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Research Article

Prime geodesic theorem for the modular surface

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Abstract

Under the generalized Lindelöf hypothesis, the exponent in the error term of the prime geodesic theorem for the modular surface is reduced to $\frac{5}{8}+\varepsilon$ outside a set of finite logarithmic measure.

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1. Introduction

Let $\Gamma = PSL\,(2,\mathbb{Z})$ be the modular group and $\mathcal{H}$ the upper half-plane equipped with the hyperbolic metric. The norms $N(P_0)$ of primitive conjugacy classes P_0 in Γ are sometimes called pseudo-primes. The length of the primitive closed geodesic on the modular surface $\Gamma \setminus \mathcal{H}$ joining two fixed points, which are the same for all representatives of P_0, equals $\log(N(P_0))$. The statement about the number $\pi_\Gamma(x)$ of classes P_0 such that $N(P_0) \le x$, for $x > 0$, is known as the prime geodesic theorem, PGT.

The main tool in the proof of PGT is the Selberg zeta function, defined by

$$Z_\Gamma(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} (1 - N(P_0)^{-s-k}),\ \mathrm{Re}(s) > 1,$$

and meromorphicaly continued to the whole complex plane.

The relationship between the prime geodesic theorem and the distribution of zeros of the Selberg zeta function resembles to a large extent the relationship between the prime number theorem and the zeros of the Riemann zeta.

The function Z_Γ satisfies the analogue of the Riemann hypothesis. The zeros $\frac{1}{2} + i\gamma = \frac{1}{2} \pm i\sqrt{\lambda - \frac{1}{4}}$ of the Selberg Z_Γ lying on $Re(s) = \frac{1}{2}$ correspond to the eigenvalues $\lambda \ge \frac{1}{4}$ of the essentially self-adjoint Laplace-Beltrami operator $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ on $\Gamma \setminus \mathcal{H}$. See, e.g., [6] for some important applications of the modular group and the modular surface $PSL(2,\mathbb{Z}) \setminus \mathcal{H}$ in physics.

It is an outstanding problem whether the error term in the prime geodesic theorem is $O(x^{\frac{1}{2}+\varepsilon})$ as it would be the case in the prime number theorem once the Riemann hypothesis be proved. The obstacles in establishing an analogue of von Koch's theorem [13, p. 84] in this setting comes from the fact that Z_Γ is a meromorphic function of order 2, while the Riemann zeta is of order 1 ([12, relation (6.14) on p. 113]).

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In the case of Fuchsian groups $\Gamma \subset PSL\,(2,\mathbb{R})$, the best estimate of the remainder term in PGT is still $O\left(\frac{x^{\frac{3}{4}}}{\log x}\right)$ obtained by Randol [18] (see also [2,7] for different proofs). We note that its analogue $O\left(x^{\frac{4d_0^2+d_0}{2d_0+1}}(\log x)^{-1}\right)$ is valid also for strictly hyperbolic manifolds of higher dimensions, where $d_0 = \frac{d-1}{2}$ and $d \geq 3$ is the dimension of a manifold [4, Theorem 2.1].

The attempts to reduce the exponent $\frac{3}{4}$ in PGT were successful only in special cases. The chronological list of improvements for the modular group $\Gamma = PSL(2,\mathbb{Z})$ includes $\frac{35}{48}+\varepsilon$ (Iwaniec [15]), $\frac{7}{10}+\varepsilon$ (Luo and Sarnak [17]), $\frac{71}{102}+\varepsilon$ (Cai [8]) and the present $\frac{25}{36}+\varepsilon$ (Soundararajan and Young [19]).

Iwaniec [14] remarked that the generalized Lindelöf hypothesis for Dirichlet L-functions would imply $\frac{2}{3}+\varepsilon$.

