*, for short), we mean a continuous map $q : X \to Y$ such that $q(X)$ is sufficiently dense in Y and the topology of X is the inverse image of Y by q.

3.1. Definition. i) A subset C of a topological space is said to be openly dense if C meets each nonemty open subset of X.

Thus we have the following implications:

Strongly dense⇒Sufficiently dense ⇒openly dense
⇓
Dense

3.2. Definition. By a Fan-Gottesman morphism (FG-morphism, for short), we mean a continuous map $q : X \to Y$ such that $q(X)$ is openly dense in Y and the topology of X is the inverse image of Y by q.We conclude that

homeomorphism ⇒quasihomeomorphism ⇒ α-homeomorphism ⇒ FG-morphism

3.3. Theorem.

(1) *The composition of two FG-morphisms is an FG-morphism.*
(2) *If $q : X \to Y$ is an FG-morphism and X is T_0, then q is injective.*
(3) *If $q : X \to Y$ is an FG-morphism and Y is T_1, then q is an onto homeomorphism.*
(4) *If $q : X \to Y$ is an FG-morphism, X is T_0 and Y is T_1, q is a homeomorphism.*

Proof. We show that (1). Let $p : X \to Y$ and $q : Y \to Z$ be two FG-morphisms. Clearly, the topology of X is the inverse image of Z by $q \circ p$. Let A be open subset of Z. Since $q^{-1}(A)$ is open in Y, the $p(x) \cap q^{-1}(A) \neq \varnothing$, so that $A \cap q(p(X)) \neq \varnothing$. Hence, $q \circ p$ is an FG-morphism.

(2) Let x_1, x_2 be two points of X with $q(x_1) = q(x_2)$. Suppose that $x_1 \neq x_2$. Then, there exists an open subset U of X such that $x_1 \in U$, $x_2 \notin U$, since X is T_0. Because there exists an open subset H of Y satisfying $q^{-1}(H) = U$, we get $q(x_1) \in H$ and $q(x_2) \notin H$, which is impossible. It follows that q is injective.

(3) Let $y \in Y$. Then, $\{y\}$ is a locally closed subset of Y. Hence, $\{y\} \cap q(X) \neq \varnothing$, since $q(X)$ is strongly dense in Y. Thus, $y \in q(X)$, hence q is an onto map.

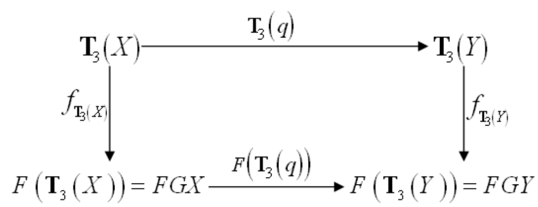
(4) It is clear that q is homeomorphism from (2) and (3). □

Now, we define FG-extensions.

3.4. Definition. A continuous map $q : X \to Y$ between T_3 spaces is said to be an FG-extension, if there is a continuous map $F(q) : FX \to FY$ such that $f_Y \circ q = F(q) \circ f_X$.

3.5. Theorem. *Let X, Y be two T_3 spaces and $q : X \to Y$ an FG-morphism. Then, q has an FG-extension which is a homeomorphism.*

Proof. We remark that diagram in the introduction commutes. Hence, $\mathbf{T}_3(q) \circ \mu_x = \mu_y \circ q$. Thus, $\mathbf{T}_3(q) \circ \mu_x$ is an FG-morphism. Now, $\mathbf{T}_3(q)$ is an FG-morphism from Proposition 2.1, since μ_x is a quasihomeomorphism. Therefore, $\mathbf{T}_3(q)$ is a homeomorphism by Proposition 2.1. It follows that $\mathbf{T}_3(q)$ has a canonical FG-extension $F(\mathbf{T}_3(q))$ which is a homeomorphism. Thus, the diagram commutes. If we denote $FG(q) = F(\mathbf{T}_3(q))$, then the diagram indicates clearly that $FG(q)$ is an FG-extension of q which is a homeomorphism.



□

It is known that if X is a T_4 space, then $FGX = wX = \beta(X)$ (the Wallman and Stone-Čech compactification, respectively)[4].

