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Research Article

Classes of harmonic starlike functions defined by Sălăgean-type q-differential operators

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Abstract

Sufficient and necessary coefficient bounds, extreme points of closed convex hulls, and distortion theorems are determined for a family of harmonic starlike functions of complex order involving Sălăgean-type q-differential operators.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions h of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{U}$.

We now recall the notion of *q-operators* or *q-difference operators* that play vital roles in the theory of hypergeometric series, quantum physics and operator theory. The application of *q-calculus* was initiated by Jackson [7] who have used the fractional *q-calculus operators* in investigations of certain classes of functions which are analytic in $\mathbb{U}$. For more details on *q-calculus* and its applications one can refer to [1, 5, 7, 13] and the references cited therein.

For $0 < q < 1$ the Jackson's *q-derivative* of a function $h \in \mathcal{S}$ is given as follows [7]

$$D_q h(z) = \begin{cases} \dfrac{h(z) - h(qz)}{(1-q)z} & for \quad z \neq 0, \\ h'(0) & for \quad z = 0, \end{cases} \tag{1.2}$$

$$D_q^2 h(z) = D_q(D_q h(z)).$$

From (1.2), we have $D_q h(z) = 1 + \sum\limits_{n=2}^{\infty} [n]_q a_n z^{n-1}$ where $[n]_q = \frac{1-q^n}{1-q}$ is sometimes called *the basic number n*. If $q \to 1^-$ then $[n]_q = [n] \to n$. For $h \in \mathcal{A}$, $m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$

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and $z \in \mathbb{U}$, Govindaraj and Sivasubramanian [5] considered the Sălăgean *q-differential operators*

$$\begin{aligned} D_q^0 h(z) &= h(z), \\ D_q^1 h(z) &= zD_q h(z), \ \ldots , \\ D_q^m h(z) &= zD_q(D_q^{m-1}h(z)) = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n \ . \end{aligned} \tag{1.3}$$

We note that if $q \to 1^-$ then

$$D^m h(z) = z + \sum_{n=2}^{\infty} [n]^m a_n z^n \quad (m \in \mathbb{N}_0, z \in \mathbb{U})$$

is the familiar Sălăgean derivative[15].

Let $\mathcal{H}$ denote the family of harmonic functions $f = h + \overline{g}$ that are orientation preserving and univalent in $\mathbb{U}$ with h as in (1.1) and g given by

$$g(z) = \sum_{n=1}^{\infty} b_n z^n, \ \ |b_1| < 1. \tag{1.4}$$

We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class $\mathcal{S}$ of normalized univalent functions if the co-analytic part of f is identically zero, i.e. $g \equiv 0$. We let $\overline{\mathcal{H}}$ be the subfamily of $\mathcal{H}$ consisting of harmonic functions $f = h + \overline{g}$ for which h and g are given by

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ \ g(z) = \sum_{n=1}^{\infty} b_n z^n, \ \ a_n \geq 0 \ and \ \ b_n \geq 0.$$

The seminal work of Clunie and Sheil-Small [4] on harmonic mappings prompted many research articles on classes of complex-valued harmonic univalent functions. In particular, [2, 6, 8, 9, 11, 12, 14, 16] have investigated properties of various subclasses of harmonic univalent functions.

For harmonic functions $f = h + \overline{g} \in \mathcal{H}$ where h and g are, respectively, given by (1.1) and (1.4), let $D_q^m h(z)$ be defined by (1.3) and $D_q^m g(z)$ be defined by

$$\begin{aligned} D_q^0 g(z) &= g(z), \\ D_q^1 g(z) &= zD_q g(z), \ \ldots , \\ D_q^m g(z) &= zD_q(D_q^{m-1}g(z)) = z + \sum_{n=2}^{\infty} [n]_q^m b_n z^n \ . \end{aligned} \tag{1.5}$$

Recently, Jahangiri [10] considered a generalized Sălăgean $q-$ differential operator $\mathcal{H}_q^m(\alpha)$ defined by

$$\Re\left(\frac{D_q^{m+1} f(z)}{D_q^m f(z)}\right) \geq \alpha; \ \ 0 \leq \alpha < 1,$$

where, $D_q^m h(z)$ and $D_q^m g(z)$ are, respectively, defined by (1.3) and (1.5) and

$$D_q^m f(z) = D_q^m h(z) + (-1)^m \overline{D_q^m g(z)}, \ \ m > -1.$$

The subfamily $\overline{\mathcal{H}}_q^m(\alpha) \subset \mathcal{H}_q^m(\alpha)$ consists of harmonic functions $f_m = h + \overline{g}_m$ for which