We proved in [3] that $\frac{2}{3}+\varepsilon$ is valid outside a set of finite logarithmic measure. In the present note, we relate the error term in the Gallagherian *PGT* (i.e., *PGT* with an error term valid outside a set of finite logarithmic measure) on $PSL(2,\mathbb{Z})$ to the subconvexity bound for Dirichlet L- functions. This enables us to replace $\frac{2}{3}+\varepsilon$ by $\frac{5}{8}+\varepsilon$ under the generalized Lindelöf hypothesis. More precisely, the main result of this paper is the following theorem.

Theorem 1.1. *Let $\Gamma = PSL(2,\mathbb{Z})$ be the modular group, $\varepsilon > 0$ arbitrarily small and θ be such that*

$$L\left(\frac{1}{2}+it,\chi_D\right) \ll (1+|t|)^A\,|D|^{\theta+\varepsilon}$$

for some fixed $A > 0$, where D is a fundamental discriminant. There exists a set B of finite logarithmic measure such that

$$\pi_\Gamma(x) = \int_0^x \frac{dt}{\log t} + O\left(x^{\frac{5}{8}+\frac{\theta}{4}+\varepsilon}\right) \quad (x\to\infty, x\notin B).$$

Inserting the Conrey-Iwaniec [9] value $\theta = \frac{1}{6}$ into Theorem 1.1, we obtain

Corollary 1.2.

$$\pi_\Gamma(x) = li\,(x) + O\left(x^{\frac{2}{3}+\varepsilon}\right) \quad (x\to\infty, x\notin B).$$

Any improvement of θ immediately results in the obvious improvement of the error term in PGT. Taking into account that the Lindelöf hypothesis allows $\theta = 0$, we get

Corollary 1.3. *Under the generalized Lindelöf hypothesis for Dirichlet L-functions in the conductor aspect, we have*

$$\pi_\Gamma(x) = li\,(x) + O\left(x^{\frac{5}{8}+\varepsilon}\right) \quad (x\to\infty, x\notin B).$$

Remark 1.4. The obtained error term in *PGT* for strictly hyperbolic Fuchsian groups is $O\left(x^{\frac{7}{10}}(\log x)^{-\frac{4}{5}}(\log\log x)^{\frac{1}{5}+\varepsilon}\right)$ outside a set of finite logarithmic measure [1]. This is in accordance with the above mentioned Luo-Sarnak unconditional exponent $\frac{7}{10}+\varepsilon$ in *PGT* for $\Gamma = PSL(2,\mathbb{Z})$. In the case of a cocompact Kleinian group or a noncompact congruence group for some imaginary quadratic number field, the respective Gallagherian bound is $O\left(x^{\frac{21}{13}}(\log x)^{-\frac{11}{13}}(\log\log x)^{\frac{2}{13}+\varepsilon}\right)$ [4, Theorem 1.2].

2. Preliminaries

The motivation for Theorem 1.1 comes from several sources, including Gallagher [11], Iwaniec [15] and Balkanova and Frolenkov [5].

Recall that $\pi_\Gamma(x) = li(x) + O\left(x^{\frac{5}{8}+\frac{\theta}{4}+\varepsilon}\right)$ is equivalent to $\psi_\Gamma(x) = x + O\left(x^{\frac{5}{8}+\frac{\theta}{4}+\varepsilon}\right)$, where $\psi_\Gamma(x) = \sum_{N(P_0)^k \le x} \log N(P_0)$ is the Γ analogue of the classical Chebyshev function ψ.

Under the Riemann hypothesis, Gallagher improved von Koch's remainder term in the prime number theorem from $\psi(x) = x + O\left(x^{\frac{1}{2}}(\log x)^2\right)$ to $\psi(x) = x + O\left(x^{\frac{1}{2}}(\log\log x)^2\right)$ outside a set of finite logarithmic measure.

Following Koyama [16], we shall apply the next lemma due to Gallagher [10] to our setting.

