Research Article GU J Sci 32(2): 649-658 (2019)

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Intuitionistic Fuzzy Congruence Relations on Intuitionistic Fuzzy Abstract Algebras

Sinem TARSUSLU YILMAZ [1,*], Mehmet CITIL[2]

[1]*Department of Mathematics, Faculty of Arts and Sciences, Mersin University, Mersin, Turkey*

[2]*Department of Mathematics, Faculty of Arts and Sciences, Kahramanmaraş Sütçü İmam University, Kahramanmaraş, Turkey*

Article Info

Received: 02/10/2017
Accepted: 16/10/2018

Keywords

Intuitionistic fuzzy sets
Intuitionistic fuzzy abstract algebra
Intuitionistic fuzzy equivalence relations
Intuitionistic fuzzy congruence relations

Abstract

In this study, the intuitionistic fuzzy congruence relations on intuitionistic fuzzy abstract algebras were examined using Zadeh extension principle. Some basic properties of intuitionistic fuzzy congruence relations on n-ary operations were obtained.

1. INTRODUCTION

Zadeh was introduced the fuzzy set theory as an extension of crisp sets [1]. Countless studies have been carried out on fuzzy set theory. Fuzzy algebraic structures like groups, rings, lattices, etc. were examined by several researchers. The generalization of universal algebras on fuzzy sets was studied by Murali, in 1987 [2]. The concept of congruence relations on fuzzy universal algebras was studied by the same author [3].

An extension of fuzzy sets which called intuitionistic fuzzy set was introduced by Atanassov in 1983 [4]. Truth value set was enlarged to the lattice $[0,1]\times[0,1]$ given in Definition 1. After the introduction of this theory, basic concepts were introduced on intuitionistic fuzzy sets. Burillo and Bustince defined intuitionistic fuzzy relation [5]. Then, they studied intuitionistic fuzzy equivalence relations [6]. The intuitionistic fuzzy congruence relations on algebraic structurs like, groups, rings, lattices, etc. were studied by different authors. In this study, we examined the intuitionistic fuzzy congruence relations on intuitionistic fuzzy abstract algebras and we obtained some properties of intuitionistic fuzzy congruence relations.

Definition 1. Let $L=[0,1]$ then $L^*=\{(a_1,a_2)\in L^2 \ : \ a_1+a_2\leq 1\}$ is a lattice with

$(a_1,a_2)\leq(b_1,b_2) :\Leftrightarrow a_1\leq b_1 \ \text{ and } a_2\geq b_2$.

The operations $\wedge$ and $\vee$ on $(L^*,\leq)$ are defned as following;

For $(a_1,b_1),(a_2,b_2)\in L^*$, $(a_1,b_1)\wedge(a_2,b_2)=(\min(a_1,a_2),\max(b_1,b_2))$

$(a_1,b_1)\vee(a_2,b_2)=(\max(a_1,a_2),\min(b_1,b_2))$

*Corresponding author, e-mail: sinemyilmaz@mersin.edu.tr

For each $J \subseteq L^*$

$\sup J = (\sup\{a : (a,b \in L),((a,b) \in J)\}, \inf\{b : (a,b \in L)((a,b) \in J)\})$ and

$\inf J = (\inf\{a : (a,b \in L)((a,b) \in J)\}, \sup\{b : (a,b \in L)((a,b) \in J)\})$.

Definition 2. [4] Let a crisp set X be fixed. An intuitionistic fuzzy set (shortly IFS) in X is an object of the following form

$A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$, where functions $\mu_A(x), (\mu_A : X \to [0,1])$ and $\nu_A(x), (\nu_A : X \to [0,1])$ are called degree of membership and the degree of non- membership of $x \in X$ to the set A, respectively, and $0 \le \mu_A(x) + \nu_A(x) \le 1$, for all $x \in X$.

$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is the definition of hesitation degree of $x \in X$.

Some basic definitions are given as follow.

Definition 3. [4] Let a set X be fixed. An IFS A is contained in an IFS B (notation $A \sqsubseteq B$) if and only if, for all $x \in X$: $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$.

Clearly, $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 4. [4] Let a set X be fixed, $A \in IFS(X)$ and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ then the complement of A defined as follow:

$A^c = \{< x, \nu_A(x), \mu_A(x) >: x \in X\}$

Atanassov introduced two modal operators over intuitionistic fuzzy sets that transforms an IFS into a fuzzy set as follows;

Definition 5. [7] Let a crisp set X be fixed and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ is an intuitionistic fuzzy set on X.

i. $\Box A = \{< x, \mu_A(x), 1 - \mu_A(x) >: x \in X\}$
ii. $\Diamond A = \{< x, 1 - \nu_A(x), \nu_A(x) >: x \in X\}$

Level sets have important role on fuzzy set theory. Atanassov defined the concept of level sets on intuitionistic fuzzy sets and studied the main properties.