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ \ g_m(z) = (-1)^m \sum_{n=1}^{\infty} b_n z^n, \ \ a_n \geq 0 \ and \ b_n \geq 0. \tag{1.6}$$

For non-zero complex number b with $|b|\leq 1$, real number γ and $0 \leq \alpha < 1$ we let $\mathcal{HS}_q^m(b,\gamma,\alpha)$ be the subclass of $\mathcal{H}$ consisting of harmonic functions $f = h + \overline{g}$ satisfying

$$\Re\left(1+\frac{1}{b}\left((1+e^{i\gamma})\frac{D_q^{m+1}f(z)}{D_q^m f(z)}-e^{i\gamma}-1\right)\right)>\alpha. \tag{1.7}$$

We also let $\overline{\mathcal{HS}}_q^m(b,\gamma,\alpha) \equiv \mathcal{HS}_q^m(b,\gamma,\alpha)\cap\overline{\mathcal{H}}$.

We note that $\mathcal{HS}_q^m(1,\gamma,\alpha) \equiv \mathcal{HR}_q^m(\gamma,\alpha)$ is generalized class of Goodman-Ronning-type harmonic starlike functions (see [14], Inequality (2), p. 46) satisfying

$$\Re\left((1+e^{i\gamma})\frac{D_q^{m+1}f(z)}{D_q^m f(z)}-e^{i\gamma}\right)>\alpha$$

and $\mathcal{HS}_q^m(b,0,\alpha) \equiv \mathcal{HR}_q^m(b,\alpha)$ is the harmonic version of generalized starlike functions of complex order (see [3], Definition 1) satisfying

$$\Re\left(1+\frac{2}{b}\left(\frac{D_q^{m+1}f(z)}{D_q^m f(z)}-1\right)\right)>\alpha.$$

It is the aim of this paper to obtain sufficient coefficient conditions, extreme points, growth theorem, and distortion bounds for harmonic functions $f = h + \overline{g}$ in $\mathcal{HS}_q^m(b,\gamma,\alpha)$. Moreover, we show that those sufficient coefficient conditions for $f \in \mathcal{HS}_q^m(b,\gamma,\alpha)$ are also necessary for $f \in \overline{\mathcal{HS}}_q^m(b,\gamma,\alpha)$.

2. Main results

The sufficient coefficient condition for $\mathcal{HS}_q^m(b,\gamma,\alpha)$ is given in the following theorem.

Theorem 2.1. *Let $f = h+\overline{g} \in \mathcal{H}$ where b is a non-zero complex number with $|b|\leq 1$, γ is a real number and $0 \leq \alpha < 1$. If*

$$\sum_{n=1}^{\infty}\left(\frac{[n]_q^m[2[n]_q-2+(1-\alpha)|b|]}{(1-\alpha)|b|}|a_n|+\frac{[n]_q^m[2[n]_q+2-(1-\alpha)|b|]}{(1-\alpha)|b|}|b_n|\right)\leq 2, \tag{2.1}$$

then f is harmonic univalent and orientation-preserving in $\mathbb{U}$ and $f \in \mathcal{HS}_q^m(b,\gamma,\alpha)$.

Proof. First we establish that f is orientation preserving in $\mathbb{U}$. In other words, we need to show that $|D_q^{m+1}h(z)| \geq |D_q^{m+1}g(z)|$. This is accomplished using the properties of absolute values and the coefficient inequality (2.1).

$$\begin{aligned}|D_q^{m+1}h(z)| &\geq 1-\sum_{n=2}^{\infty}[n]_q^{m+1}|a_n|r^{n-1} > 1-\sum_{n=2}^{\infty}[n]_q^{m+1}|a_n|\\ &\geq 1-\sum_{n=2}^{\infty}\left[\frac{2[n]_q-2+(1-\alpha)|b|}{(1-\alpha)|b|}\right][n]_q^m|a_n|\\ &\geq \sum_{n=1}^{\infty}\left[\frac{2[n]_q+2-(1-\alpha)|b|}{(1-\alpha)|b|}\right][n]_q^m|b_n|\\ &\geq \sum_{n=1}^{\infty}[n]_q^{m+1}|b_n| \geq \sum_{n=1}^{\infty}[n]_q^{m+1}|b_n|r^{n-1} \geq |D_q^{m+1}g(z)|.\end{aligned}$$

