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Exponentiated Weibull Weibull Distribution: Statistical Properties and Applications

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Abstract

We introduce a new five-parameter model related to Weibull distribution, the so called. *exponentiated Weibull Weibull* (EWW) distribution. It incluides some new and earlier distributions. Fundamental properties are deduced. We deal with *maximum likelihood* (ML) method to obtain parameter estimators. The interest of the recommended distribution is confirmed through real data sets.

1. INTRODUCTION

One of the widespread distribution for modeling lifetime data is the *Weibull* (W) distribution where it has monotone *hazard rate function* (hrf). In statistical literature, different generalizations and extensions of the W distribution were done to deal with bathtub shaped hrf. [1] and [2] pioneered and discussed the EW distribution for analyzing bathtub failure data. The modified W extension with a bathtub shaped hrf can be found in [3]. The generalized modified W distribution has been suggested in [4].

In the recent time, new generated families have attracted many of statisticians to perform new models. We list some of the generated families among many of others as: the beta-G [5, 6], gamma-G [7], Kumaraswamy-G [8], McDonald-G [9], gamma-G (Type 2) [10], transformed-transformer-G [11], W-G [12], Kumaraswamy odd log-logistic-G [13], Garhy-G [14], *exponentiated Weibull-G* (EW-G) [15] Kumaraswamy W-G [16], additive W-G [17], exponentiated extended-G [18], generalized additive W-G [19], Type II half logistic-G [20], odd Frechet-G [21] and power Lindley-G [22] among others.

The *cumulative distribution function* (cdf) of EW-G family (see [1]) is given by

$$F(x)=\left[1-e^{-\alpha\left[\frac{G(x)}{1-G(x)}\right]^{\beta}}\right]^{a} \quad ;a,\alpha,\beta>0, \tag{1}$$

where a and β (greater than zero) are the shape parameters and α (greater than zero) is the scale parameter. The *probability density function* (pdf) regarding to (1) is given by

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$$f(x)=\frac{a\alpha\beta(G(x))^{\beta-1}g(x)}{\left(1-G(x)\right)^{\beta+1}}e^{-\alpha\left\{\frac{G(x)}{1-G(x)}\right\}^{\beta}}\left[1-e^{-\alpha\left\{\frac{G(x)}{1-G(x)}\right\}^{\beta}}\right]^{a-1};a,\alpha,\beta>0. \tag{2}$$

We come up with a new five-parameter model as an interesting extension for the W distribution depending on EW-G distribution. We are motivated to study the EWW distribution because (i) it involves a number of conventional sub-models as well as it contains some new sub-models; (ii) As provided in Section 2 the EWW distribution can be considered as a mixture of W distribution as introduced in [23]; and (iii) The EWW distribution surpasses some of the recent lifetime distributions in regard to two real data examples.

The rest of the paper is outlined as follows. Section 2 defines the EWW distribution and provides its special models. Section 3 gives important representation for the EWW density and distribution functions. Furthermore, it contains basic properties of the EWW distribution. The ML method is employed to get the parameter estimators of the subject model in Section 4. The accuracy and performance of the ML estimates are checked through a simulation study in Section 5. An illustrative example is given in Section 6 to explain how a real data can be formed by EWW model. Finally, the paper is concluded in last section.

2. THE EXPONNTIATED WEIBULL WEIBULL DISTRIBUTION

We obtain the EWW distribution depending on the EW-G family. Consider the random variable *X* has the W distribution with pdf given by

$$g(x;\lambda,\gamma)=\lambda\gamma x^{\gamma-1}e^{-\lambda x^{\gamma}};\qquad x>0, \tag{3}$$

where, λ (greater than zero) is the scale parameter and γ (greater than zero) is the shape parameter. The cdf of W distribution is given by

$$G(x;\lambda,\gamma)=1-e^{-\lambda x^{\gamma}}. \tag{4}$$

We get the cdf of EWW distribution by subsituting (3) and (4) into (1) as follows

$$F(x;\kappa)=\left\{1-e^{-\alpha(e^{\lambda x^{\gamma}}-1)^{\beta}}\right\}^{a};\quad a,\alpha,\beta,\lambda,\gamma>0\quad,\quad x>0, \tag{5}$$

where, $\kappa\equiv(a,\alpha,\beta,\lambda,\gamma),$ is parameter set of distribution. A random variable *X* has (5) shall be denoted by EWW $(a,\alpha,\beta,\lambda,\gamma)$. The pdf of EWW is obtained by subsituting (3) and (4) into (2) as follows

$$f(x;\kappa)=a\alpha\beta\lambda\gamma x^{\gamma-1}[e^{\lambda x^{\gamma}}-1]^{\beta-1}\exp[-\{\alpha(e^{\lambda x^{\gamma}}-1)^{\beta}-\lambda x^{\gamma}\}]\left\langle 1-e^{-\alpha(e^{\lambda x^{\gamma}}-1)^{\beta}}\right\rangle^{a-1}. \tag{6}$$

The pdf (6) comprises some new distributions and at the same time it contains existing distributions (see Table 1).

Table 1. *Special models of the Exponentiated Weibull Weibull distribution*

	Distribution	a	α	β	γ	λ	Distribution function	Author
1	EW exponential	-	-	-	1	-	$F(x)=\left\{1-\exp(-\alpha(e^{\lambda x}-1)^{\beta})\right\}^{a}$	[24]
2	EW Rayleigh	-	-	-	2	-	$F(x)=\left\{1-\exp(-\alpha(e^{\lambda x^2}-1)^{\beta})\right\}^{a}$	
3	Exponentaited Exponential Weibull	-	-	1	-	-	$F(x)=\left\{1-\exp(-\alpha(e^{\lambda x^{\gamma}}-1))\right\}^{a}$	
4	Exponentaited Exponential exponential	-	-	1	1	-	$F(x)=\left\{1-\exp(-\alpha(e^{\lambda x}-1))\right\}^{a}$	
5	Exponentaited Exponential Rayleigh	-	-	1	2	-	$F(x)=\left\{1-\exp(-\alpha(e^{\lambda x^2}-1))\right\}^{a}$	
6	Exponentaited Rayleigh Weibull	-	-	2	-	-	$F(x)=\left\{1-\exp(-\alpha(e^{\lambda x^{\gamma}}-1)^{2})\right\}^{a}$	
7	Exponentaited Rayleigh Exponential	-	-	2	1	-	$F(x)=\left\{1-\exp(-\alpha(e^{\lambda x}-1)^{2})\right\}^{a}$	
8	Exponentaited Rayleigh Rayleigh	-	-	2	2	-	$F(x)=\left\{1-\exp(-\alpha(e^{\lambda x^2}-1)^{2})\right\}^{a}$	
9	Weibull Weibull	1	-	-	-	-	$F(x)=1-\exp(-\alpha(e^{\lambda x^{\gamma}}-1)^{\beta})$	[25]
10	Weibull Exponential	1	-	-	1	-	$F(x)=1-\exp(-\alpha(e^{\lambda x}-1)^{\beta})$	[26]
11	Weibull Rayleigh	1	-	-	2	-	$F(x)=1-\exp(-\alpha(e^{\lambda x^2}-1)^{\beta})$	[27]
12	Exponential Weibull	1	-	1	-	-	$F(x)=1-\exp(-\alpha(e^{\lambda x^{\gamma}}-1)^{\beta})$	
13	Exponential	1	-	1	1	-	$F(x)=1-\exp(-\alpha(e^{\lambda x}-1))$	
14	Exponential Rayleigh	1	-	1	2	-	$F(x)=1-\exp(-\alpha(e^{\lambda x^2}-1))$	
15	Rayleigh Weibull	1	-	2	-	-	$F(x)=1-\exp(-\alpha(e^{\lambda x^{\gamma}}-1)^{2})$	
16	Rayleigh Exponential	1	-	2	1	-	$F(x)=1-\exp(-\alpha(e^{\lambda x}-1)^{2})$	
17	Rayleigh Rayleigh	1	-	2	2	-	$F(x)=1-\exp(-\alpha(e^{\lambda x^2}-1)^{2})$	