Lemma 2.A. Let A be a discrete subset of $\mathbb{R}$ and $\eta \in (0,1)$. For any sequence $c(\nu) \in \mathbb{C}$, $\nu \in A$, let the series

$$S(u) = \sum_{\nu \in A} c(\nu) e^{2\pi i \nu u}$$

be absolutely convergent. Then

$$\int_{-U}^{U} |S(u)|^2 du \le \left(\frac{\pi\eta}{\sin \pi\eta}\right)^2 \int_{-\infty}^{+\infty} \left| \frac{U}{\eta} \sum_{t \le \nu \le t + \frac{\eta}{U}} c(\nu) \right|^2 dt.$$

Iwaniec [15] established the following explicit formula with an error term for ψ_Γ on $\Gamma = PSL(2,\mathbb{Z})$.

Lemma 2.B. For $1 \le T \le \frac{x^{\frac{1}{2}}}{(\log x)^2}$, one has

$$\psi_\Gamma(x) = x + \sum_{|\gamma| \le T} \frac{x^\rho}{\rho} + O\left(\frac{x}{T}(\log x)^2\right),$$

where $\rho = \frac{1}{2} + i\gamma$ denote zeros of Z_Γ lying on $Re(s) = \frac{1}{2}$ and counted with their multiplicities.

Recently, O. Balkanova and D. Frolenkov [5] have proved the following estimate.

Lemma 2.C.

$$\sum_{|\gamma| \le Y} x^{i\gamma} \ll \max\left(x^{\frac{1}{4}+\frac{\theta}{2}} Y^{\frac{1}{2}}, x^{\frac{\theta}{2}} Y\right) \log^3 Y,$$

$$\sum_{|\gamma| \le Y} x^{i\gamma} \ll Y \log^2 Y \text{ if } Y > \frac{x^{\frac{1}{2}+\frac{7}{6}\theta}}{\kappa(x)},$$

where $\rho = \frac{1}{2} + i\gamma$ are the zeros of Z_Γ, θ is the subconvexity exponent for Dirichlet $L-$functions, and $\kappa(x)$ is the distance from $\sqrt{x} + \frac{1}{\sqrt{x}}$ to the nearest integer.

3. Proof of Theorem 1.1

Proof. Inserting $T = \frac{x^{\frac{1}{2}}}{(\log x)^2}$ into Lemma 2.B, we obtain

$$\psi_\Gamma(x) = x + \sum_{|\gamma| \le T} \frac{x^\rho}{\rho} + O\left(x^{\frac{1}{2}}(\log x)^4\right). \tag{3.1}$$

We would like to bound the expression $\sum_{|\gamma| \le Y} \frac{x^\rho}{\rho}$, where $Y \in (0,T)$ is a parameter to be determined later on.

Let $n=\lfloor \log x\rfloor$ and $B_n=\left\{x\in\left[e^n,e^{n+1}\right):\left|\sum\limits_{|\gamma|\le Y}\frac{x^{i\gamma}}{\rho}\right|>x^{\varepsilon}Y^{\frac12}\right\}$. Looking at the logarithmic measure of B_n, we get

$$\begin{aligned}\mu^*B_n &= \int\limits_{B_n}\frac{dx}{x}=\int\limits_{B_n}x^{2\varepsilon}Y\frac{dx}{x^{1+2\varepsilon}Y}\le\int\limits_{e^n}^{e^{n+1}}\left|\sum_{|\gamma|\le Y}\frac{x^{i\gamma}}{\rho}\right|^2\frac{dx}{x^{1+2\varepsilon}Y} \\ &\le \frac{1}{e^{2n\varepsilon}Y}\int\limits_{e^n}^{e^{n+1}}\left|\sum_{|\gamma|\le Y}\frac{x^{i\gamma}}{\rho}\right|^2\frac{dx}{x}.\end{aligned}\tag{3.2}$$

After substitution $x=e^n\cdot e^{2\pi\left(u+\frac{1}{4\pi}\right)}$, the last integral becomes

$$2\pi\int\limits_{-\frac{1}{4\pi}}^{\frac{1}{4\pi}}\left|\sum_{|\gamma|\le Y}\frac{e^{\left(n+\frac12\right)i\gamma}}{\rho}e^{2\pi i\gamma u}\right|^2du.$$