To show f is univalent in $\mathbb{U}$ we use a method that was first used by Jahangiri [8]. We will show that $f(z_1) \neq f(z_2)$ when $z_1 \neq z_2$. Consider z_1 and z_2 in $\mathbb{U}$ so that $z_1 \neq z_2$. Since the unit disc $\mathbb{U}$ is simply connected and convex, we have $z(t) = (1-t)z_1 + tz_2$ in $\mathbb{U}$ for $0 \leq t \leq 1$. Then we may write

$$D_q^{m+1}f(z_2)-D_q^{m+1}f(z_1)=\int_0^1[(z_2-z_1)(D_q^{m+1}h(z(t))+\overline{(z_2-z_1)(D_q^{m+1}g(z(t))}]dt.$$

Dividing the above equation by $z_2 - z_1$ and taking the real parts we obtain

$$\Re\left(\frac{D_q^{m+1}f(z_2) - D_q^{m+1}f(z_1)}{z_2 - z_1}\right) = \int_0^1 \Re[D_q^{m+1}h(z(t)) + \frac{\overline{(z_2 - z_1)}}{z_2 - z_1}\overline{D_q^{m+1}g(z(t))}]dt \qquad (2.2)$$
$$> \int_0^1 [\Re\ (D_q^{m+1}h(z(t)) - |D_q^{m+1}g(z(t)|]dt.$$

On the other hand

$$\begin{aligned}\Re\ (D_q^{m+1}h(z(t)) - |(D_q^{m+1}g(z(t))| &\geq \Re\ (D_q^{m+1}h(z(t)) - \sum_{n=1}^{\infty}[n]_q^{m+1}|b_n|\\ &\geq 1 - \sum_{n=2}^{\infty}[n]_q^{m+1}|a_n| - \sum_{n=1}^{\infty}[n]_q^{m+1}|b_n|\\ &\geq 1 - \sum_{n=2}^{\infty}[n]_q^{m}\left[\frac{2[n]_q - 2 + (1-\alpha)|b|}{(1-\alpha)|b|}\right]|a_n|\\ &- \sum_{n=1}^{\infty}[n]_q^{m}\left[\frac{2[n]_q + 2 - (1-\alpha)|b|}{(1-\alpha)|b|}\right]|b_n|\\ &\geq 0 \text{ by } \quad (2.1).\end{aligned}$$

This together with inequality (2.2) implies the univalence of f.

Next we show that if the condition (2.1) holds then $f \in \mathcal{HS}_q^m(b,\gamma,\alpha)$. In other words, we need to show that the condition (1.7) is satisfied if (2.1) holds.

Using the fact that $\Re(w(z)) \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$ for $0 \leq \alpha < 1$ it suffices to show that

$$|(2b - \alpha b - e^{i\gamma} - 1)(\mathcal{D}_q^m h(z) + (-1)^m\overline{\mathcal{D}_q^m g(z)}) + (1+e^{i\gamma})(\mathcal{D}_q^{m+1}h(z) - (-1)^m\overline{\mathcal{D}_q^{m+1}g(z)})|$$

$$-|(1 + \alpha b + e^{i\gamma})(\mathcal{D}_q^m h(z) + (-1)^m\overline{\mathcal{D}_q^m g(z)})| - (1+e^{i\gamma})(\mathcal{D}_q^{m+1}h(z) - (-1)^m\overline{\mathcal{D}_q^{m+1}g(z)})| \geq 0.$$