Definition 6. [4] Let a set X be fixed and $A \in IFS(X)$. The $(t,s)-$ cut and strong $(t,s)-$cut of A are crisp subsets $A_{(t,s)}$ and $A_{\langle t,s \rangle}$ of the X, respectively are given by

$A_{(t,s)} = \{x : x \in X \text{ such that } \mu_A(x) \ge t, \nu_A(x) \le s\}$

$A_{\langle t,s \rangle} = \{x : x \in X \text{ such that } \mu_A(x) > t, \nu_A(x) < s\}$

where $t,s \in [0,1]$ with $t + s \le 1$.

Burille and Bustince were introduced the definitions of intuitionistic fuzzy relation and intuitionistic fuzzy equivalence relation as follows.

Definition 7. [5] An intuitionistic fuzzy relation (shortly IFR) is an intuitionistic fuzzy subset of $X \times Y$ that is, is an expression R given by

$R = \{\langle (x,y), \mu_R(x,y), \nu_R(x,y) \rangle : x \in X, y \in Y\}$

where $\mu_R : X \times Y \to [0,1]$, $\nu_R : X \times Y \to [0,1]$ with $0 \le \mu_R(x,y) + \nu_R(x,y) \le 1$ for any $(x,y) \in X \times Y$.

Definition 8. [6] Let X be a universal and $R \in IFR(X)$.

1) For every $x \in X$, $\mu_R(x,x)=1$ and $\nu_R(x,x)=0$ then R is called an intuitionistic fuzzy reflexsive.
2) For every $x,y \in X$, $\mu_R(x,y) \le \mu_R(y,x)$ and $\nu_R(x,y) \ge \nu_R(y,x)$ then R is called an intuitionistic fuzzy symmetric.
3) For every $x,y,z \in X$,
   $\mu_R(x,y) \wedge \mu_R(y,z) \le \mu_R(x,z)$ and $\nu_R(x,y) \vee \nu_R(y,z) \ge \nu_R(x,z)$

then R is called an intuitionistic fuzzy transitive.

If an intuitionistic fuzzy relation satisfies the previous properties then it is called an intuitionistic fuzzy equivalence relation ($IFE(X)$).

Theorem 1. [8] Let X be a non-empty set and $R \in IFR(X)$. Then $R \in IFE(X)$ if and only if $R_{(r,s)}$ is an equivalence relation on X for each $r,s \in [0,1]$ with $r+s \le 1$.

Definition 9. [9] Let X and Y be two non-empty sets and $f : X \to Y$ be a mapping. Let $A \in IFS(X)$ and $B \in IFS(Y)$. Then f is extended to a mapping from $IFS(X)$ to $IFS(Y)$ as

$$f(A)(y) = \left(\mu_{f(A)}(y), \nu_{f(A)}(y)\right)$$

where $\mu_{f(A)}(y) = \begin{cases} \vee\left\{\mu_A(x) : x \in f^{-1}(y)\right\} \\ 0; \text{ otherwise} \end{cases}$ and $\nu_{f(A)}(y) = \begin{cases} \wedge\left\{\nu_A(x) : x \in f^{-1}(y)\right\} \\ 1; \text{ otherwise} \end{cases}$

$f(A)$ is called the image of A under the map f Also, the pre-image of B under f is denoted by $f^{-1}(B)$ and defined as

$f^{-1}(B)(x) = \left(\mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x)\right)$ where $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ and $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$.

Abstract algebras(or algebras) have a comprehensive study area on crisp sets. Murali introduced the concept of fuzzy abstract algebra using Zadeh's extension principle [2, 10]. Fuzzy congruence relation was defined and some properties of fuzzy congruence relations were studied by same author [3]. The generalization of abstract algebra to intuitionistic fuzzy abstract algebra was studied by Çuvalcoğlu and Tarsuslu(Yılmaz) [11].

Firstly, let remember the definition of crisp abstract algebra.

Definition 10. [12] An abstract algebra (or algebra) A is a pair $[S,F]$ where S is a non-empty set and F is a specified set of operatians f_α, each mapping a power $S^{n(\alpha)}$ of S into S for some appropriate nonnegative finite integer $n(\alpha)$.