The survival function, hrf, reversed-hrf and cumulative hrf of EWW distribution are respectively given by

$$S(x;\kappa)=1-\left\langle 1-\exp(-\alpha(e^{\lambda x^{\gamma}}-1)^{\beta}\right\rangle^{a},$$

$$h(x;\kappa)=\frac{a\alpha\beta\lambda\gamma x^{\gamma-1}[e^{\lambda x^{\gamma}}-1]^{\beta-1}\exp[-\{\alpha(e^{\lambda x^{\gamma}}-1)^{\beta}-\lambda x^{\gamma}\}]\left\{1-\exp(-\alpha(e^{\lambda x^{\gamma}}-1)^{\beta}\right\}^{a-1}}{1-\left\langle 1-\exp(-\alpha(e^{\lambda x^{\gamma}}-1)^{\beta}\right\rangle^{a}},$$

$$\tau(x;\kappa)=\frac{a\alpha\beta\lambda\gamma x^{\gamma-1}[e^{\lambda x^{\gamma}}-1]^{\beta-1}\exp[-\{\alpha(e^{\lambda x^{\gamma}}-1)^{\beta}-\lambda x^{\gamma}\}]}{1-\exp(-\alpha(e^{\lambda x^{\gamma}}-1)^{\beta})},$$

and,

$$H(x;\kappa)=-\ln\left\langle 1-\left(1-\exp(-\alpha\left(e^{\lambda x^{\gamma}}-1\right)^{\beta})\right)^{a}\right\rangle.$$

Figures 1, 2, and 3 gives pdf, hrf and reversed hrf plots of the EWW distribution for certain values.

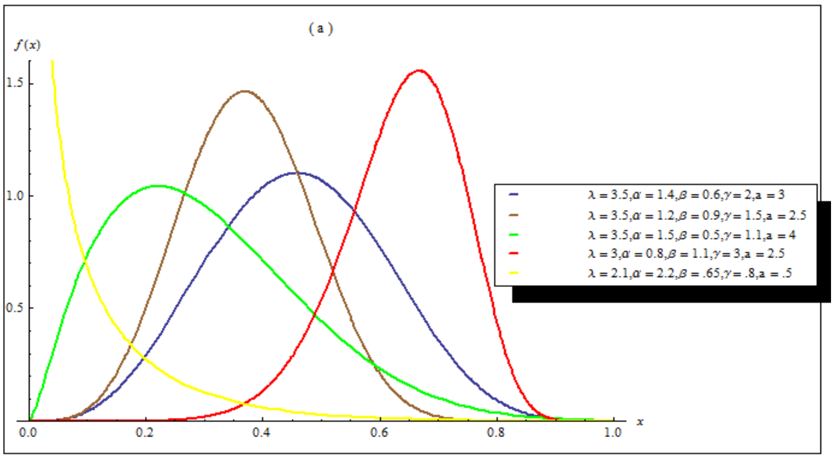


Figure 1. *The pdf plots of EWW distribution for specific parameter values*

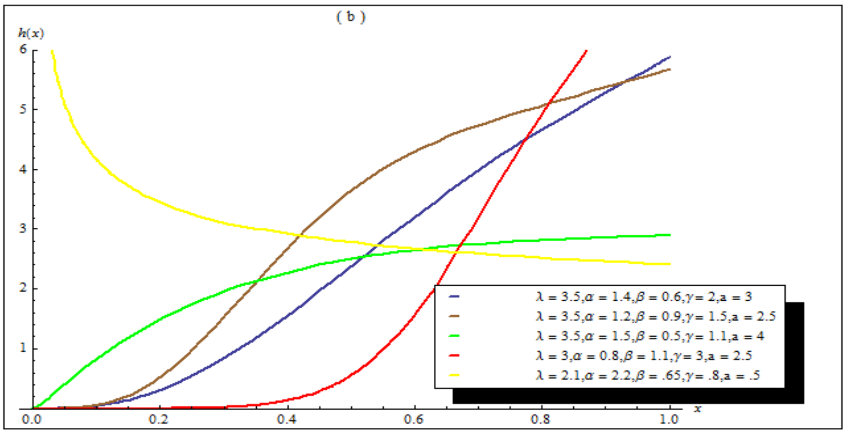


Figure 2. *The hrf plots of EWW distribution for specific parameter values*

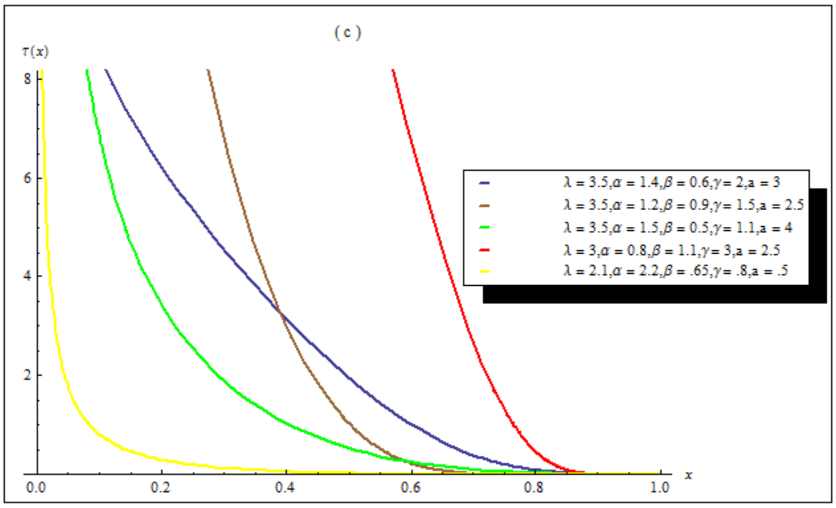


Figure 3. *The reversed hrf plots of EWW distribution for specific parameter values*

3. PRINCIPAL PROPERTIES

This section displays elementary properties of the EWW distribution.