Upon substituting for $\mathcal{D}_q^m h(z)$ and $\mathcal{D}_q^m g(z)$ we obtain

$$\begin{aligned}&|(2b - \alpha b - (1+e^{i\gamma}))\left[z + \sum_{n=2}^{\infty}[n]_q^m a_n z^n + (-1)^m\sum_{n=1}^{\infty}[n]_q^m\overline{b_n z^n}\right]\\ &+ (1+e^{i\gamma})\left[z + \sum_{n=2}^{\infty}[n]_q^{m+1}a_n z^n - (-1)^m\sum_{n=1}^{\infty}[n]_q^{m+1}\overline{b_n z^n}\right]|\\ &- |(1+\alpha b + e^{i\gamma})\left[z + \sum_{n=2}^{\infty}[n]_q^m a_n z^n + (-1)^m\sum_{n=1}^{\infty}[n]_q^m\overline{b_n z^n}\right]\\ &- (1+e^{i\gamma})\left[z + \sum_{n=2}^{\infty}[n]_q^{m+1}a_n z^n - (-1)^m\sum_{n=1}^{\infty}[n]_q^{m+1}\overline{b_n z^n}\right]|\end{aligned}$$

$$\begin{aligned}
&\geq (2-\alpha)|b||z| - \sum_{n=2}^{\infty} |(2-\alpha)b + (1+e^{i\gamma})([n]_q - 1)|[n]_q^m |a_n||z|^n \\
&\quad - \sum_{n=1}^{\infty} |(1+e^{i\gamma})([n]_q + 1) - (2-\alpha)b|[n]_q^m |b_n|\ |z|^n \\
&\quad -\alpha|b||z| - \sum_{n=2}^{\infty} |([n]_q - 1)(1+e^{i\gamma}) - \alpha b|[n]_q^m |a_n|\ |z|^n \\
&\quad - \sum_{n=1}^{\infty} |([n]_q + 1)(1+e^{i\gamma}) + \alpha b|[n]_q^m |b_n|\ |z|^n \\
&\geq 2(1-\alpha)|b||z| \left(1 - \sum_{n=2}^{\infty} [n]_q^m \left[\frac{2[2[n]_q - 2 + (1-\alpha)|b|]}{2(1-\alpha)|b|} |a_n|\right]\right) \\
&- 2(1-\alpha)|b||z| \sum_{n=1}^{\infty} [n]_q^m \left[\frac{2[2[n]_q + 2 - (1-\alpha)|b|]}{2(1-\alpha)|b|} |b_n|\right] \\
&\geq 0, \text{ by (2.1)}.
\end{aligned}$$

□

The functions

$$f(z) = z + \sum_{n=2}^{\infty} \left[\frac{(1-\alpha)|b|}{2[n]_q - 2 + (1-\alpha)|b|}\right] x_n z^n + \sum_{n=1}^{\infty} \left[\frac{(1-\alpha)|b|}{2[n]_q + 2 - (1-\alpha)|b|}\right] \overline{y}_n \overline{z}^n,$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$, shows that the coefficient bound given by (2.1) is sharp.

The next theorem shows that condition (2.1) is also necessary for $f \in \overline{\mathcal{HS}}_q^m(b,\gamma,\alpha)$.

Theorem 2.2. *Let $f_m = h + \overline{g}_m$ be given by (1.6) where b is a non-zero complex number with $|b| \leq 1$, γ is a real number and $0 \leq \alpha < 1$. Then f_m is harmonic univalent and orientation-preserving in $\mathbb{U}$ and $f_m \in \overline{\mathcal{HS}}_q^m(b,\gamma,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{[n]_q^m [2[n]_q - 2 + (1-\alpha)|b|]}{(1-\alpha)|b|} a_n + \frac{[n]_q^m [2[n]_q + 2 - (1-\alpha)|b|]}{(1-\alpha)|b|} b_n\right) \leq 2. \tag{2.3}$$

Proof. Since $\overline{\mathcal{HS}}_q^m(b,\gamma,\alpha) \subset \mathcal{HS}_q^m(b,\gamma,\alpha)$, the if part of the Theorem 2.2 follows from Theorem 2.1 . To prove the *only if* part, we will show that if (2.3) does not hold then f_m is not in $\overline{\mathcal{HS}}_q^m(b,\gamma,\alpha)$.