Unless otherwise stated, each operation f_α assigns to every $n(\alpha)-$ple $\left(x_1,...,x_{n(\alpha)}\right)$ of elements of S, a value $f_\alpha\left(x_1,...,x_{n(\alpha)}\right)$ in S, the result of performing the operation f_α on the sequence $x_1,...,x_{n(\alpha)}$. If $n(\alpha)=1$, the operation f_α is called unary; if $n(\alpha)=2$, it is called binary; if $n(\alpha)=3$, it is called ternary, etc. When $n(\alpha)=0$, the operation f_α is called nullary; it selects a fixed element of S.

$A=[S,F]$ and $B=[T,F']$ are called similar algebras if F and F' are same for each α the types of f_α and f'_α.

Definition 11. [12] Let $A=[S,F]$ and $B=[T,F']$ be two similar algebras. A function $\varphi : S \to T$ is called a homomorphism of A into B if and only if for all $f_\alpha \in F$ and $x_i \in S$, $i=1,2,...,n(a)$,

$$f'_\alpha\left(\varphi(x_1), \varphi(x_2), ..., \varphi(x_{n(a)})\right) = \varphi\left(f_\alpha\left(x_1, x_2, ..., x_{n(a)}\right)\right).$$

A crisp congruence relation on an algebraic system $A=[S,F]$ is an equivalence relation θ on $A=[S,F]$ which has the substitution property for its operations. It means that, for all $f_\alpha \in F$ and $a_i, b_i \in S$,

$i = 1, 2, ..., n(a)$,

$a_i \equiv b_i(\theta) \Rightarrow f_\alpha(a_1, a_2, ..., a_{n(a)}) \equiv f_\alpha(b_1, b_2, ..., b_{n(a)})(\theta)$.

Many algebraic structures were extended on intuitionistic fuzzy sets by several researchers and important theorems were proved. The concept of intuitionistic fuzzy abstract algebra examining on this study is defined as follows.

Definition 12. [11] Let $S = [X, F]$ be an algebra where X is a non-empty set and F is a specified set of finite operations f_α, each mapping a power $X^{n(\alpha)}$ of X into X, for some appropriate nonnegative finite integer $n(\alpha)$. For each f_α, a corresponding operation ω_α on $IFS(X)$ as follows;

ω_α : $IFS(X) \times IFS(X) \times ... \times IFS(X) \to IFS(X), \omega_\alpha(A_1, A_2, ..., A_{n(\alpha)}) = A$

such that

$$A(x) = \begin{cases} \sup\{A_1(x_1) \wedge A_2(x_2) \wedge ... \wedge A_{n(\alpha)}(x_{n(\alpha)})\}; & f_\alpha(x_1, x_2, ..., x_{n(\alpha)}) = x \\ \Theta = (0,1); & \text{other wise} \end{cases}$$

Shortly, $A = \omega_\alpha(A_1, A_2, ..., A_{n(\alpha)})$.

Let $\Omega = \{\omega_\alpha$: corresponding operation for each $f_\alpha \in F\}$ then $\square = \left[(I \times I)^X, \Omega\right]$ is called intuitionistic fuzzy abstract algebra (or intuitionistic fuzzy algebra).

If $n(\alpha) = 0$ then $f_\alpha(x) = e$ that e is a fixed element of X. So, ω_α is defined as following:

$$\omega_\alpha(A) = A_e, A_e(x) = \begin{cases} \sup_{x \in X} A(x), & x = e \\ (0,1), & x \neq e \end{cases}$$

Definition 13. [11] Let X be a non-empty set and $A \in IFS(X)$. A is called an intuitionistic fuzzy subalgebra($IF-$ subalgebra) of $\square = [IFS(X), \Omega]$ intuitionistic fuzzy algebra if and only if for nonnegative finite integer $n(\alpha), \omega_\alpha(A, A, ..., A) \sqsubseteq A$, for every ω_α .

Theorem 2. [6] Let $S = [X, F]$ be an algebra, $f_\alpha \in F$ and $A, A_1, A_2, ..., A_{n(\alpha)}$ be $IF-$subalgebras. $\omega_\alpha(A_1, A_2, ..., A_{n(\alpha)}) \sqsubseteq A$ if and only if $A\left(f_\alpha(x_1, x_2, ..., x_{n(\alpha)})\right) \geq \min_{1 \leq i \leq n(\alpha)} A_i(x_i)$

is true for every $(x_1, x_2, ..., x_{n(\alpha)}) \in X^{n(\alpha)}$.

