Research Article GU J Sci 32(2): 686-703 (2019)

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

The Integral Theorem of the Field Energy

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Article Info

Received: 22/06/2018
Accepted: 30/01/2019

Keywords

Vector field
Acceleration field
Pressure field
Relativistic uniform system

Abstract

The integral theorem of the vector field energy is derived in a covariant way, according to which under certain conditions the potential energy of the system's field turns out to be half as large in the absolute value as the field's kinetic energy associated with the four-potential of the field and the four-current of the system's particles. Thus, the integral theorem turns out to be the analogue of the virial theorem, but with respect to the field rather than to the particles. Using this theorem, it becomes possible to substantiate the fact that electrostatic energy can be calculated by two seemingly unrelated ways, either through the scalar potential of the field or through the stres energy-momentum tensor of the field. In closed systems, the theorem formulation is simplified for the electromagnetic and gravitational fields, which can act at a distance up to infinity. At the same time for the fields acting locally in the matter, such as the acceleration field and the pressure field, in the theorem formulation it is necessary to take into account the additional term with integral taken over the system's surface. The proof of the theorem for an ideal relativistic uniform system containing non-rotating and randomly moving particles shows full coincidence in all significant terms, particularly for the electromagnetic and gravitational fields, the acceleration field and the vector pressure field.

1. INTRODUCTION

In classical mechanics, the particles of an arbitrary physical system have both kinetic and potential energies. In this case, there is a relationship between the kinetic and potential energies, which is described with the help of the virial theorem. In addition to the particles, each physical system has either external fields, generated by external sources, or internal fields originating from the system's particles themselves. The fields and particles are complementary to each other and in the aggregate, they represent the main content of the physical system. Thus, we should expect that there is also some theorem for the fields that could relate the quantities equivalent to the kinetic and potential energies.

The purpose of this article is establishing such a relationship between the field energies in the most general form, which is also suitable in the curved spacetime. Although the proof is provided for the electromagnetic field, it is also valid for any vector fields that have four-potentials and the corresponding tensors.

In order to verify the derived integral theorem of the field energy, we apply it to the relativistic uniform system and show how exactly this theorem should be used. In this case our analysis will refer not only to the electromagnetic field, but also to the vector gravitational field, as well as to the acceleration field and the vector pressure field [1, 2]. In particular, the use of the integral field energy theorem makes it possible to simplify the calculation of the gravitational energy of the system, since the field energy associated with the tensor invariant can be replaced with the energy associated with the four-potential of gravitational field. Similarly, the calculation of energy of other fields is simplified.

Everywhere in our calculations we will use the metric signature of the form (+,–,–,–).

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2. THE INTEGRAL THEOREM OF THE FIELD ENERGY

Suppose that in a certain physical system there are charged particles, the motion of which is described by the charge four-current j_α. In its turn the electromagnetic field has the four-potential A_β, while the electromagnetic field tensor $F_{\alpha\beta}$ is defined by the relation:

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \partial_\alpha A_\beta - \partial_\beta A_\alpha . \tag{1}$$

The symbols ∇_α and ∂_α represent the covariant derivative and the four-gradient, respectively. The equation of the electromagnetic field with the sources is written in the standard way:

$$\nabla^\beta F_{\alpha\beta} = -\mu_0 \, j_\alpha , \tag{2}$$

where μ_0 is the magnetic constant, and the covariant derivative ∇^β with a contravariant index is used.

We will multiply the electromagnetic field tensor by the four-potential with a contravariant index and will take from this product the covariant derivative in such a way that a scalar invariant would appear. At the same time we will use (2):

$$\nabla^\beta (F_{\alpha\beta} A^\alpha) = A^\alpha \nabla^\beta F_{\alpha\beta} + F_{\alpha\beta} \nabla^\beta A^\alpha = -\mu_0 \, j_\alpha A^\alpha + F_{\alpha\beta} \nabla^\beta A^\alpha . \tag{3}$$

Let us change the places of the indices α and β in (3):

$$\nabla^\alpha (F_{\beta\alpha} A^\beta) = A^\beta \nabla^\alpha F_{\beta\alpha} + F_{\beta\alpha} \nabla^\alpha A^\beta = -\mu_0 \, j_\beta A^\beta + F_{\beta\alpha} \nabla^\alpha A^\beta . \tag{4}$$

Let us now take into account that $\nabla^\beta (F_{\alpha\beta} A^\alpha) = \nabla^\alpha (F_{\beta\alpha} A^\beta)$, $j_\alpha A^\alpha = j_\beta A^\beta$, since the scalar invariants do not depend on permutations of the indices. Also keeping in mind that the electromagnetic field tensor is antisymmetric: $F_{\alpha\beta} = -F_{\beta\alpha}$, we will sum up relations (3) and (4) and use (1):

$$2\nabla^\beta (F_{\alpha\beta} A^\alpha) = -2\mu_0 \, j_\alpha A^\alpha - F_{\alpha\beta} F^{\alpha\beta} . \tag{5}$$

The tensor product $F_{\alpha\beta} A^\alpha$ in (5) contains a contraction with respect to the index α and therefore it is equivalent to a certain four-vector B_β. For an arbitrary four-vector the following rule holds:

$$\nabla^\beta B_\beta = \nabla_\beta B^\beta = \frac{1}{\sqrt{-g}} \partial_\beta \left(\sqrt{-g} \, B^\beta \right),$$

where g is the determinant of the metric tensor $g_{\alpha\beta}$.

We will use this rule on the left-hand side of (5), and then integrate (5) with respect to the invariant four-volume, replacing $j_\alpha A^\alpha$ by $A_\alpha j^\alpha$:

$$2\int \partial_\beta \left(\sqrt{-g} \, A^\alpha F_\alpha{}^\beta \right) dx^0 dx^1 dx^2 dx^3 = -\int \left(2\mu_0 \, A_\alpha j^\alpha + F_{\alpha\beta} F^{\alpha\beta} \right) \sqrt{-g} \, dx^0 dx^1 dx^2 dx^3 . \tag{6}$$

The tensor $F_{\alpha}{}^{\beta}$ in (6) is the electromagnetic field tensor with mixed indices. We will now use the divergence theorem and transform the left-hand side in (6):

$$2\int \partial_{\beta}\left(\sqrt{-g}\,A^{\alpha}F_{\alpha}{}^{\beta}\right)dx^{0}dx^{1}dx^{2}dx^{3} = 2\int A^{\alpha}F_{\alpha}{}^{\beta}\sqrt{-g}\,dS_{\beta}, \tag{7}$$

where $dS_{\beta} = n_{\beta}dS$ is the orthonormal differential dS of the hypersurface surrounding the physical system in the four-dimensional space, n_{β} is the four-dimensional normal vector perpendicular to the hypersurface and directed outward.