For $f_m \in \overline{\mathcal{HS}}_q^m(b,\gamma,\alpha)$ we must have

$$\Re\left(1 + \frac{1}{b}\left((1+e^{i\gamma}) \frac{D_q^{m+1} h(z) - (-1)^m \overline{D_q^{m+1} g_m(z)}}{D_q^m h(z) + (-1)^m \overline{D_q^m g_m(z)}} - (e^{i\gamma} + 1)\right)\right) \geq \alpha\ .$$

Or equivalently

$$\Re\left(\frac{(1-\alpha)bz-\sum\limits_{n=2}^{\infty}[(1-\alpha)b+([n]_q-1)(1+e^{i\gamma})][n]_q^m|a_n|z^n}{b\left(z-\sum\limits_{n=2}^{\infty}[n]_q^m|a_n|z^n+(-1)^{2m}\sum\limits_{n=1}^{\infty}[n]_q^m|b_n|\overline{z}^n\right)}\right)$$

$$-\Re\left(\frac{(-1)^{2m}\sum\limits_{n=1}^{\infty}[([n]_q+1)(1+e^{i\gamma})-(1-\alpha)b][n]_q^m|b_n|\overline{z}^n}{b\left(z-\sum\limits_{n=2}^{\infty}[n]_q^m|a_n|z^n+(-1)^{2m}\sum\limits_{n=1}^{\infty}[n]_q^m|b_n|\overline{z}^n\right)}\right)$$

$$=\Re\left(\frac{(1-\alpha)|b|^2-\sum\limits_{n=2}^{\infty}[(1-\alpha)b+([n]_q-1)(1+e^{i\gamma})]\overline{b}[n]_q^m|a_n|z^{n-1}}{|b|^2\left(1-\sum\limits_{n=2}^{\infty}[n]_q^m|a_n|z^{n-1}+\frac{\overline{z}}{z}\sum\limits_{n=1}^{\infty}[n]_q^m|b_n|\overline{z}^{[n]_q-1}\right)}\right)$$

$$-\Re\left(\frac{\frac{\overline{z}}{z}\sum\limits_{n=1}^{\infty}[([n]_q+1)(1+e^{i\gamma})-(1-\alpha)b]\overline{b}[n]_q^m|b_n|\overline{z}^{n-1}}{|b|^2\left(1-\sum\limits_{n=2}^{\infty}[n]_q^m|a_n|z^{n-1}+\frac{\overline{z}}{z}\sum\limits_{n=1}^{\infty}[n]_q^m|b_n|\overline{z}^{n-1}\right)}\right)\geq 0.$$

The above condition must hold for all values of γ, $|z|=r<1$ and $0<|b|<1$. For $\gamma=0$ and $|b|=b$ let $z=r<1$ be on the positive real axis. Then the above condition becomes

$$\frac{(1-\alpha)|b|^2-\sum\limits_{n=2}^{\infty}[(2[n]_q-2)+(1-\alpha)b]|b|[n]_q^m|a_n|r^{n-1}}{|b|^2\left(1-\sum\limits_{n=2}^{\infty}[n]_q^m|a_n|r^{n-1}+\sum\limits_{n=1}^{\infty}[n]_q^m|b_n|r^{n-1}\right)}$$

$$-\frac{\sum\limits_{n=1}^{\infty}[(2[n]_q+2)-(1-\alpha)b]|b|[n]_q^m|b_n|r^{n-1}}{|b|^2\left(1-\sum\limits_{n=2}^{\infty}[n]_q^m|a_n|r^{n-1}+\sum\limits_{n=1}^{\infty}[n]_q^m|b_n|r^{n-1}\right)}\geq 0. \tag{2.4}$$

Now we observe that the numerator in the above required inequality (2.4) is negative if condition (2.3) does not hold. Thus, there exists a point $z_0=r_0$ in $(0,1)$ for which the quotient in the above inequalities are negative. This contradicts the required condition (1.7) for $f_m\in\overline{\mathcal{HS}}_q^m(b,\gamma,\alpha)$. Hence the proof is complete. □

The following theorem is a consequence of the above Theorem 2.2.

Theorem 2.3. *Let $f_m=h+\overline{g}_m$ be given by (1.6). Then $f_m\in\overline{\mathcal{HS}}_q^m(\gamma,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{[n]_q^m[2[n]_q-1-\alpha)]}{1-\alpha}a_n+\frac{[n]_q^m[2[n]_q+1+\alpha)]}{1-\alpha}b_n\right)\leq 2.$$

The extreme points of closed convex hull of $\overline{\mathcal{HS}}_q^m(b,\gamma,\alpha)$, denoted by $clco\overline{\mathcal{HS}}_q^m(b,\gamma,\alpha)$, are determined in the following theorem.