This definition coincides with [3].

2. INTUITIONISTIC FUZZY CONGRUENCE RELATIONS ON ABSTRACT ALGEBRAS

Through congruence relations, algebraic substructures are obtained which have an important role in algebra theory. The definition of intuitionistic fuzzy congruence relation on abstract algebras has introduced in [14].

Definition 14. [13] Let $S = [X, F]$ be an algebra and $f_\alpha \in F$. For any $(A_1, A_2, ..., A_{n(\alpha)}) \in IFR(X)^{n(\alpha)}$ and for any $x, y \in S$, $\overline{\omega}_\alpha(A_1, A_2, ..., A_{n(\alpha)})$ to be an element of $IFR(X)$ defined by

$$\overline{\omega}_\alpha(A_1, A_2, ..., A_{n(\alpha)})(x, y) = \sup_{x,y}\left(\min_{1 \leq i \leq n(\alpha)} A_i(x_i, y_i)\right)$$

such that the supremum is taken over all representations of $f_\alpha(x_1, x_2, ..., x_{n(\alpha)}) = x$ and $f_\alpha(y_1, y_2, ..., y_{n(\alpha)}) = y$. Therefore, $[IFR(X), \Omega]$ is an intuitionistic fuzzy algebra on intuitionistic fuzzy relations.

Definition 15. [13] Let $S = [X, F]$ be an algebra. $A \in IFE(X)$ is an intuitionistic fuzzy congruence relation on S if and only if, for each $f_\alpha \in F$, $\overline{\omega}_\alpha(A, A, ..., A) \sqsubseteq A$.

Proposition 1. [13] Let A be an intuitionistic fuzzy congruence relation on $S=[X,F]$ algebra then $A_{(r,s)}(\text{shortly } \sim)$ is a crisp congruence relation on S for each $r,s\in[0,1]$ with $r+s\leq 1$.

In this study, the properties of intuitionistic fuzzy congruence relation were examined in detailed and the following main results were obtained.

Theorem 3. Let $S=[X,F]$ be an algebra, $f_\alpha \in F$, and $A_1,A_2,...,A_{n(\alpha)},A$ be intuitionistic fuzzy relations on S.

$$\overline{\omega}_\alpha\left(A_1,A_2,...,A_{n(\alpha)}\right)\sqsubseteq A \Leftrightarrow A\left(f_\alpha\left(x_1,x_2,...,x_{n(\alpha)}\right),f_\alpha\left(y_1,y_2,...,y_{n(\alpha)}\right)\right)\geq \min_{1\leq i\leq n(\alpha)} A_i\left(x_i,y_i\right)$$

for all pairs of $n(\alpha)-$ tuples $\left(x_1,x_2,...,x_{n(\alpha)}\right)$ and $\left(y_1,y_2,...,y_{n(\alpha)}\right)$.

The proof of this theorem can be seen easily from Theorem 1 and following Corollary is clear.

Corollary 1. Let $S=[X,F]$ be an algebra. An intuitionistic fuzzy equivalence relation A on S is an intuitionistic fuzzy congruence relation on S if and only if

$$A\left(f_\alpha\left(x_1,x_2,...,x_{n(\alpha)}\right),f_\alpha\left(y_1,y_2,...,y_{n(\alpha)}\right)\right)\geq \min_{1\leq i\leq n(\alpha)} A_i\left(x_i,y_i\right)$$

for all $f_\alpha \in F$, and for all $n(\alpha)-$ tuples $\left(x_1,x_2,...,x_{n(\alpha)}\right)$, $\left(y_1,y_2,...,y_{n(\alpha)}\right)\in X^{n(\alpha)}$.

Example 1. Let consider the group $G=\{e,a,b\}$ determined by following operation *. It is clear that G is an algebra given by a binary operation, a unary operation(inversion) and a constant operation(the neutral element) satisfying well known laws.

*	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

R	e	a	b
e	(1, 0)	(0.6, 0.2)	(0.6, 0.2)
a	(0.6, 0.2)	(1, 0)	(0.6, 0.2)
b	(0.6, 0.2)	(0.6, 0.2)	(1, 0)

R is an intuitionistic fuzzy congruence relation on G. That is, R is an intuitionistic fuzzy equivalence relation on G and for any $x_1,x_2,y_1,y_2\in G$,

$R\left(x_1 * x_2, y_1 * y_2\right)\geq R\left(x_1,y_1\right)\wedge R\left(x_2,y_2\right)$ and $R\left(x_1^{-1},y_1^{-1}\right)\geq R\left(x_1,y_1\right)$

Proposition 2. Let $S=[X,F]$ be an algebra with subalgebra $\{e\}$ and A be an intuitionistic fuzzy congruence relation on S. Then $A_{\bar{e}}$ is an intuitionistic fuzzy subalgebra of S such that $\bar{e}=\{x \ : \ A(x,e)=(1,0), x\in X\}$.