3.6. Corollary. *If* $\mathbf{T}_3(X)$ *is a T_4 space, then* $FGX = w(\mathbf{T}_3(q)) = \beta(\mathbf{T}_3(q))$.

3.7. Definition. Let X be a T_3 space and Y a subspace of X.

(1) Y is called a Fan-Gottesman generator ($FG-$generator) of X, if FGY is homeomorphic to FGX.
(2) Y is called a strong Fan-Gottesman generator (sFG-generator) of X, if the canonical embedding $i : Y \to X$ has an FG-extension $FG(i)$ which is a homeomorphism.

Clearly, sFG-generator$\Rightarrow$ $FG-$generator

3.8. Theorem. *Let X, Y be two T_3 spaces and $q : X \to Y$ a continuous map. Then, the following statements are equivalent:*

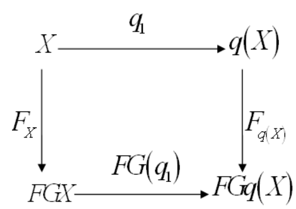
(1) q has an FG-extension which is a homeomorphism.
(2) $q(X)$ is an sFG-generator of Y and the topology of X is the inverse image of Y by q.

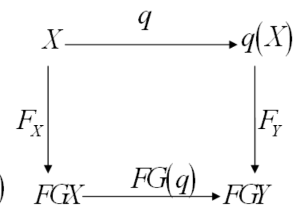
Proof. $(i) \Rightarrow (ii)$ Firstly, we show that the topology of X is the inverse image of Y by q. Let U be an open subset of X. Since $FG(q)$ is a homeomorphism, $FG(q)(U^*) = V$ is a closed subset of wY. Set $G = F_y^{-1}(V)$. We prove that $U = q^{-1}(G)$.

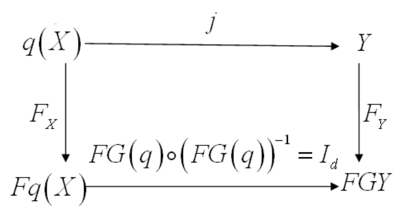
(a) Let $x \in U$. Then, $F_X(x) \in F_X(U) \subseteq U^*$. Hence, $FG(q)(F_X(x)) \in FG(q)(U^*) = V$ which gives $F_Y(q(x)) \in V$. It follows that $q(x) \in F_Y^{-1}(V) = G$. Therefore, $x \in q^{-1}(G)$.

(b) Conversely, let $x \in q^{-1}(G)$. Then, $q(x) \in G = F_X^{-1}(V)$; this means that $(F_Y \circ q)(x) \in V$, so that $FG(q)(F_X(x)) \in V = FG(q)(U^*)$. Since $FG(q)$ is bijective, $F_X(x) \in U^*$. Hence, $x \in F_X^{-1}(U^*) = U$. We have proved that $U = q^{-1}(G)$. In other words, the topology of X is the inverse image of Y by q.

Secondly, we show that $q(X)$ is an sFG-generator of Y. According to (1), the map $q_1 : X \to q(X)$ induced by q is an FG-morphism. Hence, q_1 has an FG-extension $F(q_1)$ which is a homeomorphism, by Proposition 2.1. Thus, the diagrams commute.

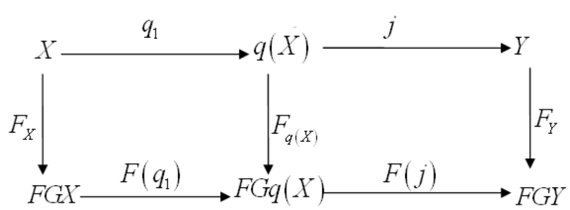






Let $j : q(X) \to Y$ be the canonical embedding. Clearly, the diagram commutes.

Therefore, j has $FG(q) \circ (FG(q))^{-1} = I_d$ as an FG-extension which is a homeomorphism. This means that $q(X)$ is an sFG generator of Y. $(ii) \Rightarrow (i)$ We assume (ii). The map $q_1 : X \to q(X)$ induced by q is an FG-morphism. Thus, according to Proposition 2.1, q_1 has an FG-extension $F(q_1)$ which is a homeomorphism. On the other hand, the canonical embedding $j : q(X) \to Y$ has an FG-extension which is a homeomorphism, by Proposition 2.1. It follows that the two diagrams commute.