In (6) and (7) we may not perform integration with respect to the time coordinate x^{0} and may consider the physical system at a fixed arbitrary time point. To this end, we will rewrite the right-hand sides of (6) and (7):

$$\begin{aligned}&-\int\left(2\mu_{0}A_{\alpha}j^{\alpha}+F_{\alpha\beta}F^{\alpha\beta}\right)\sqrt{-g}\,dx^{0}dx^{1}dx^{2}dx^{3}\\&=-\int\left[\int\left(2\mu_{0}A_{\alpha}j^{\alpha}+F_{\alpha\beta}F^{\alpha\beta}\right)\sqrt{-g}\,dx^{1}dx^{2}dx^{3}\right]dx^{0}.\end{aligned} \tag{8}$$

$$\begin{aligned}&2\int A^{\alpha}F_{\alpha}{}^{\beta}\sqrt{-g}\,dS_{\beta} = 2\int A^{\alpha}F_{\alpha}{}^{0}\sqrt{-g}\,dx^{1}dx^{2}dx^{3} + 2\int\left[\int A^{\alpha}F_{\alpha}{}^{1}\sqrt{-g}\,dx^{2}dx^{3}\right]dx^{0}\\&+2\int\left[\int A^{\alpha}F_{\alpha}{}^{2}\sqrt{-g}\,dx^{1}dx^{3}\right]dx^{0} + 2\int\left[\int A^{\alpha}F_{\alpha}{}^{3}\sqrt{-g}\,dx^{1}dx^{2}\right]dx^{0}.\end{aligned} \tag{9}$$

The right-hand sides in (8) and (9) are equal to each other as a consequence of (6). Now we will differentiate them with respect to the variable $x^{0} = ct$, where c is the speed of light, t is the coordinate time, and then equate the results to each other:

$$\begin{aligned}&-\int\left(2\mu_{0}A_{\alpha}j^{\alpha}+F_{\alpha\beta}F^{\alpha\beta}\right)\sqrt{-g}\,dx^{1}dx^{2}dx^{3} = \frac{2}{c}\frac{d}{dt}\left(\int A^{\alpha}F_{\alpha}{}^{0}\sqrt{-g}\,dx^{1}dx^{2}dx^{3}\right)\\&+2\int A^{\alpha}F_{\alpha}{}^{1}\sqrt{-g}\,dx^{2}dx^{3} + 2\int A^{\alpha}F_{\alpha}{}^{2}\sqrt{-g}\,dx^{1}dx^{3} + 2\int A^{\alpha}F_{\alpha}{}^{3}\sqrt{-g}\,dx^{1}dx^{2}.\end{aligned}$$

The last three integrals on the right-hand side can be combined into one surface integral taken over the closed two-dimensional surface S, inside which the entire system with all the particles and fields must be located. All this gives the following:

$$\begin{aligned}&-\int\left(2\mu_{0}A_{\alpha}j^{\alpha}+F_{\alpha\beta}F^{\alpha\beta}\right)\sqrt{-g}\,dx^{1}dx^{2}dx^{3} = \frac{2}{c}\frac{d}{dt}\left(\int A^{\alpha}F_{\alpha}{}^{0}\sqrt{-g}\,dx^{1}dx^{2}dx^{3}\right)\\&+2\oiint_{S} A^{\alpha}F_{\alpha}{}^{k}n_{k}\sqrt{-g}\,dS.\end{aligned} \tag{10}$$

In (10) the three-dimensional unit vector n_{k}, where $k = 1,2,3$, is the normal vector to the surface S directed outward.

In many practical cases, the right-hand side of (10) vanishes. In particular, the electromagnetic field of the system is present both inside and outside the system up to infinity. Then the last integral on the right-hand side of (10) is the surface integral over a surface of infinitely large radius. But for a closed system, in which there are only the proper fields of the system's particles, both the four-potential A^{α} and the electromagnetic field tensor $F_{\alpha}{}^{\beta}$ vanish at infinity due to the gauge of the potentials, field strength and magnetic field. Consequently, for a closed system this integral in (10) is equal to zero. If the derivative with respect to time

inside of the first integral on the right-hand side of (10) is also equal to zero, then the following relation would hold for the left-hand side:

$$\int\left(2\mu_0 A_\alpha j^\alpha + F_{\alpha\beta}F^{\alpha\beta}\right)\sqrt{-g}\,dx^1dx^2dx^3 = 0\,. \tag{11}$$

The quantities inside the integral in (11) are often used in various calculations. For example, the Lagrangian for four vector fields, including the electromagnetic field, in case of continuous medium has the following form [1]:

$$\begin{aligned} L = &\int\left(-U_\mu J^\mu - D_\mu J^\mu - A_\mu j^\mu - \pi_\mu J^\mu\right)\sqrt{-g}\,dx^1dx^2dx^3 \\ &+\int\left(\begin{array}{l} ckR - 2ck\Lambda + \dfrac{c^2}{16\pi G}\Phi_{\mu\nu}\Phi^{\mu\nu} - \dfrac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} \\ -\dfrac{c^2}{16\pi\eta}u_{\mu\nu}u^{\mu\nu} - \dfrac{c^2}{16\pi\sigma}f_{\mu\nu}f^{\mu\nu} \end{array}\right)\sqrt{-g}\,dx^1dx^2dx^3\,. \end{aligned} \tag{12}$$

where k is a certain coefficient to be determined,

R is the scalar curvature,

Λ is the cosmological constant,

$J^\mu = \rho_0 u^\mu$ is the four-vector of the mass current,

ρ_0 is the mass density in the reference frame associated with the particle,

$u^\mu = \dfrac{c\,dx^\mu}{ds}$ is the four-velocity of a point particle, dx^μ is the four-displacement, and ds is the interval,

$U_\mu = \left(\dfrac{\vartheta}{c}, -\mathbf{U}\right)$ is the four-potential of the acceleration field, where ϑ and $\mathbf{U}$ denote the scalar and vector potentials, respectively,

$D_\mu = \left(\dfrac{\psi}{c}, -\mathbf{D}\right)$ is the four-potential of the gravitational field, described through the scalar potential ψ and the vector potential $\mathbf{D}$ of this field,

$\pi_\mu = \left(\dfrac{\wp}{c}, -\mathbf{\Pi}\right)$ is the four-potential of the pressure field, consisting of the scalar potential $\wp$ and the vector potential $\mathbf{\Pi}$,

G is the gravitational constant,

$\Phi_{\mu\nu} = \nabla_\mu D_\nu - \nabla_\nu D_\mu = \partial_\mu D_\nu - \partial_\nu D_\mu$ is the gravitational tensor,

$\Phi^{\alpha\beta} = g^{\alpha\mu}g^{\nu\beta}\Phi_{\mu\nu}$ is the definition of the gravitational tensor with contravariant indices using the metric tensor $g^{\alpha\mu}$,

$A_\mu = \left(\dfrac{\varphi}{c}, -\mathbf{A}\right)$ is the four-potential of the electromagnetic field, defined by the scalar potential φ and the vector potential $\mathbf{A}$ of this field,

$j^\mu = \rho_{0q}u^\mu$ is the four-vector of the charge current,

ρ_{0q} is the charge density in the reference frame associated with the particle,

$u_{\mu\nu} = \nabla_\mu U_\nu - \nabla_\nu U_\mu = \partial_\mu U_\nu - \partial_\nu U_\mu$ is the acceleration field tensor, calculated using the curl of the four-potential of the acceleration field,

η is the acceleration field coefficient,

$f_{\mu\nu} = \nabla_\mu \pi_\nu - \nabla_\nu \pi_\mu = \partial_\mu \pi_\nu - \partial_\nu \pi_\mu$ is the pressure field tensor,
σ is the pressure field coefficient.