Proof: Let $f_\alpha \in F$ and $\left(x_1,x_2,...,x_{n(\alpha)}\right)\in X^{n(\alpha)}$. Then,

$$\begin{aligned} A_{\bar{e}}\left(f_\alpha\left(x_1,x_2,...,x_{n(\alpha)}\right)\right) &= A\left(f_\alpha\left(x_1,x_2,...,x_{n(\alpha)}\right),e\right) \\ &= A\left(f_\alpha\left(x_1,x_2,...,x_{n(\alpha)}\right),f_\alpha\left(e,e,...,e\right)\right) \\ &\geq \min_{1\leq i\leq n(\alpha)} A\left(x_i,e\right) \\ &= \min_{1\leq i\leq n(\alpha)} A_{\bar{e}}\left(x_i\right) \end{aligned}$$

So, $f_\alpha\left(A_{\bar{e}},A_{\bar{e}},...,A_{\bar{e}}\right)\leq A_{\bar{e}}$.

Proposition 2: Let $S=[X,F]$ be an algebra. If A is an intuitionistic fuzzy congruence relation on S then so are $\Box A$ and $\Diamond A$.

Proof: Let A be an intuitionistic fuzzy congruence relation.

(i) For all $a\in X$,

$\mu_{\Box A}(a,a)=\mu_A(a,a)=1$ and $\nu_{\Box A}(a,a)=1-\mu_A(a,a)=0$

intuitionistic fuzzy reflexive property has been provided.

(ii) Let $a,b\in X$,

$\mu_{\Box A}(a,b)=\mu_A(a,b)=\mu_A(b,a)=\mu_{\Box A}(b,a)$ and

$\nu_{\Box A}(a,b)=1-\mu_A(a,b)=1-\mu_A(b,a)=\nu_{\Box A}(b,a)$

intuitionistic fuzzy symmetric has been provided.
(iii) Let $a,b\in X$, and A is an intuitionistic fuzzy congruence relation then

$\sup\limits_{z\in X}\{\mu_A(a,c)\wedge\mu_A(c,b)\}\le\mu_A(a,b)\Rightarrow\inf\limits_{c\in X}\{1-\mu_A(a,c)\vee 1-\mu_A(c,b)\}\ge 1-\mu_A(a,b)$

is proved.
(iv) For each $\overline{\omega}_\alpha$,

$A(a,b)\ge\overline{\omega}_\alpha(A,A,...,A)(a,b)=\sup\limits_{a,b}\left(\min\limits_{1\le i\le n(\alpha)}A(a_i,b_i)\right)$ and

$\sup\limits_{a,b}\left(\min\limits_{1\le i\le n(\alpha)}\mu_A(a_i,b_i)\right)\le\mu_A(a,b)\Rightarrow\inf\limits_{a,b}\left(\max\limits_{1\le i\le n(\alpha)}1-\mu_A(a_i,b_i)\right)\ge 1-\mu_A(a,b)\Rightarrow\overline{\omega}_\alpha(\Box A,\Box A,...,\Box A)\le\Box A$

So, proof is completed. The operator $\Diamond A$ can be examined similarly.

We obtained some results about intuitionistic fuzzy congruence relations under homomorphsim.

Theorem 4. Let $S=[X,F]$, $T=[Y,F]$ be two similar algebras and ϕ be a homomorphism of S into T If B is an intuitionistic fuzzy congruence relation on T then $\phi^{-1}(B)$ is an intuitionistic fuzzy congruence relation on S.

Proof: (i) For all $a\in X$,

$$\phi^{-1}(B)(a,a)=B(\phi(a),\phi(a))$$
$$=(\mu_B(\phi(a),\phi(a)),\nu_B(\phi(a),\phi(a)))=(1,0)$$

$\phi^{-1}(B)$ is intuitionistic fuzzy reflexsive.