3.1. Expansions

Important mixture expressions for the pdf and cdf of the EWW distribution are displayed. So, we rewrite the pdf (6) as follows

$$f(x;\kappa)=\frac{a\alpha\beta\lambda\gamma x^{\gamma-1}e^{-\lambda x^{\gamma}}}{\left[e^{-\lambda x^{\gamma}}\right]^{\beta+1}}\left[1-e^{-\lambda x^{\gamma}}\right]^{\beta-1}\exp\left\{-\alpha\left[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}\right]^{\beta}\right\}\left[1-\exp\left\{-\alpha\left[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}\right]^{\beta}\right\}\right]^{a-1}.$$

Since, the generalized binomial expansion

$$\left(1-m\right)^{c-1}=\sum_{r=0}^{\infty}(-1)^{r}\binom{c-1}{r}m^{r}. \tag{7}$$

where c (greater than zero) is real and $|m|<1$, then by using (7), the EWW pdf reduces to

$$f(x;\kappa)=\frac{a\alpha\beta\lambda\gamma x^{\gamma-1}e^{-\lambda x^{\gamma}}}{\left[e^{-\lambda x^{\gamma}}\right]^{\beta+1}}\left[1-e^{-\lambda x^{\gamma}}\right]^{\beta-1}\sum_{i=0}^{\infty}[-1]^{i}\binom{a-1}{i}\exp\left\{-\alpha(i+1)\left[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}\right]^{\beta}\right\}. \tag{8}$$

By using the following relation

$$\exp\left\{-\alpha(i+1)\left[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}\right]^{\beta}\right\}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\alpha^{j}(i+1)^{j}}{j!}\left[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}\right]^{\beta j}. \tag{9}$$

Inserting the expansion (9) in (8) we have

$$f(x;\kappa)=a\beta\lambda\gamma x^{\gamma-1}\,\mathrm{e}^{-\lambda x^{\gamma}}\sum_{i,j=0}^{\infty}\frac{[-1]^{i+j}\,\alpha^{j+1}(i+1)^{j}}{j!}\binom{a-1}{i}\frac{\left[1-\mathrm{e}^{-\lambda x^{\gamma}}\right]^{\beta(j+1)-1}}{\left[1-\left[1-\mathrm{e}^{-\lambda x^{\gamma}}\right]\right]^{\beta(j+1)+1}}. \qquad (10)$$

Using (7), then we write

$$\left[1-\left[1-\mathrm{e}^{-\lambda x^{\gamma}}\right]\right]^{-[\beta(j+1)+1]}=\sum_{k=0}^{\infty}\binom{\beta(j+1)+k}{k}\left[1-\mathrm{e}^{-\lambda x^{\gamma}}\right]^{k}.$$

Then, (10) reduces to

$$f(x;\kappa)=a\beta\lambda\gamma x^{\gamma-1}\,\mathrm{e}^{-\lambda x^{\gamma}}\sum_{i,j,k=0}^{\infty}\frac{[-1]^{i+j}\,\alpha^{j+1}(i+1)^{j}}{j!}\binom{a-1}{i}\binom{\beta(j+1)+k}{k}\left[1-\mathrm{e}^{-\lambda x^{\gamma}}\right]^{\beta(j+1)+k-1}.$$

Again, using the binomial theorem another time, then the pdf is written as infinite linear combination of W distribution, that is

$$f(x;\kappa)=\sum_{i,j,k,\ell_1=0}^{\infty}\eta_{i,j,k,\ell_1}x^{\gamma-1}e^{-\lambda(\ell_1+1)x^{\gamma}}, \qquad (11)$$

where,

$$\eta_{i,j,k,\ell_1}=\frac{a\beta\lambda\gamma[-1]^{i+j+\ell_1}\,\alpha^{j+1}(i+1)^{j}}{j!}\binom{a-1}{i}\binom{\beta(j+1)+k}{k}\binom{\beta(j+1)+k-1}{\ell_1}.$$

Furtheremore, an expansion for the cdf; $[F(x;\kappa)]^{h}$ is derived. Using binomial expansion for cdf (5)

$$[F(x;\kappa)]^{h}=\left\langle 1-\exp\left\{-\alpha\left[\frac{1-\mathrm{e}^{-\lambda x^{\gamma}}}{\mathrm{e}^{-\lambda x^{\gamma}}}\right]^{\beta}\right\}\right\rangle^{ah},$$

where h is an integer and a (greater than zero), gets :

$$[F(x;\kappa)]^{h}=\sum_{q=0}^{\infty}[-1]^{q}\binom{ah}{q}\exp\left\{-\alpha q\left[\frac{1-\mathrm{e}^{-\lambda x^{\gamma}}}{\mathrm{e}^{-\lambda x^{\gamma}}}\right]^{\beta}\right\}.$$

Applying exponential expansion for the cdf in the previous Equation, we get

$$[F(x;\kappa)]^h = \sum_{q,t=0}^{\infty} \frac{[-1]^{q+t}(\alpha q)^t \mathrm{e}^{-\lambda x^{\gamma}}}{t!} \binom{ah}{q} \frac{\left[1-\mathrm{e}^{-\lambda x^{\gamma}}\right]^{\beta t}}{\left[1-\left[1-\mathrm{e}^{-\lambda x^{\gamma}}\right]\right]^{\beta t+1}}.$$

Using the relation (9) in the preceding Equation, we have

$$[F(x;\kappa)]^h = \sum_{q,t,m=0}^{\infty} \frac{[-1]^{q+t}(\alpha q)^t \mathrm{e}^{-\lambda x^{\gamma}}}{t!} \binom{ah}{q} \binom{\beta t+m}{m} \left[1-\mathrm{e}^{-\lambda x^{\gamma}}\right]^{\beta t+m}.$$

Using the relation (7) in the preceding Equation, we get

$$[F(x;\kappa)]^h = \sum_{q,t,m,\ell_2=0}^{\infty} \eta_{q,t,m,\ell_2} e^{-\lambda(\ell_2+1)x^{\gamma}}, \tag{12}$$

where, $\eta_{q,t,m,\ell_2} = \dfrac{[-1]^{q+t+\ell_2}(\alpha q)^t}{t!} \dbinom{ah}{q} \dbinom{\beta t+m}{m} \dbinom{\beta t+m}{\ell_2}.$