In (12) the gravitational field is considered as a vector field in the framework of the covariant theory of gravitation. If (11) holds true, then in (12) the term $-\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$ is half as large as the term $-A_\mu j^\mu$, and has the opposite sign.

As another example we will give the expression for the relativistic energy of a physical system of particles and four vector fields, also in the approximation of continuous medium [1]:

$$E_{rel} = \frac{1}{c}\int \left(\rho_0 \vartheta + \rho_0 \psi + \rho_{0q} \varphi + \rho_0 \wp \right) u^0 \sqrt{-g}\, dx^1 dx^2 dx^3$$
$$-\int \left(\begin{array}{l} \dfrac{c^2}{16\pi G} \Phi_{\mu\nu} \Phi^{\mu\nu} - \dfrac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} \\ -\dfrac{c^2}{16\pi\eta} u_{\mu\nu} u^{\mu\nu} - \dfrac{c^2}{16\pi\sigma} f_{\mu\nu} f^{\mu\nu} \end{array} \right) \sqrt{-g}\, dx^1 dx^2 dx^3. \qquad (13)$$

If (11) is satisfied in such a system, then the integral of the quantity $\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$ over the infinite volume in (13) can be replaced by the integral of the quantity $-\frac{1}{2} A_\alpha j^\alpha$ over the volume occupied by the matter. This would allow us to significantly simplify the calculation of the system's energy.

The classical virial theorem for the kinetic energy E_k of the system's particles and the potential energy W of these particles is written as follows: $2E_k + W \approx 0$. Comparison of (13) and (11) shows that in some cases a quantitatively opposite relation of the form $E_{kf} + 2W_f \approx 0$ is satisfied for the electromagnetic field. In this case

$$E_{kf} = \int A_\alpha j^\alpha \sqrt{-g}\, dx^1 dx^2 dx^3, \qquad W_f = \frac{1}{4\mu_0}\int F_{\alpha\beta} F^{\alpha\beta} \sqrt{-g}\, dx^1 dx^2 dx^3,$$

so that E_{kf} plays the role of the kinetic energy of the field in the interaction of the four-potential A_α with the charge four-current j^α of the particles, and W_f characterizes the potential energy of the field, not that of the particles.

3. THE INTEGRAL THEOREM OF ENERGY FOR OTHER VECTOR FIELDS

In the covariant theory of gravitation [1] the description of the gravitational field occurs in the same way as it is done for the electromagnetic field. This means similarity of the equations of both fields, and we can immediately write the integral theorem of energy for the gravitational field, replacing in (10) the notation of the four-current, four-potential and field tensor, and replacing μ_0 by $-\frac{4\pi G}{c^2}$:

$$-\int\left(-\frac{8\pi G}{c^2}D_\alpha J^\alpha+\Phi_{\alpha\beta}\Phi^{\alpha\beta}\right)\sqrt{-g}\,dx^1dx^2dx^3=\frac{2}{c}\frac{d}{dt}\left(\int D^\alpha\Phi_\alpha{}^0\sqrt{-g}\,dx^1dx^2dx^3\right)$$
$$+2\oiint_S D^\alpha\Phi_\alpha{}^k n_k\sqrt{-g}\,dS. \tag{14}$$

If the physical system is closed, then in (14) the last surface integral on the right-hand side would vanish as a consequence of the field gauge at infinity, where the four-potential D^α and the gravitational field tensor $\Phi_\alpha{}^\beta$ of the system must be equal to zero.

Similarly, we can proceed with the acceleration field and with the vector pressure field [2], for which the integral theorem of the field energy is written as follows:

$$-\int\left(\frac{8\pi\eta}{c^2}U_\alpha J^\alpha+u_{\alpha\beta}u^{\alpha\beta}\right)\sqrt{-g}\,dx^1dx^2dx^3=\frac{2}{c}\frac{d}{dt}\left(\int U^\alpha u_\alpha{}^0\sqrt{-g}\,dx^1dx^2dx^3\right)$$
$$+2\oiint_S U^\alpha u_\alpha{}^k n_k\sqrt{-g}\,dS. \tag{15}$$

$$-\int\left(\frac{8\pi\sigma}{c^2}\pi_\alpha J^\alpha+f_{\alpha\beta}f^{\alpha\beta}\right)\sqrt{-g}\,dx^1dx^2dx^3=\frac{2}{c}\frac{d}{dt}\left(\int\pi^\alpha f_\alpha{}^0\sqrt{-g}\,dx^1dx^2dx^3\right)$$
$$+2\oiint_S \pi^\alpha f_\alpha{}^k n_k\sqrt{-g}\,dS. \tag{16}$$

However, the acceleration field and the pressure field differ substantially from the electromagnetic and gravitational fields, because they act only within the limits of matter. Therefore, in (15) and (16) the surface integrals on the right-hand side should be taken over the outer surface of the volume occupied by the system's matter.

4. APPLICATION OF THE INTEGRAL THEOREM OF ENERGY IN THE RELATIVISTIC UNIFORM SYSTEM

A relativistic uniform system is a suitable object for testing many physical laws. Thus in [3] we studied the virial theorem and found out the difference from the classical approach due to taking into account the relativistic corrections. In [4] we applied the formulas, derived for a relativistic uniform system, to planets and stars and found good agreement with the results of other authors. Besides we assumed that the matter is in random motion, the matter particles do not have proper rotation and there are no directed fluxes of matter. As a result, in this system under consideration both the global vector potentials of all the fields and the global solenoidal field vectors vanish. For example, for the electromagnetic field this means that both the global vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B}$ are equal to zero. A more thorough analysis shows that each charged moving typical particle has its own small vector potential $\mathbf{A}_p$, which is proportional to the instantaneous velocity $\mathbf{v}'$ of the particle and to the proper electric scalar potential φ_p of the particle, as well as its own magnetic field $\mathbf{B}_p=\nabla\times\mathbf{A}_p$. The contribution from $\mathbf{A}_p$ and $\mathbf{B}_p$ in the subsequent calculations is small due to the small value of the charge of each particle, therefore it can be neglected in the first approximation. The same also applies to the corresponding values for the gravitational field.

Let us now consider how the integral theorem of energy is satisfied for the electromagnetic field in the case of a relativistic uniform system. We will assume that the system is closed, has a spherical shape and is held in equilibrium under the action of the forces from gravitational attraction and the repulsion forces from the electromagnetic field and the pressure field. The acceleration field also contributes to the equilibrium of forces, since at random motion inside the sphere the particles experience a centripetal force from the

particles' velocity component, which is tangential with respect to the sphere's radius. We will use the approximation of continuous medium, so the intervals between typical particles are minimal and we can assume that the sphere's volume consists of the volumes of particles.

In order to simplify the subsequent calculations we will consider the situation within the framework of the special theory of relativity, in which $\sqrt{-g}=1$.