Theorem 2.4. *Let $f_m\in clco\overline{\mathcal{HS}}_q^m(b,\gamma,\alpha)$ if and only if*

$$f_m(z)=\sum_{n=1}^{\infty}(X_nh_n+Y_ng_{m_n}) \tag{2.5}$$

where

$$h_1(z)=z, h_n(z)=z-\frac{(1-\alpha)|b|}{[n]_q^m[2[n]_q-2+(1-\alpha)|b|]}z^n,\ \ n=2,3,\ldots;$$

$$g_{m_n}(z) = z + (-1)^m \frac{(1-\alpha)|b|}{[n]_q^m[2[n]_q + 2 - (1-\alpha)|b|]}\overline{z}^n, \;\; n = 1, 2, \ldots;$$

$\sum\limits_{n=1}^{\infty}(X_n + Y_n) = 1,\ X_n \geq 0$ *and* $Y_n \geq 0.$

In particular, the extreme points of $clco\overline{\mathcal{H}}\mathcal{S}_q^m(b, \gamma, \alpha)$ *are* $\{h_n\}$ *and* $\{g_{m_n}\}$.

Proof. For functions of the form (2.5), we have

$$\begin{aligned} f_m(z) &= \sum_{n=1}^{\infty}(X_n h_n + Y_n g_{m_n}) \\ &= \sum_{n=1}^{\infty}(X_n + Y_n)\, z - \sum_{n=2}^{\infty}\frac{(1-\alpha)|b|}{[n]_q^m[2[n]_q - 2 + (1-\alpha)|b|]}X_n z^n \\ &+(-1)^m \sum_{n=1}^{\infty}\frac{(1-\alpha)|b|}{[n]_q^m[2[n]_q + 2 - (1-\alpha)|b|]}Y_n \overline{z}^n. \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{n=2}^{\infty}\frac{[n]_q^m[2[n]_q - 2 + (1-\alpha)|b|]}{(1-\alpha)|b|}\left(\frac{(1-\alpha)|b|}{[n]_q^m[2[n]_q - 2 + (1-\alpha)|b|]}\right)X_n \\ &\quad + \sum_{n=1}^{\infty}\frac{[n]_q^m[2[n]_q + 2 - (1-\alpha)|b|]}{(1-\alpha)|b|}\left(\frac{(1-\alpha)|b|}{[n]_q^m[2[n]_q + 2 - (1-\alpha)|b|]}\right)Y_n \\ &= \sum_{n=2}^{\infty}X_n + \sum_{n=1}^{\infty}Y_n = 1 - X_1 \leq 1. \end{aligned}$$

Thus, $f_m \in clco\overline{\mathcal{H}}\mathcal{S}_q^m(b, \gamma, \alpha)$. Conversely, suppose that $f_m \in clco\overline{\mathcal{H}}\mathcal{S}_q^m(b, \gamma, \alpha)$. Set

$$X_n = \frac{[n]_q^m[2[n]_q - 2 + (1-\alpha)|b|]}{(1-\alpha)|b|}|a_n|, n = 2, 3, \ldots,$$

and

$$Y_n = \frac{[n]_q^m[2[n]_q + 2 - (1-\alpha)|b|]}{(1-\alpha)|b|}|b_n|, n = 1, 2, \ldots,$$

where $\sum\limits_{n=1}^{\infty}(X_n + Y_n) = 1$. Then

$$\begin{aligned} f_m(z) &= z - \sum_{n=2}^{\infty}a_n z^n + (-1)^m\sum_{n=1}^{\infty}b_n\overline{z}^n \\ &= z - \sum_{n=2}^{\infty}\frac{(1-\alpha)|b|}{[n]_q^m[2[n]_q - 2 + (1-\alpha)|b|]}X_n z^n + (-1)^m\sum_{n=1}^{\infty}\frac{(1-\alpha)|b|}{[n]_q^m[2[n]_q + 2 - (1-\alpha)|b|]}Y_n\overline{z}^n \\ &= z - \sum_{n=2}^{\infty}[X_n(h_n(z) - z)] + \sum_{n=1}^{\infty}[Y_n(g_{m_n}(z) - z)] \\ &= \sum_{n=1}^{\infty}(X_n h_n + Y_n g_{m_n}). \end{aligned}$$

Now from Theorem 2.2, we can deduce that $0 \leq X_n \leq 1$, $(n \geq 2)$ and $0 \leq Y_n \leq 1$, $(n \geq 1)$. Therefore $X_1 = 1 - \sum\limits_{n=2}^{\infty}X_n - \sum\limits_{n=1}^{\infty}Y_n \geq 0$. Thus $\sum\limits_{n=1}^{\infty}(X_n h_n + Y_n g_{m_n}) = f_m(z)$ as required in the theorem. □

Finally, we determine the distortion theorem for the family $\overline{\mathcal{H}}\mathcal{S}_q^m(b, \gamma, \alpha)$.