3.2. Quantile and Median

For *X* has EWW distribution, its quantile function, say $Q(u)=F^{-1}(u)$ is yielded by inverting (5) as follows

$$Q(u) = \sqrt[\gamma]{\ln\left\{1+(\ln(1-(u)^{\frac{1}{a}})^{\frac{-1}{\alpha}})^{\frac{1}{\beta}}\right\}^{\frac{1}{\lambda}}}, \tag{13}$$

where, *u* is a uniform random variable on (0,1). For $u = 0.5$ the median of distribution is as follows

$$Median = Q(u) = \sqrt[\gamma]{\ln\left\{1+(\ln(1-(0.5)^{\frac{1}{a}})^{\frac{-1}{\alpha}})^{\frac{1}{\beta}}\right\}^{\frac{1}{\lambda}}}.$$

3.3. Moments

The *r*th moment of EWW distribution is obtained as follows

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx. \tag{14}$$

Substituting (11) into (14) yields:

$$\mu_r' = E(X^r) = \sum_{i,j,k,\ell_1=0}^{\infty} \eta_{i,j,k,\ell_1} \int_0^{\infty} x^{r+\gamma-1} e^{-\lambda(\ell_1+1)x^{\gamma}} dx.$$

Let $y=\lambda(\ell_1+1)x^{\gamma}$ then, μ_r' becomes

$$\mu_r' = \sum_{i,j,k,\ell_1=0}^{\infty} \eta_{i,j,k,\ell_1} \int_0^{\infty} x^{r+\gamma-1} e^{-\lambda(\ell_1+1)x^{\gamma}} dx .$$

Generally, the moment generating function of the EWW distribution is obtained as

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r)=\sum_{r,j,k,\ell_1=0}^{\infty}\frac{t^r}{r!}\frac{\eta_{j,k,\ell_1}\Gamma(r/\gamma+1)}{\gamma}\left[\lambda\left[\ell_1-\beta(k+1)\right]\right]^{\frac{-r}{\gamma}-1}.$$

3.4. Incomplete and Conditional Moments

The *s*th incomplete moment, say $\varphi_s(t),$ is defined by

$$\varphi_s(t)=\int_0^t x^s f(x)dx \quad .$$

Using (11), then $\varphi_s(t)$ will be as follows

$$\varphi_s(t)=\sum_{i,j,k,\ell_1=0}^{\infty}\frac{\eta_{i,j,k,\ell_1}}{\gamma}\nu\left(s/\gamma+1,\lambda(\ell_1+1)t^{\gamma}\right)\left(\lambda(\ell_1+1)\right)^{\frac{-s}{\gamma}-1}, \tag{15}$$

where $\nu(s,t)=\int_0^t x^{s-1}e^{-x}dx$ is the lower incomplete gamma function. Further, the *s*th conditional moment, say $\pi_s(t),$ is defined by

$$\pi_s(t)=\int_t^{\infty} x^s f(x)dx \quad .$$

Hence, by using pdf (11), we get

$$\pi_s(t)=\sum_{i,j,k,\ell_1=0}^{\infty}\frac{\eta_{i,j,k,\ell_1}}{\gamma}\Gamma\left(s/\gamma+1,\lambda(\ell_1+1)t^{\gamma}\right)\left(\lambda(\ell_1+1)\right)^{\frac{-s}{\gamma}-1},$$

where $\Gamma(s,t)=\int_t^{\infty} x^{s-1}e^{-x}dx$ is the upper incomplete gamma function. Additionally, the mean deviation can be calculated by using

$$\delta_1(X)=2\mu F(\mu)-2J(\mu) \qquad \text{and} \qquad \delta_2(X)=\mu-2J(\mathrm{M}),$$

where, $J(q)$ is the first incomplete moment and is obtained from (15), so

$$J(\mu)=\sum_{i,j,k,\ell_1=0}^{\infty}\frac{\eta_{i,j,k,\ell_1}}{\gamma}\nu\left(1/\gamma+1,\lambda(\ell_1+1)\mu^{\gamma}\right)\left(\lambda(\ell_1+1)\right)^{\frac{-1}{\gamma}-1},$$

and,

$$J(M)=\sum_{i,j,k,\ell_1=0}^{\infty}\frac{\eta_{i,j,k,\ell}}{\gamma}\nu\left(1/\gamma+1,\lambda(\ell_1+1)M^{\gamma}\right)\left(\lambda(\ell_1+1)\right)^{\frac{-1}{\gamma}-1}.$$

3.5. Residual Life Function

The *n*th moment of the residual life function (RLF) of *X* is given by

$$m_n(t)=\frac{1}{S(t)}\int_t^{\infty}\left(x-t\right)^n f(x)dx.$$

Employing the binomial expansion for $\left(x-t\right)^n$ we have

$$m_n(t)=\frac{1}{S(t;\kappa)}\sum_{i,j,k,\ell_1=0}^{\infty}\sum_{d=0}^{n}\left(-t\right)^d\binom{n}{d}\eta_{i,j,k,\ell_1}\frac{\Gamma\left(n-d/\gamma+1,\lambda(\ell_1+1)t^{\gamma}\right)}{\gamma\left(\lambda(\ell_1+1)\right)^{\frac{n-d}{\gamma}+1}},\tag{16}$$

The *n*th moment of the reversed RLF of *X* is given by

$$M_n(t)=\frac{1}{F(t)}\int_0^{t}\left(t-x\right)^n f(x)dx.$$

Again, employing the binomial expansion for $\left(x-t\right)^n$, we have

$$M_n(t)=\frac{1}{F(t;\kappa)}\sum_{i,j,k,\ell_1=0}^{\infty}\sum_{d=0}^{n}\left(-t\right)^d\binom{n}{d}\eta_{i,j,k,\ell_1}\frac{\nu\left(n-d/\gamma+1,\lambda(\ell_1+1)t^{\gamma}\right)}{\gamma\left(\lambda(\ell_1+1)\right)^{\frac{n-d}{\gamma}+1}},$$