For a closed system the surface integral in (10) vanishes and for the electromagnetic field we have the following:

$$-\int\left(2\mu_0 A_\alpha j^\alpha + F_{\alpha\beta}F^{\alpha\beta}\right)dx^1dx^2dx^3 = \frac{2}{c}\frac{d}{dt}\left(\int A^\alpha F_\alpha{}^0\, dx^1dx^2dx^3\right). \tag{17}$$

Since $A_\alpha = \left(\frac{\varphi}{c}, -\mathbf{A}\right)$, and in the first approximation we consider that in the system under consideration $\mathbf{A}=0$, then in order to calculate the four-potential we also need to know the distribution of the global electric potential φ in the system. As was found in [5], the electric potential inside the sphere depends on the sinusoidal functions:

$$\varphi_i = \frac{c^2\rho_{0q}\gamma_c}{4\pi\varepsilon_0\eta\rho_0 r}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right) - r\cos\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right] \approx \frac{\rho_{0q}\gamma_c(3a^2-r^2)}{6\varepsilon_0}. \tag{18}$$

In (18) ε_0 is the electric constant, γ_c is the Lorentz factor of the particles at the center of the sphere, a is the radius of the sphere. For the charge four-current we have: $j^\alpha = \rho_{0q}u^\alpha$, while the four-velocity is $u^\alpha = (\gamma' c, \gamma'\mathbf{v}')$, where $\gamma' = \frac{1}{\sqrt{1-v'^2/c^2}}$ is the Lorentz factor for the particles, $\mathbf{v}'$ is the particles' velocity. The dependence of γ' on the current radius r is as follows [6]:

$$\gamma' = \frac{c\gamma_c}{r\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right) \approx \gamma_c - \frac{2\pi\eta\rho_0 r^2\gamma_c}{3c^2}. \tag{19}$$

With this in mind, for the scalar product of the four-vectors we find: $A_\alpha j^\alpha = \rho_{0q}\varphi_i\gamma'$. Now, using (18) and (19), we can calculate the first term on the left-hand side of (17):

$$-2\mu_0\int A_\alpha j^\alpha\, dx^1dx^2dx^3 = -2\mu_0\rho_{0q}\int\varphi_i\gamma'\,dx^1dx^2dx^3$$

$$= -\frac{\mu_0 c^3\rho_{0q}^2\gamma_c^2}{2\pi\varepsilon_0\eta\rho_0\sqrt{4\pi\eta\rho_0}}\int\frac{1}{r^2}\left[\begin{array}{l}\dfrac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\dfrac{r}{c}\sqrt{4\pi\eta\rho_0}\right)\\ -r\cos\left(\dfrac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\end{array}\right]\sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right)dx^1dx^2dx^3.$$

In the spherical coordinates $dx^1dx^2dx^3 = r^2\,dr\,d\varphi\sin\theta\,d\theta$, and this integral will equal:

$$
\begin{aligned}
&-2\mu_0 \int A_\alpha \, j^\alpha \, dx^1 dx^2 dx^3 \\
&= -\frac{\mu_0 c^4 \rho_{0q}^2 \gamma_c^2}{2\pi\varepsilon_0 \eta^2 \rho_0^2}\left[\frac{a}{2} - \frac{3c}{4\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{2a}{c}\sqrt{4\pi\eta\rho_0}\right) + a\cos^2\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right] \\
&\approx -\frac{3\mu_0 q^2 \gamma_c^2}{5\pi\varepsilon_0 a}\left(1 - \frac{4\eta m}{7ac^2}\right).
\end{aligned}
\tag{20}
$$

In (20), the charge q is the product of the particles' invariant charge density ρ_{0q} by the sphere's volume, and likewise, the mass m is the product of the particles' invariant mass density ρ_0 by the sphere's volume. The quantities q and m have a technical character and they are not equal to the sphere's total charge q_b and the gravitational mass m_g, respectively. In particular, according to [5],

$$
\begin{aligned}
&q_b = \rho_{0q}\int \gamma' dV = \frac{c^2 \rho_{0q}\gamma_c}{\eta\rho_0}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right) - a\cos\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right] \\
&\approx q\gamma_c\left(1 - \frac{3\eta m}{10ac^2}\right).
\end{aligned}
$$

$$
\begin{aligned}
&m_g = \rho_0\int \gamma' dV = \frac{c^2\gamma_c}{\eta}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right) - a\cos\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right] \\
&\approx m\gamma_c\left(1 - \frac{3\eta m}{10ac^2}\right).
\end{aligned}
\tag{21}
$$

In view of (18) and the equality of the vector potential to zero, that is, $\mathbf{A}_i = 0$, the electric field strength inside the sphere can be determined by the formula:

$$
\begin{aligned}
&\mathbf{E}_i = -\nabla\varphi_i - \frac{\partial \mathbf{A}_i}{\partial t} = \frac{c^2\rho_{0q}\gamma_c \mathbf{r}}{4\pi\varepsilon_0\eta\rho_0 r^3}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right) - r\cos\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right)\right] \\
&\approx \frac{\rho_{0q}\gamma_c \mathbf{r}}{3\varepsilon_0}\left(1 - \frac{2\pi\eta\rho_0 r^2}{5c^2}\right).
\end{aligned}
\tag{22}
$$

Similarly, the electric field strength outside the sphere is equal to:

$$
\begin{aligned}
&\mathbf{E}_o = \frac{c^2\rho_{0q}\gamma_c \mathbf{r}}{4\pi\varepsilon_0\eta\rho_0 r^3}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right) - a\cos\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right] \\
&= \frac{q_b \mathbf{r}}{4\pi\varepsilon_0 r^3} \approx \frac{q\gamma_c \mathbf{r}}{4\pi\varepsilon_0 r^3}\left(1 - \frac{3\eta m}{10ac^2}\right).
\end{aligned}
\tag{23}
$$

In the general case, the electromagnetic field tensor $F_{\alpha\beta}$ contains the components of the electric field strength vector $\mathbf{E}$ and the magnetic field vector $\mathbf{B}$. In the system under consideration, on the average

$\mathbf{B}=0$, while in the Cartesian space coordinates $F_{01}=-F_{10}=-F^{01}=F^{10}=\frac{E_x}{c}$, $F_{02}=-F_{20}=-F^{02}=F^{20}=\frac{E_y}{c}$, $F_{03}=-F_{30}=-F^{03}=F^{30}=\frac{E_z}{c}$, and the remaining components of the tensors $F_{\alpha\beta}$ and $F^{\alpha\beta}$ are assumed to be equal to zero. Therefore, in this case $F_{\alpha\beta}F^{\alpha\beta}=-\frac{2}{c^2}E^2$. The integral over the volume inside the sphere taken for the second term on the left-hand side of (17), in view of (22), in the spherical coordinates is equal to:

$$-\int\limits_{r=0}^{a} F_{\alpha\beta}F^{\alpha\beta}r^2\,dr\,d\varphi\sin\theta\,d\theta=\frac{8\pi}{c^2}\int\limits_{r=0}^{a}E_i^2\,r^2\,dr$$
$$=\frac{c^2\rho_{0q}^2\gamma_c^2}{2\pi\varepsilon_0^2\eta^2\rho_0^2}\int\limits_{r=0}^{a}\frac{1}{r^2}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right)-r\cos\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right)\right]^2 dr.$$