Applying Lemma 2.A, with $\eta=U=\frac{1}{4\pi}$ and $c_\gamma=\frac{e^{\left(n+\frac12\right)i\gamma}}{\rho}$ for $|\gamma|\le Y$, $c_\gamma=0$ otherwise, we get

$$\int\limits_{-\frac{1}{4\pi}}^{\frac{1}{4\pi}}\left|\sum_{|\gamma|\le Y}\frac{e^{\left(n+\frac12\right)i\gamma}}{\rho}e^{2\pi i\gamma u}\right|^2du\le\left(\frac{\frac14}{\sin\frac14}\right)^2\int\limits_{-\infty}^{+\infty}\left(\sum_{\substack{t<\gamma\le t+1\\|\gamma|\le Y}}\frac{1}{|\rho|}\right)^2dt.\tag{3.3}$$

Note that $\sum_{t<\gamma\le t+1}\frac{1}{|\rho|}=O\left(1\right)$ since $\#\left\{\gamma:t<|\gamma|\le t+1\right\}=O\left(t\right)$ by the Weyl law. Thus,

$$\int\limits_{-\infty}^{+\infty}\left(\sum_{\substack{t<\gamma\le t+1\\|\gamma|\le Y}}\frac{1}{|\rho|}\right)^2dt=O\left(\int\limits_{\frac12}^{Y}dt\right)=O\left(Y\right).\tag{3.4}$$

The relations (3.2), (3.3) and (3.4) imply $\mu^*B_n\ll\frac{Y}{e^{2n\varepsilon}Y}=\frac{1}{e^{2n\varepsilon}}$. Hence, the set $B=\cup B_n$ has a finite logarithmic measure.

For $x\notin B$, we have $\left|\sum\limits_{|\gamma|\le Y}\frac{x^{i\gamma}}{\rho}\right|\le x^{\varepsilon}Y^{\frac12}$, i.e.

$$\left|\sum_{|\gamma|\le Y}\frac{x^{\rho}}{\rho}\right|\le x^{\frac12+\varepsilon}Y^{\frac12}.\tag{3.5}$$

Now, we rely on Lemma 2.C to estimate $\left|\sum\limits_{Y<|\gamma|\le T}\frac{x^{\rho}}{\rho}\right|$. Let us put $S\left(x,T\right)=\sum\limits_{|\gamma|\le T}x^{i\gamma}$. By Abel's partial summation, we have

$$\sum_{Y<|\gamma|\le T}\frac{x^{i\gamma}}{\rho}=\frac{S\left(x,T\right)}{\frac12+iT}-\frac{S\left(x,Y\right)}{\frac12+iY}+i\int\limits_{Y}^{T}\frac{S\left(x,u\right)}{\left(\frac12+iu\right)^2}du.$$

Multiplying the last relation by $x^{\frac12}$ and recalling that Lemma 2.C yields $\sum\limits_{|\gamma|\le Y}x^{i\gamma}\ll x^{\frac14+\frac{\theta}{2}+\varepsilon}Y^{\frac12}$ for $Y<T=\frac{x^{\frac12}}{\left(\log x\right)^2}$, we get

$$\left|\sum_{Y<|\gamma|\leq T}\frac{x^{\rho}}{\rho}\right| \ll \frac{x^{\frac{3}{4}+\frac{\theta}{2}+\varepsilon}}{T^{\frac{1}{2}}}+\frac{x^{\frac{3}{4}+\frac{\theta}{2}+\varepsilon}}{Y^{\frac{1}{2}}}+\int_{Y}^{T}\frac{x^{\frac{3}{4}+\frac{\theta}{2}+\varepsilon}u^{\frac{1}{2}}}{u^{2}}du \ll \frac{x^{\frac{3}{4}+\frac{\theta}{2}+\varepsilon}}{Y^{\frac{1}{2}}}. \tag{3.6}$$

Combining (3.5) and (3.6), we see that the optimal choice for the parameter Y is $Y \approx x^{\frac{1}{4}+\frac{\theta}{2}}$. Then, $\sum\limits_{|\gamma|\leq T}\frac{x^{\rho}}{\rho} = O\left(x^{\frac{1}{2}+\varepsilon}Y^{\frac{1}{2}}\right) = O\left(x^{\frac{5}{8}+\frac{\theta}{4}+\varepsilon}\right)$ for $x \notin B$.