Theorem 2.5. *Let* $f_m \in \overline{\mathcal{H}}\mathcal{S}_q^m(b,\gamma,\alpha)$ *where* $|z| = r < 1$. *Then*

$$|f_m(z)| \leq (1+b_1)r + \left(\frac{(1-\alpha)|b|}{[2]_q^m[2[2]_q - 2 + (1-\alpha)|b|]} - \frac{4-(1-\alpha)|b|}{[2]_q^m[2[2]_q - 2 + (1-\alpha)|b|]}|b_1|\right) r^2$$

and

$$|f_m(z)| \geq (1-b_1)r - \left(\frac{(1-\alpha)|b|}{[2]_q^m[2[2]_q - 2 + (1-\alpha)|b|]} - \frac{4-(1-\alpha)|b|}{[2]_q^m[2[2]_q - 2 + (1-\alpha)|b|]}|b_1|\right) r^2.$$

Proof. We will prove the right hand inequality. The proof for the left hand inequality will be similar and is omitted. Let $f_m(z) \in \overline{\mathcal{H}}\mathcal{S}_q^m(b,\gamma,\alpha)$. Upon taking the absolute value of f_m, we obtain

$$\begin{aligned}
|f_m(z)| &\leq (1+|b_1|)r + \sum_{n=2}^{\infty}[|a_n| + |b_n|]\ [n]_q^m)r^n \\
&\leq (1+|b_1|)r + r^2\sum_{n=2}^{\infty}(|a_n| + |b_n|)[n]_q^m \\
&= (1+|b_1|)r + \frac{(1-\alpha)|b|r^2}{[2]_q^m[2[n]_q - 2 + (1-\alpha)|b|]} \\
&\quad \times \sum_{n=2}^{\infty}[2]_q^m\left(\frac{2[2]_q - 2 + (1-\alpha)|b|}{(1-\alpha)|b|}|a_n| + \frac{2[2]_q - 2 + (1-\alpha)|b|}{(1-\alpha)|b|}|b_n|\right) \\
&\leq (1+|b_1|)r + \frac{(1-\alpha)|b|r^2}{[2]_q^m[2[2]_q - 2 + (1-\alpha)|b|]} \\
&\quad \times \sum_{n=2}^{\infty}[n]_q^m\left(\frac{2[n]_q - 2 + (1-\alpha)|b|}{(1-\alpha)|b|}|a_n| + \frac{2[n]_q + 2 - (1-\alpha)|b|}{(1-\alpha)|b|}|b_n|\right) \\
&\leq (1+|b_1|)r + \frac{(1-\alpha)|b|}{[2]_q^m[2[2]_q - 2 + (1-\alpha)|b|]}\left(1 - \frac{[4-(1-\alpha)|b|]}{(1-\alpha)|b|}|b_1|\right)r^2 \\
&\leq (1+|b_1|)r + \left(\frac{(1-\alpha)|b|}{[2]_q^m[2[2]_q - 2 + (1-\alpha)|b|]} - \frac{4-(1-\alpha)|b|}{[2]_q^m[2[2]_q - 2 + (1-\alpha)|b|]}|b_1|\right)r^2.
\end{aligned}$$

The result is sharp for

$$f(z) = z + |b_1|\overline{z} + \left(\frac{(1-\alpha)|b|}{[2]_q^m[2[2]_q - 2 + (1-\alpha)|b|]} - \frac{4-(1-\alpha)|b|}{[2]_q^m[2[2]_q - 2 + (1-\alpha)|b|]}|b_1|\right)\overline{z}^2,$$

where $|b_1| \leq \frac{(1-\alpha)|b|}{4-(1-\alpha)|b|}$. □

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