This integral must be taken by parts, placing $\frac{1}{r^2}$ under the differential sign in the form $\frac{1}{r^2}dr=-d\frac{1}{r}$. This gives the following:

$$-\int\limits_{r=0}^{a} F_{\alpha\beta}F^{\alpha\beta}r^2\,dr\,d\varphi\sin\theta\,d\theta$$
$$=\frac{c^2\rho_{0q}^2\gamma_c^2}{2\pi\varepsilon_0^2\eta^2\rho_0^2}\left[\frac{a}{2}+\frac{c}{4\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{2a}{c}\sqrt{4\pi\eta\rho_0}\right)-\frac{c^2}{4\pi\eta\rho_0 a}\sin^2\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right]. \quad (24)$$

Let us now calculate the integral over the volume outside the sphere taken for the second term on the left-hand side of (17), in view of (23), in the spherical coordinates:

$$-\int\limits_{r=a}^{\infty} F_{\alpha\beta}F^{\alpha\beta}r^2\,dr\,d\varphi\sin\theta\,d\theta=\frac{8\pi}{c^2}\int\limits_{r=a}^{\infty}E_o^2\,r^2\,dr$$
$$=\frac{c^2\rho_{0q}^2\gamma_c^2}{2\pi\varepsilon_0^2\eta^2\rho_0^2 a}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)-a\cos\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right]^2. \quad (25)$$

The sum of (24) and (25) gives the integral over the entire space:

$$-\int F_{\alpha\beta}F^{\alpha\beta}\,dx^1dx^2dx^3$$
$$=\frac{c^2\rho_{0q}^2\gamma_c^2}{2\pi\varepsilon_0^2\eta^2\rho_0^2}\left[\frac{a}{2}-\frac{3c}{4\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{2a}{c}\sqrt{4\pi\eta\rho_0}\right)+a\cos^2\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right]. \quad (26)$$

If we take into account the identity $\mu_0\varepsilon_0c^2=1$ and substitute (20) and (26) into (17), then we can see that the left-hand side of (17) becomes equal to zero. Therefore, in the given physical system the right-hand side of (17) must also be equal to zero:

$$\frac{2}{c}\frac{d}{dt}\left(\int A^{\alpha} F_{\alpha}{}^{0}\, dx^1 dx^2 dx^3\right) = 0. \tag{27}$$

And this is true, since the space components of the four-potential A^{α} are assumed to be equal to zero, that is, $\mathbf{A} = 0$. At the same time, the time component of the electromagnetic field tensor is equal to zero due to antisymmetry of the tensor, $F_0{}^0 = 0$. Consequently, the product $A^{\alpha} F_{\alpha}{}^{0} = 0$, and equation (27) is satisfied.

For the gravitational field the situation is completely analogous to that of the electromagnetic field. For a closed system, within the framework of the special theory of relativity, in (14) the surface integral vanishes and we have the following:

$$-\int\left(-\frac{8\pi G}{c^2} D_{\alpha} J^{\alpha} + \Phi_{\alpha\beta}\Phi^{\alpha\beta}\right) dx^1 dx^2 dx^3 = \frac{2}{c}\frac{d}{dt}\left(\int D^{\alpha}\Phi_{\alpha}{}^{0}\, dx^1 dx^2 dx^3\right). \tag{28}$$

Since in the relativistic uniform system the global vector potential of the gravitational field is equal to zero, $\mathbf{D} = 0$, both the product $D^{\alpha}\Phi_{\alpha}{}^{0}$ and the right-hand side of (28) are equal to zero. As for the left-hand side of (28), in order to calculate it we need the global scalar gravitational potential ψ_i inside the sphere and the gravitational field strengths inside and outside the sphere [5]:

$$\psi_i = -\frac{G c^2 \gamma_c}{\eta r}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right) - r\cos\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right] \approx -\frac{2\pi G\rho_0\gamma_c(3a^2 - r^2)}{3}. \tag{29}$$

$$\begin{aligned}\mathbf{\Gamma}_i &= -\nabla\psi_i - \frac{\partial \mathbf{D}_i}{\partial t} = -\frac{G c^2\gamma_c \mathbf{r}}{\eta r^3}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right) - r\cos\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right)\right] \\ &\approx -\frac{4\pi G\rho_0\gamma_c\mathbf{r}}{3}\left(1 - \frac{2\pi\eta\rho_0 r^2}{5c^2}\right).\end{aligned} \tag{30}$$

$$\begin{aligned}\mathbf{\Gamma}_o &= -\nabla\psi_o - \frac{\partial \mathbf{D}_o}{\partial t} = -\frac{G c^2\gamma_c \mathbf{r}}{\eta r^3}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right) - a\cos\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right] \\ &= -\frac{G m_g \mathbf{r}}{r^3} \approx -\frac{G m\gamma_c \mathbf{r}}{r^3}\left(1 - \frac{3\eta m}{10 a c^2}\right).\end{aligned} \tag{31}$$

We obtain the product $D_{\alpha} J^{\alpha}$ as follows: $D_{\alpha} J^{\alpha} = \rho_0\psi_i\gamma'$. Then taking into account (19) and (29), for the first term on the left-hand side of (28) we find:

$$\begin{aligned}\frac{8\pi G}{c^2}\int D_{\alpha} J^{\alpha}\, dx^1 dx^2 dx^3 &= \frac{8\pi G\rho_0}{c^2}\int \psi_i\gamma' dx^1 dx^2 dx^3 \\ &= -\frac{8\pi G^2 c^2\gamma_c^2}{\eta^2}\left[\frac{a}{2} - \frac{3c}{4\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{2a}{c}\sqrt{4\pi\eta\rho_0}\right) + a\cos^2\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right].\end{aligned} \tag{32}$$

For the second term on the left-hand side of (28), in view of (30) and (31) we have the following:

$$-\int \Phi_{\alpha\beta}\Phi^{\alpha\beta}\,dx^1dx^2dx^3=\frac{8\pi}{c^2}\int\limits_{r=0}^{a}\Gamma_i^2\,r^2\,dr+\frac{8\pi}{c^2}\int\limits_{r=a}^{\infty}\Gamma_o^2\,r^2\,dr$$
$$=\frac{8\pi G^2c^2\gamma_c^2}{\eta^2}\left[\frac{a}{2}-\frac{3c}{4\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{2a}{c}\sqrt{4\pi\eta\rho_0}\right)+a\cos^2\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right]. \quad (33)$$

The sum of (32) and (33) vanishes according to (28), where the right-hand side is equal to zero.

5. THE ACCELERATION FIELD

Let us begin with clarification of what should be meant by the four-potential of the acceleration field of a certain physical system in the general case. According to [2], the four-potential of any vector field, the vector potential of which is equal to zero in its proper reference frame, that is, in the center-of-momentum frame, in case of rectilinear motion in the laboratory reference frame can be defined by the following formula:

$$L_\alpha=\frac{k_f\,\varepsilon}{\rho_0c^2}u_{\alpha L}, \quad (34)$$

where $k_f=\dfrac{\rho_0}{\rho_{0q}}$ for the electromagnetic field and $k_f=1$ for the remaining fields; ε is the invariant energy density of interaction, which is the product of the four-potential of the field by the corresponding 4-current; $u_{\alpha L}$ is the four-velocity with a covariant index that defines the motion of the center of momentum of the physical system in the laboratory reference frame.