The relation (3.1) becomes

$$\psi_{\Gamma}(x) = x + O\left(x^{\frac{5}{8}+\frac{\theta}{4}+\varepsilon}\right) \quad (x\to\infty,\, x\notin B),$$

as asserted. □

References

[1] M. Avdispahić, *Prime geodesic theorem of Gallagher type*, arXiv:1701.02115, 2017.

[2] M. Avdispahić, *On Koyama's refinement of the prime geodesic theorem*, Proc. Japan Acad. Ser. A Math. Sci. **94** (3), 21–24, 2018.

[3] M. Avdispahić, *Gallagherian PGT on $PSL(2,\mathbb{Z})$*, Funct. Approx. Comment. Math. **58** (2), 207–213, 2018.

[4] M. Avdispahić, *Errata and addendum to "On the prime geodesic theorem for hyperbolic 3-manifolds" Math. Nachr. **291** (2018), no. 14-15, 2160–2167*, Math. Nachr. **292** (4), 691–693, 2019.

[5] O. Balkanova and D. Frolenkov, *Bounds for a spectral exponential sum*, J. London Math. Soc. **99** (2), 249–272, 2019.

[6] R. Blumenhagen and E. Plauschinn, *Introduction to conformal field theory: With applications to string theory*, in: Lect. Notes Phys. **779**, Springer, Berlin Heidelberg, 2009.

[7] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Progress in Mathematics **106**, Birkhäuser, Boston-Basel-Berlin, 1992.

[8] Y. Cai, *Prime geodesic theorem*, J. Théor. Nombres Bordeaux, **14** (1), 59–72, 2002.

[9] J.B. Conrey and H. Iwaniec, *The cubic moment of central values of automorphic L-functions*, Ann. of Math. (2) **151** (3), 1175–1216, 2000.

[10] P.X. Gallagher, *A large sieve density estimate near $\sigma = 1$*, Invent. Math. **11**, 329–339, 1970.

[11] P.X. Gallagher, *Some consequences of the Riemann hypothesis*, Acta Arith. **37**, 339–343, 1980.

[12] D.A. Hejhal, *The Selberg trace formula for $PSL(2,R)$. Vol I*, Lecture Notes in Mathematics **548**, Springer, Berlin, 1976.

[13] A.E. Ingham, *The distribution of prime numbers*, Cambridge University Press, 1932.

[14] H. Iwaniec, *Non-holomorphic modular forms and their applications*, in: Modular forms (Durham, 1983), Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., 157–196, Horwood, Chichester, 1984.

[15] H. Iwaniec, *Prime geodesic theorem*, J. Reine Angew. Math. **349**, 136–159, 1984.

[16] S. Koyama, *Refinement of prime geodesic theorem*, Proc. Japan Acad. Ser A Math. Sci. **92** (7), 77–81, 2016.

[17] W. Luo and P. Sarnak, *Quantum ergodicity of eigenfunctions on $PSL_2(Z)\backslash H^2$*, Inst. Hautes Études Sci. Publ. Math. **81**, 207–237, 1995.

[18] B. Randol, *On the asymptotic distribution of closed geodesics on compact Riemann surfaces*, Trans. Amer. Math. Soc. **233**, 241–247, 1977.

[19] K. Soundararajan and M.P. Young, *The prime geodesic theorem*, J. Reine Angew. Math. **676**, 105–120, 2013.