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APPROXIMATION BY NÖRLUND AND RIESZ MEANS IN WEIGHTED LEBESGUE SPACE WITH VARIABLE EXPONENT

AHMET TESTICI

Abstract. We investigate the approximation properties of Nörlund and Riesz means of trigonometric Fourier series are investigated in the subset of weighted Lebesgue space with variable exponent.

1. Introduction and Main results

Let $\mathbb{T} := [0, 2\pi]$ and let $p(\cdot) : \mathbb{T} \to [1, \infty)$ be a Lebesgue measurable 2π periodic function. We suppose that the considered exponent functions $p(\cdot)$ satisfy the condition

$$1 < p_- := ess \inf_{x\in\mathbb{T}} p(x) \le ess \sup_{x\in\mathbb{T}} p(x) := p^+ < \infty.$$

In addition to this requirement if there exist a positive constant c such that

$$|p(x) - p(y)| \ln(1/|x-y|) \le c, \quad x, y \in \mathbb{T}, \quad 0 < |x-y| \le 1/2$$

then we say that $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{T})$ is defined as the set of all Lebesgue measurable 2π periodic functions f such that $\rho_{p(\cdot)}(f) := \int_0^{2\pi} |f(x)|^{p(x)} dx < \infty$. Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \le 1\right\}$$

$L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ becomes a Banach space. The fundamental properties of Lebesgue spaces with variable exponent are explained in monographs [3, 4, 5].

For a given weight ω we define the weighted variable Lebesgue space $L^{p(\cdot)}_\omega(\mathbb{T})$ as the set of all measurable 2π periodic functions f such that $f\omega \in L^{p(\cdot)}(\mathbb{T})$. The norm of $L^{p(\cdot)}_\omega(\mathbb{T})$ can be defined as $\|f\|_{p(\cdot),\omega} := \|f\omega\|_{p(\cdot)}$.

If $p(\cdot)$=*constant*, then $L^{p(\cdot)}_\omega(\mathbb{T})$ coincides with the weighted Lebesgue spaces $L^p_\omega(\mathbb{T})$. In this case A_p Muckenhoupt class becomes important point. In order

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to prove boundedness of some operators and crucial theorems of harmonic analysis in weighted Lebesgue spaces it is needed that weight function ω belongs to A_p Muckenhoupt class. Similar situations are valid in the weighted Lebesgue space with variable exponent $L_\omega^{p(\cdot)}(\mathbb{T})$. In our investigations we will use the Muckenhoupt weights class $A_{p(\cdot)}(\mathbb{T})$ defined as

Definition 1. *For a given exponent $p(\cdot)$ we say that $\omega \in A_{p(\cdot)}(\mathbb{T})$ if*

$$\sup_{B_j} |B_j|^{-1} \left\| \omega \chi_{B_j} \right\|_{p(\cdot)} \left\| \omega^{-1} \chi_{B_j} \right\|_{p'(\cdot)} < \infty, \quad 1/p(\cdot) + 1/p'(\cdot) = 1,$$

where supremum is taken over all open intervals $B_j \subset \mathbb{T}$ with the characteristic functions χ_{B_j}.

Let $f \in L^1(\mathbb{T})$ and let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{1}$$

be Fourier series of f where

$$a_k := a_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cos ktdt \quad \text{and} \quad b_k := b_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sin ktdt$$

are Fourier coefficients of f. Let also

$$u_0(f)(x) := \frac{a_0}{2}, \;\; u_k(f)(x) := a_k \cos kx + b_k \sin kx \;, \quad k = 1, 2, ..., n$$

where a_k and b_k are Fourier coefficients of f. We denote the $n-$th partial sums of the series (1) by

$$S_n(f)(x) := \sum_{k=0}^{n} u_k(f)(x) \;\;, \quad n = 0, 1, 2, ... \;\; .$$

Let $(p_n)_{n=0}^{\infty}$ be sequence of positive real numbers. We define the Nörlund and Riesz means of the series (1), respectively,

$$N_n(f)(x) := \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} u_m(f)(x),$$

and

$$R_n(f)(x) := \frac{1}{P_n} \sum_{m=0}^{n} p_m u_m(f)(x).$$

where $P_n = \sum_{m=0}^{n} p_m$ and $P_{-1} = p_{-1} := 0$. In the case of $p_n = 1$ for all $n = 0, 1, 2, ..,$ the both of $N_n(f)$ and $R_n(f)$ means coincide with the Cesàro mean $\sigma_n(f)$, defined as

$$\sigma_n(f)(x) := \frac{1}{n+1}\sum_{m=0}^{n} u_m(f)(x).$$

Definition 2. *Let $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. We define the modulus of smoothness as*

$$\Omega(f,\delta)_{p(\cdot),\omega} := \sup_{|h|\leq\delta}\left\|\frac{1}{h}\int_0^h [f(x+t)-f(x)]\,dt\right\|_{p(\cdot),\omega}, \quad \delta>0.$$