In the proper reference frame $u_{\alpha L}=\left(c\dfrac{dt}{d\tau},\,0,0,0\right)$, and the vector potential as the space component L_α vanishes according to (34). However, some physical systems, even if their center of momentum is fixed, have not only a scalar potential but also a vector field potential within the system. Therefore, the more general expression for the four-potential of the field in the laboratory reference frame is as follows:

$$L_\alpha=M_\alpha{}^\beta L'_\beta, \quad (35)$$

where $M_\alpha{}^\beta$ is a matrix connecting the coordinates and time of two reference frames, one of which is the laboratory reference frame and the other moves together with the center of momentum of the physical system under consideration, so that it has the four-potential L'_β of the field in it. In the special case of the system's motion at the constant velocity $M_\alpha{}^\beta$ represents the Lorentz transformation matrix [7].

As a typical example we will consider a neutron star consisting mainly of neutrons and to some extent of protons and electrons. Both the star itself and the nucleons have fast rotation and strong magnetic fields. Despite the absence of charge, each neutron has a complex internal electromagnetic structure and a magnetic moment. Suppose that it is required to model a star as a relativistic uniform system and to specify the four-potential of the field of an arbitrary moving nucleon as a typical particle. To do this, we must use formula (35), since in (34) it is assumed that there is no vector potential in the nucleon's center-of-momentum frame. Really, a nucleon may not move in space, but due to proper rotation and complex internal structure in the nucleon there are nonzero vector potentials of almost all the fields.

In order to simplify the calculations, we will further assume that the physical system under consideration does not have general rotation, the system's typical particles move randomly and have neither proper rotation, nor proper vector potentials in the center-of-momentum frames of the particles. In this case, we can use a simpler formula (34) instead of (35).

In a fixed sphere, the energy density in the volume of each particle is $\varepsilon = \gamma' \rho_0 c^2$, and for the acceleration field in case of rectilinear motion of the sphere in the laboratory reference frame, according to (34), the four-potential will equal $U_{\alpha L} = \gamma' u_{\alpha L}$. This means that if for an observer inside the sphere with particles within the relativistic uniform model the quantity γ' is an invariantly determined Lorentz factor as a certain function of coordinates and time, then for an observer in the laboratory reference frame, in which the sphere's center has the four-velocity $u_{\alpha L}$, the four-potential of the acceleration field for each point inside the moving sphere will equal $U_{\alpha L}$.

In the ideal case, when the system of particles is an absolutely solid body and the particles inside the system are motionless, it should be $\gamma' = 1$, and then the four-potential of the acceleration field would coincide with the four-velocity of the system's center of momentum, $U_{\alpha L} = u_{\alpha L}$. A material point is a tiny physical system, and if we do not delve deeply into the structure of the internal motion of its matter and consider this point as a solid body, then the four-potential of the acceleration field of such a point would be equal to the four-velocity of its rectilinear motion.

By definition, the four-potential of the acceleration field is a four-vector $U_\alpha = \left(\frac{\vartheta}{c}, -\mathbf{U} \right)$, where ϑ and $\mathbf{U}$ denote the scalar and vector potentials, respectively. In view of (34) and the relation $\varepsilon = \gamma' \rho_0 c^2$, it turns out that in the relativistic uniform system under consideration in the form of a fixed sphere the scalar potential will be $\vartheta = \gamma' c^2$. As for the global vector potential of the acceleration field $\mathbf{U}$, it is equal to zero due to the randomness of motion of the matter particles. On the other hand, inside each typical particle there is always a small vector potential $\mathbf{U}_p$ of the acceleration field, which is proportional to the instantaneous velocity $\mathbf{v}'$ of the particle. This changes to some extent the form of the effectively acting four-potential of the acceleration field inside the sphere.

Let an arbitrary typical particle move inside the sphere, and its four-velocity within the framework of the special theory of relativity $u_\alpha = (\gamma' c, -\gamma' \mathbf{v}')$, where $\mathbf{v}'$ is the velocity of the particle, γ' is the Lorentz factor of the particle. This particle, in turn, can be considered as a relativistic uniform system, in which subparticles with the Lorentz factor γ_p move randomly relative to the particle's center of momentum. Then, according to (34), the four-potential of the acceleration field for this moving particle will be written as $U_\alpha = \gamma_p u_\alpha = (\gamma_p \gamma' c, -\gamma_p \gamma' \mathbf{v}')$. Comparison with the expression $U_\alpha = \left(\frac{\vartheta_p}{c}, -\mathbf{U}_p \right)$ allows us to determine the acceleration field potentials inside each moving particle of the sphere: $\vartheta_p = \gamma_p \gamma' c^2$, $\mathbf{U}_p = \gamma_p \gamma' \mathbf{v}'$. In this case it turns out that $\vartheta_p > \vartheta$, that is, the motion of subparticles inside the particle with the Lorentz factor γ_p increases the scalar potential of the moving particle up to the value ϑ_p.

Due to the smallness of the local vector potential $\mathbf{U}_p$, we will not use it in our calculations. As a result, for the four-potential of the acceleration field inside the sphere we can write the following:

$$U_\alpha = \left(\frac{\vartheta}{c}, 0, 0, 0 \right) = (\gamma' c, 0, 0, 0) = U^\alpha . \tag{36}$$

This means that we do not take into account the internal motion of subparticles in individual particles, assuming that $\gamma_p = 1$, so that the scalar potential of the particles will be equal to ϑ and will coincide with the acceleration field potential inside the fixed sphere.

5.1. Calculation for the Acceleration Field

Given that the mass four-current is $J^\alpha = \rho_0 u^\alpha = \rho_0 (\gamma' c, \gamma' \mathbf{v}')$, and the effective four-potential of the acceleration field inside the sphere is determined in (36), we find that $U_\alpha J^\alpha = \rho_0 \gamma'^2 c^2$.

Now we can write the first part of the integral on the left-hand side of (15) and in view of (19) we can perform integration in the spherical coordinates:

$$\begin{aligned} &-\frac{8\pi\eta}{c^2} \int U_\alpha J^\alpha \, dx^1 dx^2 dx^3 = -8\pi\eta\rho_0 \int \gamma'^2 r^2 \, dr d\varphi \sin\theta d\theta \\ &= -8\pi c^2 \gamma_c^2 \left[\frac{a}{2} - \frac{c}{4\sqrt{4\pi\eta\rho_0}} \sin\left(\frac{2a}{c} \sqrt{4\pi\eta\rho_0} \right) \right]. \end{aligned} \tag{37}$$

Since the acceleration tensor is defined by the expression $u_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu$, then in view of (36) the tensor invariant has the following form: $u_{\alpha\beta} u^{\alpha\beta} = -\frac{2}{c^2} S^2$.