Therefore, $F(j) \circ F(q_1)$ is an FG-extension of $q : X \to Y$ which is a homeomorphism. $\square$

Theorem 3.4 seems us to the following classical fact about the Stone-Ćech compactification $e_X : X \to \beta X$ of a Tychonoff space X .

Consider any continuous mapping $p : X \to Y$, where Y is also Tychonoff. Then, the map $\beta(p) : \beta X \to \beta Y$ is a homeomorphism if and only if p is a dense C^*-embedding.

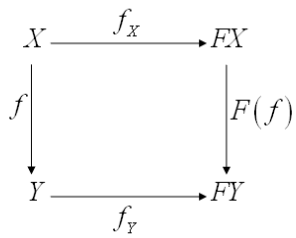
We can mention this analogy in our paper.

3.9. Theorem. *If X and Y are Tychonoff spaces, then the following are equivalent for a map $f : X \to Y$;*

(1) $F(f)[FX \backslash X]$ is contained in $FY \backslash Y$.
(2) The diagram

is pullback.

Proof. (1) $\Rightarrow$ (2) Suppose that $h : Z \to FX$ and $g : Z \to Y$ are mapping such that $F(f) \circ h = f_Y \circ g$. Since $f_Y \circ g[Z]$ is contained in FY and $F(f)$ sends $FX \backslash X$ into $FY \backslash Y$, we have that $h[Z]$ is contained in X. Hence, defining $I : Z \to X$ by $I(z) = h(z)$,



it is shown that the square is pullback.(2) $\Rightarrow$ (1) Choose p in FX and assume that $F(f)(p) = y$ belongs to Y. Then, let h be the map which embeds $\{p\}$ into FX and g be the map from the subspace $\{p\}$ which sends p to $F(f)(p)$. Then, $F(f) \circ h = f_Y \circ g$ so that there exist a map $I : \{p\} \to X$ such that $h = f_x \circ I$. Hence, p belongs to X. $\square$

References

[1] M. Baran, Separation Properties, Indian J. Pure and Appl. Math. 23, 333-342 (1992).

[2] M. Baran and H. Altındiş, T_2-objects in topological categories, Acta Math. Hungarica 71, 41-48 (1996).

[3] K. Belaid and O. Echi, $T_{(\alpha,\beta)}$-Space and the Wallman Compactification, International Journal of Mathematics and Mathematical Sciences Vol.2004, Issue 68, 3717-3735 (2004).

[4] C.Elmalı and T.Uğur, Fan-Gottesman Compactification of some specific space is Wallman-type compactification, Chaos,Soliton and Fractals, Vol.42, no.1, 17-19 (2009).

[5] K. Fan and N. Gottesman, On compactifications of Freudenthal and Wallman, Indag. Math. 14, 504-510 (1952).

[6] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove and D. S. Scott, Continuous Lattices and Domains, Cambridge University Press, Cambridge, England (2003).

[7] A. Grothendieck and J. Dieudonné, Éléments de Géometrie Algébrique, Die Grundlehren der mathematischen Wissenchaften, vol.166, Springer-Verlag, New York, (1971).

[8] D. Harris, The Wallman compactification is an epireflection, Proc. Amer. Math. Soc. 31, 265-267 (1972).

[9] H. Herrlich, On the concept of reflections in general topology, Contributions to Extension Theory of Topological Structures (Proc. Sympos., Berlin, 1967), Deutsch. Verlag Wissensch., Berlin, 105-114 (1969).

[10] S. Maclane, Categories for the Working Mathematician, Springer-Verlag, New York, (1971).

[11] S.Maclane and S.Eilenberg, Eilenberg-Maclane:collected works, Academic Press Inc., Orlando, FL, (1986).

[12] K.Morita, Category-isomorphism and endomorphism rings of modules, Trans. Amer. Math. Soc. 103, 451-469 (1962).

[13] K.Morita, Localization in categories of modules I. Math. Z. 114, 121-144 (1970).

[14] K.Morita, Localization in categories of modules II. J. Reine Angew. Math. 242, 163-169 (1970).

[15] K.Morita, Localization in categories of modules III. Math. Z. 119, 313-320 (1971).

[16] M. H. Stone, Applications of Boolean algebras to general topology, Bull. Amer. Math. Soc. 41, 375-481 (1997).

[17] H. Wallman, Lattices and topological spaces, Ann. of Math. (2) 39, 112-126 (1938).