The correctness of this definition follows from the boundedness of the maximal operator

$$M : f \to Mf(x) := \sup_{B\ni x}\frac{1}{|B|}\int_B |f(t)|\,dt$$

in the $L_{\omega}^{p(\cdot)}(\mathbb{T})$, where B is any open subinterval of $\mathbb{T}$ (see, [7]). So we have that if $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$, then the maximal operator M is bounded in $L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$. In this case there exist a positive constant $c_1(p)$ such that the inequality

$$\|Mf\|_{p(\cdot),\omega} \leq c_1(p)\,\|f\|_{p(\cdot),\omega} \tag{2}$$

holds for every $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$. By this fact if $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$, then there exists a positive constant $c_2(p)$ such that

$$\Omega(f,\delta)_{p(\cdot),\omega} \leq c_2(p)\,\|f\|_{p(\cdot),\omega}\ . \tag{3}$$

Moreover, it can be shown that if $f,g \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, then

$\Omega(f+g,\delta)_{p(\cdot),\omega} \leq \Omega(f,\delta)_{p(\cdot),\omega} + \Omega(g,\delta)_{p(\cdot),\omega}$ and also $\lim_{\delta\to 0}\Omega(f,\delta)_{p(\cdot),\omega} = 0.$

$W_{\omega}^{p(\cdot),r}(\mathbb{T})$, $r = 1,2,...$, denotes the class of all Lebesgue measurable 2π periodic and $r-1$ times continuously differentiable functions such that $f^{(r)} \in L_{\omega}^{p(\cdot)}(\mathbb{T})$.

We define the variable exponent Lipschitz class $Lip_r^{p(\cdot)}(\alpha,\omega)$, $0<\alpha\leq 1$, as

$$Lip_r^{p(\cdot)}(\alpha,\omega) := \left\{ f \in W_{\omega}^{p(\cdot),r}(\mathbb{T}) : \Omega\left(f^{(r)},\delta\right)_{p(\cdot),\omega} = \mathcal{O}(\delta^{\alpha}), \quad \delta>0 \right\}.$$

In the classical case the approximation properties of $\sigma_n(f)$ in classical Lipschitz classes where $1 \leq p < \infty$ and $0<\alpha\leq 1$ were investigated by Quade in [8]. The Quade's results were generalized by Mohapatra and Russel [9], Chandra [10, 11] and Leindler [12]. In [11] under the some conditions related with the sequence $(p_n)_{n=0}^{\infty}$ Chandra proved satisfactory results about approximation by the $N_n(f)$ and $R_n(f)$ means in in classical Lipschitz classes where $1 \leq p < \infty$ and $0<\alpha\leq 1$. Guven carried and extended the results obtained in [11] to weighted Lipschitz classes where $1<p<\infty$ (see, [13, 14]). In the Lebesgue space with variable exponent space Guven and Israfilov investigated the approximation properties of $N_n(f)$ and

$R_n(f)$ means for Lipschitz classes in [15]. After that Guven extended this results to triangular matrix transforms in [16]. In weighted Lebesgue space with variable exponent Israfilov and Testici were investigated the approximation properties of matrix transform of Fourier series in [18]. In the Lebesgue space with variable exponent approximation Nörlund and Riesz submethods were studied in [20]. In [21] the results obtained in [13] generalize to weighted Lorentz space for the derivatives of functions.

In this work we investigate the approximation properties of the Nörlund and Riesz means of the Fourier series in $Lip_r^{p(\cdot)}(\alpha,\omega)$, $0<\alpha\leq 1$ where $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$, $\omega\in A_{p(\cdot)}(\mathbb{T})$, also it is important to emphasize that obtained results in this work can be considered that generalizations of the given results in [15].

A sequence of positive real numbers $(p_n)_{n=0}^{\infty}$ is called almost monotone increasing if there exists a constant K, depending only on the sequence $(p_n)_{n=0}^{\infty}$ such that for all $n\geq m$ the inequality

$$p_m\leq Kp_n$$

holds. Almost monotone increasing sequences are denoted by $(p_n)_{n=0}^{\infty}\in AMIS$.