3.6. Inequality Measures

Lorenz, Bonferroni and Zenga curves are inequality measures which are extensively used in income and wealth distributions (see [31]). They are obtained, respectively, as below

$$L_F(t)=\frac{\int_0^{t}xf(x;\kappa)dt}{E(X)}=\frac{\sum_{i,j,k,\ell_1=0}^{\infty}\eta_{i,j,k,\ell_1}\nu\left(1/\gamma+1,\lambda(\ell_1+1)t^{\gamma}\right)\left(\lambda(\ell_1+1)\right)^{\frac{-1}{\gamma}-1}}{\sum_{i,j,k,\ell_1=0}^{\infty}\eta_{i,j,k,\ell_1}\Gamma(1/\gamma+1)\left[\lambda(\ell_1+1)\right]^{\frac{-1}{\gamma}-1}},$$

$$B_F(t)=\frac{L_F(t)}{F(t;\kappa)}=\frac{\sum_{i,j,k,\ell_1=0}^{\infty}\eta_{i,j,k,\ell_1}\nu\left(1/\gamma+1,\lambda(\ell_1+1)t^{\gamma}\right)\left(\lambda(\ell_1+1)\right)^{\frac{-1}{\gamma}-1}}{\sum_{i,j,k,\ell_1=0}^{\infty}\eta_{i,j,k,\ell_1}\Gamma(1/\gamma+1)\left[\lambda(\ell_1+1)\right]^{\frac{-1}{\gamma}-1}(1-e^{-\alpha(e^{\lambda t^{\gamma}}-1)^{\beta}})^{a}},$$

and

$$A_F(t)=1-\frac{\mu^{-}(t)}{\mu^{+}(t)},$$

where

$$\mu^{-}(t)=\frac{\varphi_1(t)}{E(T)}=\frac{\sum_{i,j,k,\ell_1=0}^{\infty}\eta_{i,j,k,\ell_1}\nu\left(1/\gamma+1,\lambda(\ell_1+1)t^{\gamma}\right)\left(\lambda(\ell_1+1)\right)^{\frac{-1}{\gamma}-1}}{\sum_{i,j,k,\ell_1=0}^{\infty}\eta_{i,j,k,\ell_1}\Gamma(1/\gamma+1)\left(\lambda(\ell_1+1)\right)^{\frac{-1}{\gamma}-1}},$$

and

$$\mu^{+}(t)=\frac{\varphi_1(t)}{1-F(t;\kappa)}=\frac{\sum_{i,j,k,\ell_1=0}^{\infty}\eta_{i,j,k,\ell_1}\nu\left(1/\gamma+1,\lambda(\ell_1+1)t^{\gamma}\right)\left(\lambda(\ell_1+1)\right)^{\frac{-1}{\gamma}-1}}{[1-(1-e^{-\alpha(e^{\lambda t^{\gamma}}-1)^{\beta}})^{a}]\sum_{i,j,k,\ell_1=0}^{\infty}\eta_{i,j,k,\ell_1}\Gamma(1/\gamma+1)\left[\lambda(\ell_1+1)\right]^{\frac{-1}{\gamma}-1}}.$$

3.7. Rényi and q-entropies

The Rényi entropy of EWW is formally given by

$$I_{\delta}(X)=(1-\delta)^{-1}\log\int_{-\infty}^{\infty}f(x)^{\delta}dx,\quad \delta>0 \text{ and } \delta\neq 1.$$

We rewrite the pdf $f(x;\kappa)^{\delta}$ by using the binomial expansion (7) in (6) as follows

$$f(x;\kappa)^{\delta}=\sum_{j,k,m,\ell_1=0}^{\infty}W_{j,k,m,\ell_1}x^{\delta(\gamma-1)}e^{-\lambda(\ell_1+\delta)x^{\gamma}},$$

where,

$$W_{j,k,m,\ell_1}=\frac{(ab\beta\lambda\gamma)^{\delta}\alpha^{k+\delta}[-1]^{j+k+\ell_1}(j+\delta)^{k}}{k!}\binom{\delta(a-1)}{j}\binom{\beta(k+\delta)+m}{m}\binom{\beta(k+\delta)+m-1}{\ell_1}.$$

Hence, $I_{\delta}(X)$ of the EWW distribution is specified by

$$I_{\delta}(X)=(1-\delta)^{-1}\log\left[\sum_{j,k,\ell_1=0}^{\infty}\frac{W_{j,k,\ell_1}}{\gamma}\Gamma(\delta+(1/\gamma)-(\delta/\gamma))\left[\lambda\left(\ell_1+\delta\right)\right]^{-\delta-\frac{1}{\gamma}+\frac{\delta}{\gamma}}\right].$$

Furthermore, the q-entropy takes the form

$$H_q(X)=(1-q)^{-1}\log\left(1-\int_{-\infty}^{\infty}f(x)^q dx\right), q>0 \text{ and } q\neq 1.$$

Therefore, the q-entropy of the EWW distribution takes the form

$$H_q(X)=(1-q)^{-1}\log\left\{1-\left[\sum_{j,k,\ell_1=0}^{\infty}\frac{W_{j,k,\ell_1}}{\gamma}\Gamma(q+(1/\gamma)-(q/\gamma))\left[\lambda\left(\ell_1+q\right)\right]^{-q-(1/\gamma)+(q/\gamma)}\right]\right\}.$$

3.8. The Probability Weighted Moments

In general, the probability weighted moments (PWM) method is employed for estimating the parameters of distributions in which they inverse form are not in explicit form (see [32]). It is specified by

$$\tau_{r,s}=E[X^r\,\mathrm{F}(x)^s]=\int_{-\infty}^{\infty}x^r\,\mathrm{f}(x)(\mathrm{F}(x))^s dx. \tag{17}$$

By substituting (11) and (12) into (17), replacing *h* with *s*, leads to:

$$\tau_{r,s}=\sum_{i,j,k,\ell_1=0}^{\infty}\sum_{q,t,m,\ell_2=0}^{\infty}\eta_{q,t,m,\ell_2}\eta_{i,j,k,\ell_1}\int_0^{\infty}x^{r+\gamma-1}e^{-\lambda(\ell_1+\ell_2+2)x^{\gamma}}dx.$$

Hence, the PWM of the EWW distribution will be

$$\tau_{r,s}=\sum_{i,j,k,\ell_1=0}^{\infty}\sum_{q,t,m,\ell_2=0}^{\infty}\frac{\eta_{q,t,m,\ell_2}\eta_{i,j,k,\ell_1}\Gamma(r/\gamma+1)}{\gamma}\left[\lambda(\ell_1+\ell_2+2)\right]^{\frac{-r}{\gamma}-1}.$$

3.9. Order Statistics

Given $X_{1:n}<X_{2:n}<\ldots,<X_{n:n}$ be the order statistics of a random sample of size *n* has the EWW distribution, then,the pdf of the *k*th order statistics can be written as follows

$$f_{X_{k:n}}(x)=\frac{f(x)}{B(k,n-k+1)}\sum_{\nu=0}^{n-k}(-1)^{\nu}\binom{n-k}{\nu}F(x)^{\nu+k-1}, \tag{18}$$

where, *B*(.,.) is the beta function. By substituting (11) and (12) in (18), changing *h* with $\nu+k$-1,gives

$$f_{X_{k:n}}(x)=\frac{1}{B(k,n-k+1)}\sum_{v=0}^{n-k}\sum_{i,j,k,\ell_1=0}^{\infty}\sum_{q,t,m,\ell_2=0}^{\infty}\eta^{*}x^{\gamma-1}e^{-\lambda(\ell_1+\ell_2+2)x^{\gamma}},\tag{19}$$

where $\eta^{*}=(-1)^{v}\binom{n-k}{v}\eta_{i,j,k,\ell_1}\eta_{q,t,m,\ell_2}$.