(ii) Let $a,b\in X$,

$$\phi^{-1}(B)(a,b)=B(\phi(a),\phi(b))=(\mu_B(\phi(a),\phi(b)),\nu_B(\phi(a),\phi(b)))$$
$$=(\mu_B(\phi(b),\phi(a)),\nu_B(\phi(b),\phi(a)))$$
$$=B(\phi(b),\phi(a))=\phi^{-1}(B)(b,a)$$

$\phi^{-1}(B)$ is intuitionistic fuzzy symmetric.
(iii) Let $a,b\in X$,

$$\left(\phi^{-1}(B)\circ\phi^{-1}(B)\right)(a,b)=\sup_{c\in X}\left\{\phi^{-1}(B)(a,c)\wedge\phi^{-1}(B)(c,b)\right\}=\sup_{c\in X}\left\{B\left(\phi(a),\phi(c)\right)\wedge B\left(\phi(c),\phi(b)\right)\right\}$$

$$=\begin{pmatrix}\sup_{c\in X}\left\{\mu_B\left(\phi(a),\phi(c)\right)\wedge\mu_B\left(\phi(c),\phi(b)\right)\right\}, \\ \inf_{c\in X}\left\{\nu_B\left(\phi(a),\phi(c)\right)\vee\nu_B\left(\phi(c),\phi(b)\right)\right\}\end{pmatrix}$$

$$\leq\begin{pmatrix}\sup_{y\in Y}\left\{\mu_B\left(\phi(a),y\right)\wedge\mu_B\left(y,\phi(b)\right)\right\}, \\ \inf_{y\in Y}\left\{\nu_B\left(\phi(a),y\right)\vee\nu_B\left(y,\phi(b)\right)\right\}\end{pmatrix}$$

$$=(B\circ B)\left(\phi(a),\phi(b)\right)=\phi^{-1}(B)(a,b)$$

So, $\phi^{-1}(B)$ is intuitionistic fuzzy equivalence relation.

(iv) Let $f_\alpha\in F$ and $a,b\in X$,

$$\varpi_\alpha\left(\phi^{-1}(B),\phi^{-1}(B),...,\phi^{-1}(B)\right)(a,b)=\sup_{\substack{a=f_\alpha\left(a_1,a_2,...,a_{n(\alpha)}\right)\\ b=f_\alpha\left(b_1,b_2,...,b_{n(\alpha)}\right)}}\left\{\min_{1\leq i\leq n(\alpha)}\left(\phi^{-1}(B)\right)(a_i,b_i)\right\}$$

$$=\sup_{\substack{a=f_\alpha\left(a_1,a_2,...,a_{n(\alpha)}\right)\\ b=f_\alpha\left(b_1,b_2,...,b_{n(\alpha)}\right)}}\left\{\min_{1\leq i\leq n(\alpha)}B\left(\phi(a_i),\phi(b_i)\right)\right\}$$

$$=\begin{pmatrix}\sup_{\substack{a=f_\alpha\left(a_1,a_2,...,a_{n(\alpha)}\right)\\ b=f_\alpha\left(b_1,b_2,...,b_{n(\alpha)}\right)}}\left\{\min_{1\leq i\leq n(\alpha)}\mu_B\left(\phi(a_i),\phi(b_i)\right)\right\}, \\ \inf_{\substack{a=f_\alpha\left(a_1,a_2,...,a_{n(\alpha)}\right)\\ b=f_\alpha\left(b_1,b_2,...,b_{n(\alpha)}\right)}}\left\{\max_{1\leq i\leq n(\alpha)}\nu_B\left(\phi(a_i),\phi(b_i)\right)\right\}\end{pmatrix}$$

$$\leq\begin{pmatrix}\sup_{\substack{\phi(a)=f_\alpha\left(\phi(a_1),\phi(a_2),...,\phi(a_{n(\alpha)})\right)\\ \phi(b)=f_\alpha\left(\phi(b_1),\phi(b_2),...,\phi(b_{n(\alpha)})\right)}}\left\{\min_{1\leq i\leq n(\alpha)}\mu_B\left(\phi(a_i),\phi(b_i)\right)\right\}, \\ \inf_{\substack{\phi(a)=f_\alpha\left(\phi(a_1),\phi(a_2),...,\phi(a_{n(\alpha)})\right)\\ \phi(b)=f_\alpha\left(\phi(b_1),\phi(b_2),...,\phi(b_{n(\alpha)})\right)}}\left\{\max_{1\leq i\leq n(\alpha)}\nu_B\left(\phi(a_i),\phi(b_i)\right)\right\}\end{pmatrix}$$

$$=\varpi_\alpha(B,B,...,B)\left(\phi(a),\phi(b)\right)\leq B\left(\phi(a),\phi(b)\right)=\phi^{-1}(B)(a,b)$$

the substitution property has been provided.