Along this work we will use the notations

$$\Delta g_n:=g_n-g_{n+1}\ ,\quad \Delta_m g(n,m):=g(n,m)-g(n,m+1)\,,$$

and $f=\mathcal{O}(g)$ means that there exists some positive constant c such that $f\leq cg$. Moreover $c(\cdot)$, $c_1(\cdot)$, $c_2(\cdot)$, ..., denote the constants (in general different in the different relations) depending in general on the parameters given in the brackets and independent of n.

Our main results are following:

Theorem 3. *Let $p(\cdot)\in\mathcal{P}_0(\mathbb{T}),\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T}),0<\alpha<1$, and let $(p_n)_{n=0}^{\infty}$ be a sequence of positive real numbers such that $(p_n)_{n=0}^{\infty}\in AMIS$ and*

$$(n+1)^{r+1}p_n=\mathcal{O}(P_n)\,. \tag{4}$$

If $f\in Lip_r^{p(\cdot)}(\alpha,\omega)$, then the estimate

$$\|f-N_n(f)\|_{p(\cdot),\omega}=\mathcal{O}\left(n^{-(\alpha+r)}\right),\ n=1,2,...,$$

holds.

Theorem 4. *Let $p(\cdot)\in\mathcal{P}_0(\mathbb{T}),\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T})$ and let $(p_n)_{n=0}^{\infty}$ be a sequence of positive real numbers such that*

$$\sum_{k=0}^{n-1}|\Delta p_k|=\mathcal{O}\left(\frac{P_n}{n^{r+1}}\right). \tag{5}$$

If $f\in Lip_r^{p(\cdot)}(1,\omega)$, then the estimate

$$\|f-N_n(f)\|_{p(\cdot),\omega}=\mathcal{O}\left(n^{-(1+r)}\right),\ n=1,2,...,$$

holds.

Theorem 5. *Let* $p(\cdot) \in \mathcal{P}_0(\mathbb{T}), \omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$, *and let* $(p_n)_{n=0}^{\infty}$ *be a sequence of positive real numbers such that*

$$\sum_{m=0}^{n-1} \left| \Delta \left(\frac{P_m}{m+1} \right) \right| = \mathcal{O} \left(\frac{P_n}{(n+1)^{r+1}} \right). \tag{6}$$

If $f \in Lip_r^{p(\cdot)}(\alpha, \omega)$, *then the estimate*

$$\| f - R_n(f) \|_{p(\cdot),\omega} = \mathcal{O}\left(n^{-(1+r)} \right) \ , \quad n = 1, 2, ...,$$

holds.

2. Auxiliary results

In weighted Lebesgue space with variable exponent the approximation problems were studied using some different type modulus of smoothness in [17], [1], [2]. In these works the weight function ω satisfies the condition that $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ for some $1 < p_0 < p_-$. After that under the more intelligible condition, namely $\omega \in A_{p(\cdot)}$, the direct and inverse theorems of approximation theory in the weighted Lebesgue space with variable exponent were proved in [18] and [19], respectively. For the formulations of the results obtained in this work we need some auxiliary results proved in [18, 19].

Let Π_n be the class of trigonometric polynomials of degree not exceeding n. The *best approximation number* of $f \in L_\omega^{p(\cdot)}(\mathbb{T})$ is defined as

$$E_n(f)_{p(\cdot),\omega} := \inf \left\{ \| f - T_n \|_{p(\cdot),\omega} : T_n \in \Pi_n \right\} \ , \quad n = 0, 1, 2, .. \ ,$$

and if $E_n(f)_{p(\cdot),\omega} = \| f - T_n^* \|_{p(\cdot),\omega}$, then $T_n^* \in \Pi_n$ is called *the best approximation trigonometric polynomial to* f in $L_\omega^{p(\cdot)}(\mathbb{T})$.

Lemma 6. [18] *Let* $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ *and* $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. *Then there exists a positive constant* $c_3(p)$ *such that the inequality*

$$\| S_n(f) \|_{p(\cdot),\omega} \leq c_3(p) \| f \|_{p(\cdot),\omega} \ , \ n = 1, 2, ...,$$

holds for every $f \in L_\omega^{p(\cdot)}(\mathbb{T})$.

Lemma 7. [19] *Let* $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ *and* $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. *Then there exists a positive constant* $c_4(p)$ *such that the inequality*

$$\| \sigma_n(f) \|_{p(\cdot),\omega} \leq c_4(p) \| f \|_{p(\cdot),\omega} \ , \ n = 1, 2, ...,$$

holds for every $f \in L_\omega^{p(\cdot)}(\mathbb{T})$.