Moments of order statistics is given by:

$$E(X_{k:n}^{r})=\frac{1}{B(k,n-k+1)}\sum_{v=0}^{n-k}\sum_{i,j,k,\ell_1=0}^{\infty}\sum_{q,t,m,\ell_2=0}^{\infty}\frac{\eta^{*}\Gamma(r/\gamma+1)}{\gamma}\left[\lambda(\ell_1+\ell_2+2)\right]^{\frac{-r}{\gamma}-1}.$$

4. MAXIMUM LIKLIHOOD ESTIMATION

The ML estimators of the population parameters for the EWW distribution are derived in case of complete samples. Let $x_1,\dots,x_n$ be the observed values from the EWW distribution with set of parameters $\kappa=(a,\alpha,\beta,\lambda,\gamma)^{T}$. The total log-likelihood function of κ is

$$\ln L(\kappa)=n\ln a+n\ln\alpha+n\ln\beta+n\ln\lambda+n\ln\gamma+(\gamma-1)\sum_{i=1}^{n}\ln\left(x_i\right)+(\beta-1)\sum_{i=1}^{n}\ln\left(e^{\lambda x_i^{\gamma}}-1\right)$$
$$+\lambda\sum_{i=1}^{n}x_i^{\gamma}-\alpha\sum_{i=1}^{n}\left[\left(e^{\lambda x_i^{\gamma}}-1\right)\right]^{\beta}+(a-1)\sum_{i=1}^{n}\ln\left[1-\exp\left[-\alpha\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}\right]\right].$$

The elements of the score function $U(\kappa)=(U_a,U_\alpha,U_\beta,U_\lambda,U_\gamma)$ are given by

$$U_a=n/a+\sum_{i=1}^{n}\ln\left[1-\exp\left[-\alpha\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}\right]\right],\tag{20}$$

$$U_\alpha=n/\alpha-\sum_{i=1}^{n}\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}+(a-1)\sum_{i=1}^{n}\frac{\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}\exp[-\alpha\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}]}{1-\exp[-\alpha\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}]},\tag{21}$$

$$U_\beta=n/\beta+\sum_{i=1}^{n}\ln\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}-\alpha\sum_{i=1}^{n}\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}\ln\left(e^{\lambda x_i^{\gamma}}-1\right)$$
$$+\alpha(a-1)\sum_{i=1}^{n}\frac{\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}\exp\left[-\alpha\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}\right]\ln\left(e^{\lambda x_i^{\gamma}}-1\right)}{1-\exp\left[-\alpha\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}\right]},\tag{22}$$

$$U_\lambda=n/\lambda+(\beta-1)\sum_{i=1}^{n}\frac{x_i^{\gamma}e^{\lambda x_i^{\gamma}}}{e^{\lambda x_i^{\gamma}}-1}+\alpha\beta(a-1)\sum_{i=1}^{n}\frac{x_i^{\gamma}e^{\lambda x_i^{\gamma}}\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta-1}e^{-\alpha\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}}}{1-\exp\left[-\alpha\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}\right]}+\sum_{i=1}^{n}x_i^{\gamma}$$
$$-\alpha\beta\sum_{i=1}^{n}x_i^{\gamma}\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta-1}e^{\lambda x_i^{\gamma}},\tag{23}$$

and

$$U_{\gamma}=n/\gamma+\sum_{i=1}^{n}\ln\left(x_i^{\gamma}\right)+\left(\beta-1\right)\sum_{i=1}^{n}\frac{x_i^{\gamma}e^{\lambda x_i^{\gamma}}}{e^{\lambda x_i^{\gamma}}-1}+\lambda\sum_{i=1}^{n}x_i^{\gamma}\ln\left(x_i\right)-\alpha\beta\lambda\sum_{i=1}^{n}x_i^{\gamma}\ln\left(x_i\right)\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta-1}e^{\lambda x_i^{\gamma}}$$
$$+\alpha\beta\lambda(a-1)\sum_{i=1}^{n}\frac{x_i^{\gamma}\ln\left(x_i\right)e^{\lambda x_i^{\gamma}}\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta-1}1-\exp\left[-\alpha\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}\right]}{1-\exp\left[-\alpha\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}\right]}. \tag{24}$$

Equations [20-24] are setted by zeros, then we get the ML estimators of the parameters. It is very hard to solve these Equations, so Newton-Raphson's iteration method is employed.

5. SIMULATION ILLUSTRATION

A comprehnsive numerical inspection is achieved to evaluate the behaviour of ML estimates (MLEs) for the EWW model. The biases, *mean square errors* (MSEs) and variance are calculated, for different sample sizes, to evalute performance of ML estimates. The simulation procedure is done via Mathematica (9) and described as follows:

- ❖ We generate 10000 random samples of sizes n = 20, 30, 50 and 100 from the EWW distribution.
- ❖ Select parameter values of $a, \alpha, \beta, \lambda, \gamma$ as Set1 (1.2, 2, 0.5, 1.5, 1), Set2 (1.5, 2, 0.5, 1.5, 1), Set3 (1.8, 2, 0.5, 1.5, 1), and Set4 (2, 2,0.5, 1.5, 1).
- ❖ The biases, MSE, mean and variances of MLEs at each sample size are calculated. Outcomes of simulation are sorted in Tables 2 to 5.

***Table 2**. Biases, Mean, MSEs, and Variances of MLEs for EWW distribution of Set1= (1.2,2,0.5,1.5,1)*

n	MLE	Mean	Bias	MSE	Variance
20	$\hat{a}$	1.2732	0.0732	0.0809	0.0756
	$\hat{\alpha}$	2.1281	0.1281	0.3411	0.3247
	$\hat{\beta}$	0.5477	0.0477	0.0283	0.0260
	$\hat{\lambda}$	1.5499	0.0499	0.0775	0.0750
	$\hat{\gamma}$	1.1095	0.1095	0.1352	0.1232
30	$\hat{a}$	1.2677	0.0677	0.0575	0.0529
	$\hat{\alpha}$	2.0523	0.0523	0.1866	0.1838
	$\hat{\beta}$	0.5482	0.0482	0.0211	0.0188
	$\hat{\lambda}$	1.5249	0.0249	0.0412	0.0405
	$\hat{\gamma}$	1.1077	0.1077	0.1085	0.0969
50	$\hat{a}$	1.2270	0.0270	0.0403	0.0396
	$\hat{\alpha}$	2.0844	0.0844	0.1322	0.1251
	$\hat{\beta}$	0.5223	0.0223	0.0128	0.0123
	$\hat{\lambda}$	1.5327	0.0327	0.0317	0.0307
	$\hat{\gamma}$	1.0436	0.0436	0.0599	0.0580
100	$\hat{a}$	1.2118	0.0118	0.0172	0.0170
	$\hat{\alpha}$	2.0486	0.0486	0.0697	0.0673
	$\hat{\beta}$	0.5080	0.0080	0.0058	0.0057
	$\hat{\lambda}$	1.5150	0.0150	0.0186	0.0184
	$\hat{\gamma}$	1.0159	0.0159	0.0240	0.0237