Theorem 5. Let $S=[X,F]$, $T=[Y,F]$ be two similar algebras and ϕ be a monomorphism of S into T. If A is an intuitionistic fuzzy congruence relation on S then $\phi(A)$ is an intuitionistic fuzzy congruence relation on T.

Proof: (i) For all $b\in Y$, $\phi(A)(b,b)=\sup_{a,c\in\phi^{-1}(b)}A(a,c)=(1,0)$

$\phi(A)$ is intuitionistic fuzzy reflexsive and the symmetric relation is clear.

(ii) Let $b_1,b_2\in Y$,

$$\left(\phi(A)\circ\phi(A)\right)(b_1,b_2)=\sup_{b_3\in Y}\left(\phi(A)(b_1,b_3)\wedge\phi(A)(b_3,b_2)\right)$$

$$=\sup_{b_3\in Y}\left(\left(\sup_{\substack{a_1\in\phi^{-1}(b_1)\\ a_3\in\phi^{-1}(b_3)}} A(a_1,a_3)\right)\wedge\left(\sup_{\substack{a_2\in\phi^{-1}(b_2)\\ a_3\in\phi^{-1}(b_3)}} A(a_3,a_2)\right)\right)$$

$$\le\sup_{b_3\in Y}\left(\sup_{\substack{a_1\in\phi^{-1}(b_1),a_2\in\phi^{-1}(b_2)\\ a_3\in\phi^{-1}(b_3)}} A(a_1,a_3)\wedge A(a_3,a_2)\right)$$

$$\le\sup_{\substack{a_1\in\phi^{-1}(b_1)\\ a_2\in\phi^{-1}(b_2)}}\left(\sup_{a_3\in X} A(a_1,a_3)\wedge A(a_3,a_2)\right)$$

$$=\sup_{\substack{a_1\in\phi^{-1}(b_1)\\ a_2\in\phi^{-1}(b_2)}} A(a_1,a_2)=\phi(A)(b_1,b_2)$$

So, $\phi(A)$ is intuitionistic fuzzy transitive.

(iii) Let $f_\alpha\in F$ and $a',b'\in Y$,

$$\varpi_\alpha(\phi(A),\phi(A),...,\phi(A))(a',b')=\sup_{\substack{a'=f_\alpha(a_1',a_2',...,a_{n(\alpha)}')\\ b'=f_\alpha(b_1',b_2',...,b_{n(\alpha)}')}}\left(\min_{1\le i\le n(\alpha)}\phi(A)(a_i',b_i')\right)$$

$$=\sup_{\substack{a'=f_\alpha(a_1',a_2',...,a_{n(\alpha)}')\\ b'=f_\alpha(b_1',b_2',...,b_{n(\alpha)}')}}\left(\min_{1\le i\le n(\alpha)}\left(\sup_{\substack{a_i\in\phi^{-1}(a_i')\\ b_i\in\phi^{-1}(b_i')}} A(a_i,b_i)\right)\right)\le\sup_{\substack{a'=f_\alpha(\phi(a_1),\phi(a_2),...,\phi(a_{n(\alpha)}))\\ b'=f_\alpha(\phi(b_1),\phi(b_2),...,\phi(b_{n(\alpha)}))}}\left(\min_{1\le i\le n(\alpha)} A(a_i,b_i)\right)$$

$$=\sup_{\substack{a'=\phi(f_\alpha(a_1,a_2,...,a_{n(\alpha)}))\\ b'=\phi(f_\alpha(b_1,b_2,...,b_{n(\alpha)}))}}\left(\min_{1\le i\le n(\alpha)} A(a_i,b_i)\right)=\varpi_\alpha(A,A,...,A)(\phi^{-1}(a'),\phi^{-1}(b')).$$

$$\le A(\phi^{-1}(a'),\phi^{-1}(b'))\le\phi(A)(a_i,b_i).$$

Therefore $\phi(A)$ is an intuitionistic fuzzy congruence relation on T.

Theorem 6. Let $\{A_j\ :\ j\in J\}$ be a non-empty family of intuitionistic fuzzy congruence relation on algebra $S=[X,F]$. Then

$$A=\inf_{j\in J} A_j=\bigwedge_{j\in J} A_j$$

is an intuitionistic fuzzy congruence relation on S.