The acceleration field strength inside the sphere is calculated in terms of the scalar and vector potentials [2], and since $\vartheta = \gamma' c^2$, $\mathbf{U} = 0$, according to (36), then in view of (19) we obtain:

$$\begin{aligned} &\mathbf{S} = -\nabla\vartheta - \frac{\partial \mathbf{U}}{\partial t} = -c^2 \nabla\gamma' \\ &= \frac{c^2 \gamma_c \mathbf{r}}{r^3} \left[\frac{c}{\sqrt{4\pi\eta\rho_0}} \sin\left(\frac{r}{c} \sqrt{4\pi\eta\rho_0} \right) - r\cos\left(\frac{r}{c} \sqrt{4\pi\eta\rho_0} \right) \right]. \end{aligned} \tag{38}$$

Using (38) we will calculate the following integral over the sphere's volume:

$$\begin{aligned} &-\int u_{\alpha\beta} u^{\alpha\beta} \, dx^1 dx^2 dx^3 = \frac{2}{c^2} \int S^2 \, dx^1 dx^2 dx^3 \\ &= 2c^2 \gamma_c^2 \int \frac{1}{r^4} \left[\frac{c}{\sqrt{4\pi\eta\rho_0}} \sin\left(\frac{r}{c} \sqrt{4\pi\eta\rho_0} \right) - r\cos\left(\frac{r}{c} \sqrt{4\pi\eta\rho_0} \right) \right]^2 dx^1 dx^2 dx^3 . \end{aligned}$$

This integral can be calculated similarly to (24) in the spherical coordinates:

$$-\int u_{\alpha\beta}u^{\alpha\beta}\,dx^1dx^2dx^3$$
$$=8\pi c^2\gamma_c^2\left[\frac{a}{2}+\frac{c}{4\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{2a}{c}\sqrt{4\pi\eta\rho_0}\right)-\frac{c^2}{4\pi\eta\rho_0 a}\sin^2\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right]. \tag{39}$$

Let us now go over to the right-hand side of (15), for which it is necessary to calculate the product $U^\alpha u_\alpha{}^0$ inside the sphere. If according to (36) the four-potential has the components $U^\alpha=(\gamma' c,0,0,0)$, then the time components of the acceleration tensor in the Cartesian space coordinates are $u_0{}^0=0$, $u_1{}^0=-\frac{S_x}{c}$, $u_2{}^0=-\frac{S_y}{c}$, $u_3{}^0=-\frac{S_z}{c}$. Consequently, $U^\alpha u_\alpha{}^0=0$, and the first integral on the right-hand side of (15) is equal to zero.

We have also to calculate the surface integral on the right-hand side of (15). If we introduce the vector $\mathbf{F}=(F_x,\,F_y,\,F_z)=(U^\alpha u_\alpha{}^1,\,U^\alpha u_\alpha{}^2,\,U^\alpha u_\alpha{}^3)$, then we see that the surface integral reduces to a doubled flux of this vector through the spherical surface of the system.

The radial component of the vector $\mathbf{F}$ is defined by the expression $F_r=\mathbf{F}\cdot\mathbf{e}_r$, where $\mathbf{e}_r$ is a unit vector directed along the radius. To determine the doubled flux of the vector $\mathbf{F}$ it suffices to multiply the value F_r, calculated at $r=a$, by the doubled area of the sphere:

$$2\oiint_S U^\alpha u_\alpha{}^k n_k\,dS=8\pi a^2F_r(a). \tag{40}$$

Since according to (36) the four-potential of the acceleration field inside the sphere has the components $U^\alpha=(\gamma' c,0,0,0)$, and the nonzero components of the acceleration tensor in the Cartesian space coordinates equal $u_0{}^1=-\frac{S_x}{c}$, $u_0{}^2=-\frac{S_y}{c}$, $u_0{}^3=-\frac{S_z}{c}$, then it should be:

$$\mathbf{F}=-\gamma'\mathbf{S}, \qquad F_r=\mathbf{F}\cdot\mathbf{e}_r=-\gamma' S_r.$$

In view of (19) and (38), we find:

$$F_r=-\frac{c^3\gamma_c^2}{r^3\sqrt{4\pi\eta\rho_0}}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right)-r\cos\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right)\right]\sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right).$$

At $r=a$ this expression gives $F_r(a)$, and then the surface integral (40) is calculated as follows:

$$2\oiint_S U^\alpha u_\alpha{}^k n_k\,dS=8\pi a^2F_r(a)$$
$$=-\frac{8\pi c^3\gamma_c^2}{a\sqrt{4\pi\eta\rho_0}}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin^2\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)-\frac{a}{2}\sin\left(\frac{2a}{c}\sqrt{4\pi\eta\rho_0}\right)\right]. \tag{41}$$

Substituting (37), (39), and (41) into (15), we can see that the theorem of energy for the acceleration field is exactly satisfied.

6. THE PRESSURE FIELD

In the physical system under consideration the vector potential $\mathbf{\Pi}$ is assumed to be equal to zero, and then the four-potential of the pressure field inside the sphere in the approximation of the special theory of relativity will be written as follows:

$$\pi_\alpha = \left(\frac{\wp}{c}, 0, 0, 0 \right) = \pi^\alpha . \tag{42}$$

The scalar potential of the pressure field was calculated in [6]:

$$\wp = \wp_c - \frac{\sigma c^2 \gamma_c}{\eta} + \frac{\sigma c^3 \gamma_c}{r\eta\sqrt{4\pi\eta\rho_0}} \sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0} \right) \approx \wp_c - \frac{2\pi\sigma\rho_0 r^2 \gamma_c}{3}. \tag{43}$$

The mass four-current has the following form: $J^\alpha = \rho_0 u^\alpha = \rho_0 (\gamma' c, \gamma' \mathbf{v}')$. With this in mind $\pi_\alpha J^\alpha = \rho_0 \wp \gamma'$, and we can write the first integral on the left-hand side of (16):

$$-\frac{8\pi\sigma}{c^2} \int \pi_\alpha J^\alpha \, dx^1 dx^2 dx^3 = -\frac{8\pi\sigma\rho_0}{c^2} \int \wp \gamma' \, dx^1 dx^2 dx^3 .$$

Substituting here (43) and γ' from (19), we find:

$$\begin{aligned} &-\frac{8\pi\sigma}{c^2} \int \pi_\alpha J^\alpha \, dx^1 dx^2 dx^3 \\ &= -\frac{8\pi\sigma\gamma_c}{\eta} \left(\begin{aligned} &\left(\wp_c - \frac{\sigma c^2 \gamma_c}{\eta} \right) \left[\frac{c}{\sqrt{4\pi\eta\rho_0}} \sin\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0} \right) - a\cos\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0} \right) \right] \\ &+ \frac{\sigma c^2 \gamma_c}{\eta} \left[\frac{a}{2} - \frac{c}{4\sqrt{4\pi\eta\rho_0}} \sin\left(\frac{2a}{c}\sqrt{4\pi\eta\rho_0} \right) \right] \end{aligned} \right). \end{aligned} \tag{44}$$