Table 3. *Biases, Mean, MSEs, and Variances of MLEs for EWW distribution of Set2 =(1.5,2,0.5,1.5,1)*

n	MLE	Mean	Bias	MSE	Variance
20	$\hat{a}$	1.5780	0.0780	0.1450	0.1389
	$\hat{\alpha}$	2.1658	0.1658	0.5358	0.5083
	$\hat{\beta}$	0.5509	0.0509	0.0398	0.0372
	$\hat{\lambda}$	1.5583	0.0583	0.0946	0.0912
	$\hat{\gamma}$	1.1145	0.1145	0.2012	0.1881
30	$\hat{a}$	1.5478	0.0478	0.0861	0.0838
	$\hat{\alpha}$	2.1025	0.1025	0.2535	0.2430
	$\hat{\beta}$	0.5308	0.0308	0.0209	0.0200
	$\hat{\lambda}$	1.5405	0.0405	0.0609	0.0592
	$\hat{\gamma}$	1.0668	0.0668	0.0949	0.0904
50	$\hat{a}$	1.5333	0.0333	0.0498	0.0487
	$\hat{\alpha}$	2.0550	0.0550	0.1378	0.1348
	$\hat{\beta}$	0.5195	0.0195	0.0113	0.0110
	$\hat{\lambda}$	1.5217	0.0217	0.0378	0.0373
	$\hat{\gamma}$	1.0425	0.0425	0.0493	0.0475
100	$\hat{a}$	1.5175	0.0175	0.0238	0.0235
	$\hat{\alpha}$	2.0235	0.0235	0.0617	0.0611
	$\hat{\beta}$	0.5092	0.0092	0.0050	0.0049
	$\hat{\lambda}$	1.5090	0.0090	0.0191	0.0190
	$\hat{\gamma}$	1.0208	0.0208	0.0216	0.0212

Table 4. *Biases, Mean, MSEs, and Variances of MLEs for EWW distribution of Set3= (1.8,2,0.5,1.5,1)*

n	MLE	Mean	Bias	MSE	Variance
20	$\hat{a}$	1.8909	0.0909	0.2028	0.1945
	$\hat{\alpha}$	2.2181	0.2181	0.7847	0.7371
	$\hat{\beta}$	0.5535	0.0535	0.0399	0.0371
	$\hat{\lambda}$	1.5780	0.0780	0.1190	0.1129
	$\hat{\gamma}$	1.1156	0.1156	0.2008	0.1875
30	$\hat{a}$	1.8624	0.0624	0.1283	0.1244
	$\hat{\alpha}$	2.1299	0.1299	0.3671	0.3503
	$\hat{\beta}$	0.5339	0.0339	0.0216	0.0205
	$\hat{\lambda}$	1.5511	0.0511	0.0786	0.0760
	$\hat{\gamma}$	1.0749	0.0748	0.1043	0.0987
50	$\hat{a}$	1.8367	0.0367	0.0712	0.0698
	$\hat{\alpha}$	2.0746	0.0746	0.1752	0.1696
	$\hat{\beta}$	0.5198	0.0198	0.0112	0.0108
	$\hat{\lambda}$	1.5318	0.0317	0.0470	0.0460
	$\hat{\gamma}$	1.0415	0.0414	0.0490	0.0473
100	$\hat{a}$	1.8188	0.0188	0.0341	0.0337
	$\hat{\alpha}$	2.0320	0.0320	0.0749	0.0738
	$\hat{\beta}$	0.5089	0.0089	0.0049	0.0048
	$\hat{\lambda}$	1.5138	0.0138	0.0228	0.0226
	$\hat{\gamma}$	1.0202	0.0202	0.0217	0.0213

***Table 5**. Biases, Mean, MSEs, and Variances of MLEs for EWW distribution of Set4 = (2,2,0.5,1.5,1)*

n	MLE	Mean	Bias	MSE	Variance
20	$\hat{a}$	2.1077	0.1077	0.2602	0.2486
	$\hat{\alpha}$	2.2382	0.2382	1.0230	0.9662
	$\hat{\beta}$	0.5533	0.0533	0.0391	0.0362
	$\hat{\lambda}$	1.5835	0.0835	0.1346	0.1276
	$\hat{\gamma}$	1.1232	0.1231	0.2794	0.2642
30	$\hat{a}$	2.0762	0.0762	0.1607	0.1549
	$\hat{\alpha}$	2.1315	0.1314	0.3773	0.3600
	$\hat{\beta}$	0.5350	0.0350	0.0220	0.0208
	$\hat{\lambda}$	1.5519	0.0519	0.0870	0.0843
	$\hat{\gamma}$	1.0787	0.0787	0.1059	0.0997
50	$\hat{a}$	2.0432	0.0432	0.0899	0.0881
	$\hat{\alpha}$	2.0838	0.0838	0.2053	0.1982
	$\hat{\beta}$	0.5202	0.0202	0.0113	0.0109
	$\hat{\lambda}$	1.5350	0.0350	0.0546	0.0534
	$\hat{\gamma}$	1.0432	0.0432	0.0511	0.0492
100	$\hat{a}$	2.0212	0.0212	0.0415	0.0411
	$\hat{\alpha}$	2.0371	0.0371	0.0814	0.0801
	$\hat{\beta}$	0.5101	0.0101	0.0051	0.0050
	$\hat{\lambda}$	1.5171	0.0171	0.0255	0.0252
	$\hat{\gamma}$	1.0213	0.0213	0.0221	0.0217

Generally, it can be seen from above tables that the MSEs of parameter estimates of a,α,β,λ and γ decrease as n increases.

6. DATA ANALYSIS

Here, two real data are utilized to explain the advantage of the EWW distribution compared with some sub-models; namely, WW, *exponential exponential* (EE), *Rayleigh W* (RW), *exponential W* (EW) and *Rayleigh Rayleigh* (RR) dsitributions.