Proof: (i) For all $a\in X$,

$$A(a,a)=\inf_{j\in J} A_j(a,a)=\left(\inf_{j\in J}\mu_{A_j}(a,a),\sup_{j\in J}\nu_{A_j}(a,a)\right)=(1,0)$$

(ii) Let $a,b\in X$,

$$A(a,b)=\inf_{j\in J} A_j(a,b)=\left(\inf_{j\in J}\mu_{A_j}(a,b),\sup_{j\in J}\nu_{A_j}(a,b)\right)$$

$$=\left(\inf_{j\in J}\mu_{A_j}(b,a),\sup_{j\in J}\nu_{A_j}(b,a)\right)=\inf_{j\in J} A_j(b,a)=A(b,a)$$

(iii) Let $a,b\in X$,

$$(A\circ A)(a,b)=\sup_{c\in X}\big(A(a,c)\wedge A(c,b)\big)=\sup_{c\in X}\left(\inf_{j\in J} A_j(a,c)\wedge\inf_{j\in J} A_j(c,b)\right)$$

$$=\sup_{c\in X}\left(\left(\inf_{j\in J}\mu_{A_j}(a,c),\sup_{j\in J}\nu_{A_j}(a,c)\right)\wedge\left(\inf_{j\in J}\mu_{A_j}(c,b),\sup_{j\in J}\nu_{A_j}(c,b)\right)\right)$$

$$=\sup_{c\in X}\left(\inf_{j\in J}\mu_{A_j}(a,c)\wedge\inf_{j\in J}\mu_{A_j}(c,b),\sup_{j\in J}\nu_{A_j}(a,c)\vee\sup_{j\in J}\nu_{A_j}(c,b)\right)$$

$$=\left(\sup_{c\in X}\left(\inf_{j\in J}\mu_{A_j}(a,c)\wedge\inf_{j\in J}\mu_{A_j}(c,b)\right),\inf_{z\in X}\left(\sup_{j\in J}\nu_{A_j}(a,c)\vee\sup_{j\in J}\nu_{A_j}(c,b)\right)\right)$$

$$=\left(\sup_{c\in X}\left(\inf_{j\in J}\left(\inf_{k\in J}\mu_{A_j}(a,c)\wedge\mu_{A_k}(c,b)\right)\right),\inf_{c\in X}\left(\sup_{j\in J}\left(\sup_{k\in J}\nu_{A_j}(a,c)\vee\nu_{A_k}(c,b)\right)\right)\right)$$

$$\le\left(\sup_{c\in X}\left(\inf_{j\in J}\left(\mu_{A_j}(a,c)\wedge\mu_{A_j}(c,b)\right)\right),\inf_{c\in X}\left(\sup_{j\in J}\left(\nu_{A_j}(a,c)\vee\nu_{A_j}(c,b)\right)\right)\right)$$

$$=\left(\inf_{j\in J}\mu_{A_j}(a,b),\sup_{j\in J}\nu_{A_j}(a,b)\right)=A(a,b)$$

(iv) Let $f_\alpha\in F$ and $\left(a_1,a_2,...,a_{n(\alpha)}\right)$, $\left(b_1,b_2,...,b_{n(\alpha)}\right)\in X^{n(\alpha)}$.

By corollary,

$$A\left(f_\alpha\left(a_1,a_2,...,a_{n(\alpha)}\right),f_\alpha\left(b_1,b_2,...,b_{n(\alpha)}\right)\right)=\wedge_{j\in J} A_j\left(f_\alpha\left(a_1,a_2,...,a_{n(\alpha)}\right),f_\alpha\left(b_1,b_2,...,b_{n(\alpha)}\right)\right)$$

$$=\inf_{j\in J}\left(A_j\left(f_\alpha\left(a_1,a_2,...,a_{n(\alpha)}\right),f_\alpha\left(b_1,b_2,...,b_{n(\alpha)}\right)\right)\right)\ge\inf_{j\in J}\left(\min_{1\le i\le n(\alpha)} A_j(a_i,b_i)\right)$$

$$=\min_{1\le i\le n(\alpha)}\left(\inf_{j\in J} A_j(a_i,b_i)\right)=\min_{1\le i\le n(\alpha)} A(a_i,b_i)$$

3. CONCLUSION

The properties of the intuitionistic fuzzy congruence relation extended over the intuitionistic fuzzy algebras have been studied. In the following studies, the concept of intuitionistic fuzzy free algebra can be introduced and congruence relations can be defined on intuitionistic fuzzy free algebras. Same or different type properties can be examined on intuitionistic fuzzy free algebras.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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