Since we assumed that in the system under consideration the vector potential of the pressure field is absent, then the solenoidal vector $\mathbf{I}$ of the pressure field, calculated as the curl of the vector potential [2], will also be equal to zero. In this case, the pressure field tensor $f_{\alpha\beta}$ will depend only on the field strength $\mathbf{C}$, so that the tensor invariant will equal $f_{\alpha\beta} f^{\alpha\beta} = -\frac{2}{c^2} C^2$. The pressure field strength is determined by the formula:

$$\begin{aligned} \mathbf{C} &= -\nabla \wp - \frac{\partial \mathbf{\Pi}}{\partial t} = \frac{\sigma c^2 \gamma_c \mathbf{r}}{\eta r^3} \left[\frac{c}{\sqrt{4\pi\eta\rho_0}} \sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0} \right) - r\cos\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0} \right) \right] \\ &\approx \frac{4\pi\sigma\rho_0\gamma_c \mathbf{r}}{3} \left(1 - \frac{2\pi\eta\rho_0 r^2}{5c^2} \right). \end{aligned} \tag{45}$$

Now we can write the second integral on the left-hand side of (16) in the spherical coordinates:

$$-\int f_{\alpha\beta} f^{\alpha\beta}\, dx^1 dx^2 dx^3 = \frac{8\pi}{c^2}\int\limits_{r=0}^{a} C^2 r^2\, dr$$

$$= \frac{8\pi\sigma^2 c^2 \gamma_c^2}{\eta^2}\int\limits_{r=0}^{a}\frac{1}{r^2}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right) - r\cos\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right)\right]^2 dr.$$

This integral is calculated in the same way as (24):

$$-\int f_{\alpha\beta} f^{\alpha\beta}\, dx^1 dx^2 dx^3$$

$$= \frac{8\pi\sigma^2 c^2 \gamma_c^2}{\eta^2}\left[\frac{a}{2} + \frac{c}{4\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{2a}{c}\sqrt{4\pi\eta\rho_0}\right) - \frac{c^2}{4\pi\eta\rho_0 a}\sin^2\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right]. \quad (46)$$

On the right-hand side of (16) there is a product $\pi^\alpha f_\alpha{}^0$, and the pressure field tensor components are the following: $f_0{}^0 = 0$, $f_1{}^0 = -\frac{C_x}{c}$, $f_2{}^0 = -\frac{C_y}{c}$, $f_3{}^0 = -\frac{C_z}{c}$. If we take into account the components of the four-potential π^α according to (42), then we can see that $\pi^\alpha f_\alpha{}^0 = 0$.

Now we will turn to the product $\pi^\alpha f_\alpha{}^k$ on the right-hand side of (16), where $k = 1,2,3$. Since the four-potential π^α contains only the time component with the index $\alpha = 0$, we will write out all the nonzero components of $f_0{}^k$: $f_0{}^1 = -\frac{C_x}{c}$, $f_0{}^2 = -\frac{C_y}{c}$, $f_0{}^3 = -\frac{C_z}{c}$. Consequently, the product $\pi^\alpha f_\alpha{}^k = -\frac{\wp}{c^2}\left(C_x, C_y, C_z\right) = -\frac{\wp}{c^2}\mathbf{C}$ is a radial vector directed oppositely to the pressure field strength vector $\mathbf{C}$. The fact that $\pi^\alpha f_\alpha{}^k$ is a radial vector allows us immediately to find the surface integral on the right-hand side of (16). To calculate this integral, we need to assume $r = a$ in the field strength $\mathbf{C}$ (45) and in the scalar potential $\wp$ (43), which are part of $\pi^\alpha f_\alpha{}^k$, and then to multiply $\mathbf{C}$ by the normal vector n_k, and multiply the obtained result by the area of the sphere's surface:

$$2\oiint\limits_S \pi^\alpha f_\alpha{}^k n_k\, dS = -\frac{8\pi a^2}{c^2}\wp(a) C_r(a)$$

$$= -\frac{8\pi\sigma\gamma_c}{\eta}\left[\wp_c - \frac{\sigma c^2\gamma_c}{\eta} + \frac{\sigma c^3\gamma_c}{\eta a\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right]$$

$$\times\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right) - a\cos\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right]. \quad (47)$$

Substituting the expressions from (44), (46) and (47) into (16), we find that the theorem of energy for the pressure field is satisfied, since all the terms in (16) completely cancel out with each other.

7. CONCLUSION

By the example of the electromagnetic field we derived the integral theorem of the field energy in relation (10). In addition, we introduced the concepts of the kinetic energy E_{kf} and the potential energy W_f of the electromagnetic field:

$$E_{kf} = \int A_\alpha j^\alpha \sqrt{-g}\, dx^1 dx^2 dx^3, \qquad W_f = \frac{1}{4\mu_0}\int F_{\alpha\beta}F^{\alpha\beta}\sqrt{-g}\, dx^1 dx^2 dx^3. \tag{48}$$

In (48), the energy E_{kf} is related to the energy of interaction of the field and particles and is calculated in terms of the product of the four-potential A_α of the field and the charge four-current j^α of the particles, and the energy W_f is expressed in terms of the volume integral of the tensor invariant $F_{\alpha\beta}F^{\alpha\beta}$ of the electromagnetic field. From (10) and (48) we obtain the following relation:

$$E_{kf} + 2W_f = -\frac{1}{\mu_0 c}\frac{d}{dt}\left(\int A^\alpha F_\alpha{}^0 \sqrt{-g}\, dx^1 dx^2 dx^3\right) - \frac{1}{\mu_0}\oiint_S A^\alpha F_\alpha{}^k n_k \sqrt{-g}\, dS. \tag{49}$$

For a closed system, the surface integral on the right-hand side of (49) vanishes due to the gauge of the four-potential A^α and the electromagnetic field tensor $F_\alpha{}^k$ at infinity. In the relativistic uniform system the product $A^\alpha F_\alpha{}^0$ also vanishes, and then (49) reduces to a simple relation $E_{kf} + 2W_f = 0$. This relation for the field resembles the classical virial theorem for particles of the form $2E_k + W \approx 0$, where E_k is kinetic energy, and W is the potential energy of the particles. The relation $E_{kf} + 2W_f = 0$ is often used in electrostatics, allowing to determine the electrical energy of the system in two different ways – either with the charge density and the electric potential, or with the field strength, which is part of the electromagnetic tensor. However, in the general case there was no proof of existence of a relationship between the energies (48) in the presence of electric currents and magnetic fields. Now we see that such a relationship in (49) is the consequence of the integral theorem of the field energy.

In (14) we presented the integral theorem of energy for the vector gravitational field in the framework of the covariant theory of gravitation, and in (15) and (16) – the integral theorem of energy for the acceleration field and the pressure field, respectively. By analogy with (48), for these fields we can also introduce the concepts of the kinetic energy and the potential energy of the field. In particular, in [8] for closed static uniform systems it was found that a relation of the form $E_{kf} + 2W_f = 0$ holds in them for the gravitational field.

For all the four vector fields in Sections 4, 5 and 6 we showed by direct calculation of all the terms in the formulation of the integral theorem of energy how exactly this theorem is satisfied in the case of a relativistic uniform system. These calculations prove that the integral theorem of the field energy is exactly satisfied, confirming the validity of the theorem.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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