MLEs of parameters and their related *standard errors* (S.E.) are computed. Citeria like; *minus of log-likelihood function* (-2 ln *L*), *Kolmogorov-Smirnov* (K-S) statistic, *Akaike information criterion* (AIC), *correct AIC* (CAIC), *Hannan-Quinn IC* (HQIC) and *Bayesian IC* (BIC) are considered to compare the distribution models. For each data set, we plot the histogram and the estimated pdf of the EWW,WW, EE, RW, EW and RR models.

Example 6.1. The data represent 30 successive values of March precipitation (in inches) in Minneapolis/St Paul (see [33]). MLEs of models parameter and their S.E in parenthesis are placed in Table 6. The results of the mentioned measures are placed in Table 7.

Table 6. *The MLEs of model parameters and S.Es for first data*

Model	MLEs				
	$\hat{a}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$
EWW	78.61 (0.14836)	79.35 (0.561)	20.486 (0.131)	0.624 (0.024)	0.014 (0.148)
WW	-	39.853 (0.414)	3.154 (0.518)	0.196 (0.102)	0.5 (0.072)
EE	-	42.659 (0.35762)	-	0.014 (.00499)	-
RW	-	104.304 (0.50776)	-	0.018 (.0085)	1.81 (0.156)
EW	-	3.918 (0.03031)	-	.0002904 (0.011)	5.511 (0.148)
RR	-	100.351 (0.21297)	-	0.014 (.002292)	-

Table 7. *Values of -2LnL, AIC, BIC, CAIC, HQIC and K-S for first data*

Distribution	-2Ln*L*	AIC	CAIC	BIC	HQIC	K-S
EWW	129.022	139.022	141.522	136.407	141.263	0.113
WW	138.194	146.194	147.794	145.623	149.819	0.07549
EE	178.758	182.758	183.202	181.712	183.654	0.234
RW	212.093	218.093	219.016	216.525	219.438	0.384
EW	304.274	310.274	311.197	308.705	311.619	0.715
RR	243.627	247.627	248.071	247.341	249.439	0.427

From Table 7, it can be observed that the EWW distribution has the smallest values of proposed measures compared to other models. So, it suitable model for these data than their special sub-models. Figure 4 provides plots of the fitted densities and the histogram.

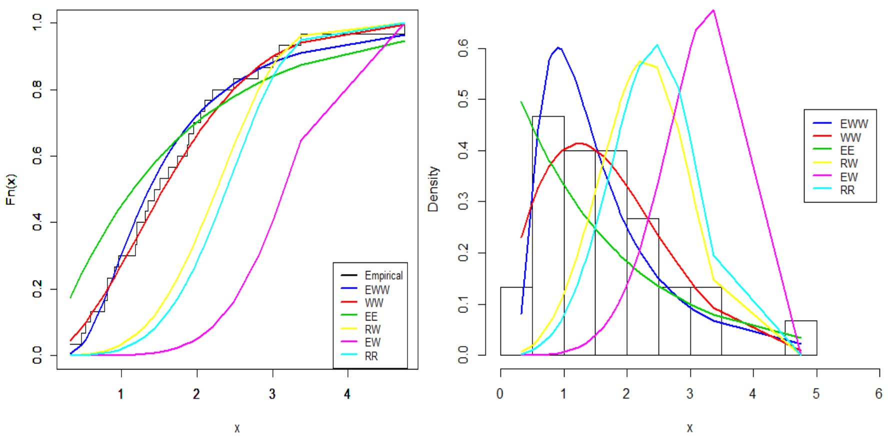


Figure 4. *Estimated cdf and estimated pdf for the first data*

Example 6.2. The data are obtained from [34]. Data represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli.

Table 8 gives MLEs of the models parameter and their S.E. Results of considered measures are presented in Table 9.

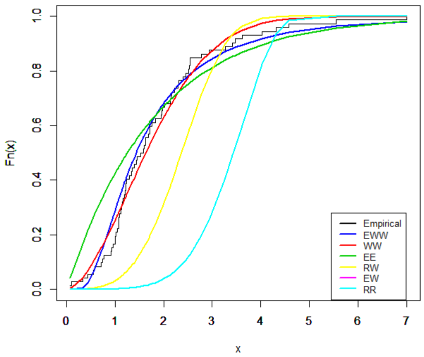
Table 8. *The MLEs of model parameters and S.Es for second data*

Model	MLEs				
	$\hat{a}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$
EWW	115.001 (0.0787)	125.918 (0.361)	19.125 (0.085)	0.61 (0.015)	0.013 (0.093)
WW	-	48.725 (0.27)	2.947 (0.324)	0.162 (0.063)	0.546 (0.047)
EE	-	42.093 (0.23097)	-	0.013 (.0032)	-
RW	-	86.137 (0.32826)	-	0.019 (.0055)	1.751 (0.099)
EW	-	3.816 (0.01957)	-	.00029 (.0073)	5.149 (0.096)
RR	-	25.417 (0.13753)	-	0.022 (.001468)	-

Table 9. *Values of -2LnL, AIC, BIC, CAIC, HQIC and K-S for second data*

Distribution	-2Ln*L*	AIC	CAIC	BIC	HQIC	K-S
EWW	302.076	312.076	312.972	311.363	316.608	0.134
WW	337.304	345.304	345.901	344.734	348.93	0.10975
EE	429.411	433.411	433.585	433.126	435.224	0.2939
RW	502.013	508.013	508.366	507.585	510.732	0.413
EW	732.507	738.507	738.86	738.079	741.226	0.729
RR	552.88	556.88	557.054	556.595	558.693	0.507

We observe from Table 9 that the EWW distribution has the smallest values of considered measures compared to other models. So, it suitable model for these data than their special sub-models. Figure 5 provides plots of the fitted densities and the histogram



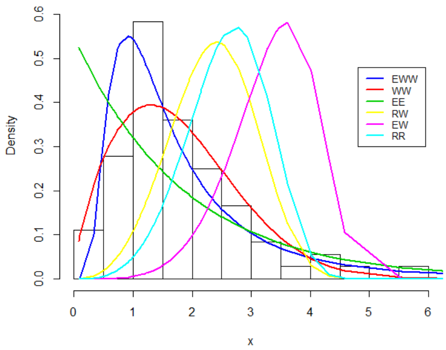


Figure 5. *Estimated cdf and estimated pdf for the second data set*

7. CONCLUSION

We introduce a new five-parameter, the so called exponentiated Weibull Weibull distribution. The main properties are provided. The EWW distribution contains some usuall distributions which obtained in [24-27] besides, it contains some new distributions. The simulation study is conducted to evaluate the behaviour of the maximum likelihood estimates of EWW parameters. The practical importance of the EWW distribution is demonstrated in two applications to show its superiority compared to other existing lifetime distributions. Application appeared that the EWW model can be employed rather than other considred distributions.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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