Lemma 8. [18] *If* $f\in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$, $\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T})$, *then the estimate*

$$E_n(f)_{p(\cdot),\omega}=\mathcal{O}\left(\Omega(f,1/n)_{p(\cdot),\omega}\right)\ ,\ n=1,2,...,$$

holds.

Lemma 9. [19] *If* $f\in W_{\omega}^{p(\cdot),1}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$, $\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T})$, *then there exist a positive constant* $c_5(p)$ *such that the inequality*

$$\|f-S_n(f)\|_{p(\cdot),\omega}\leq\frac{c_5(p)}{n}E_n(f')_{p(\cdot),\omega}\ ,\ n=1,2,...,$$

holds.

Lemma 10. *Let* $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$, $\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T})$ *and* $0<\alpha\leq 1$. *If* $f\in Lip_r^{p(\cdot)}(\alpha,\omega)$, *then the estimate*

$$E_n(f)_{p(\cdot),\omega}=\mathcal{O}\left(n^{-(\alpha+r)}\right),\ n=1,2,...,$$

holds.

Proof. Let $f\in Lip_r^{p(\cdot)}(\alpha,\omega)$. Since $f\in W_{\omega}^{p(\cdot),r}(\mathbb{T})$ and $r=1,2,...,$ Lemma 9 implies that

$$E_n(f)_{p(\cdot),\omega}\leq\frac{c_5(p)}{n}E_n(f')_{p(\cdot),\omega}\ ,\ n=1,2,...,$$

Thus, consecutively r times, using this inequality, we have

$$E_n(f)_{p(\cdot),\omega}\leq\frac{c_6(p)}{n^r}E_n\left(f^{(r)}\right)_{p(\cdot),\omega}\ . \tag{7}$$

By Lemma 8 and (7) we obtain

$$E_n(f)_{p(\cdot),\omega}\leq\frac{c_6(p)}{n^r}E_n\left(f^{(r)}\right)_{p(\cdot),\omega}\leq\frac{c_7(p)}{n^r}\Omega\left(f^{(r)},1/n\right)_{p(\cdot),\omega}=\mathcal{O}\left(n^{-(\alpha+r)}\right).$$

□

Lemma 11. *Let* $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$, $\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T})$ *and* $0<\alpha\leq 1$. *If* $f\in Lip_r^{p(\cdot)}(\alpha,\omega)$, *then the estimate*

$$\|f-S_n(f)\|_{p(\cdot),\omega}=\mathcal{O}\left(n^{-(\alpha+r)}\right)\ ,\quad n=1,2,3...,$$

holds.

Proof. Let $Lip_r^{p(\cdot)}(\alpha,\omega)$, $r=1,2,...,$ and let T_n^* $(n=0,1,2,...)$ be the best approximation trigonometric polynomial to f in $L_{\omega}^{p(\cdot)}(\mathbb{T})$. Applying Lemma 10 we have

$$\|f-T_n^*\|_{p(\cdot),\omega}=E_n(f)_{p(\cdot),\omega}=\mathcal{O}\left(n^{-(\alpha+r)}\right)\ .$$

By Lemma 6 for $n=1,2,3...,$ we obtain

$$\begin{aligned}\|f-S_n(f)\|_{p(\cdot),\omega} &\leq \|f-T_n^*\|_{p(\cdot),\omega}+\|T_n^*-S_n(f)\|_{p(\cdot),\omega}\\ &= \|f-T_n^*\|_{p(\cdot),\omega}+\|S_n(T_n^*)-S_n(f)\|_{p(\cdot),\omega}\end{aligned}$$

$$\begin{aligned} &= \mathcal{O}\left(\|f-T_n^*\|_{p(\cdot),\omega}\right) \\ &= \mathcal{O}\left(n^{-(\alpha+r)}\right). \end{aligned}$$

□

Lemma 12. *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T}), \omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. If $f \in Lip_r^{p(\cdot)}(1,\omega)$, then the estimate*

$$\|S_n(f)-\sigma_n(f)\|_{p(\cdot),\omega} = \mathcal{O}\left(n^{-(1+r)}\right) \ , \quad n=1,2,3...,$$

holds.

Proof. Let $f \in Lip_r^{p(\cdot)}(1,\omega)$, $r=1,2,...,$ and let T_n^* $(n=0,1,2,...)$ be the best approximation trigonometric polynomial to f in $L_\omega^{p(\cdot)}(\mathbb{T})$. Since $f \in L_\omega^{p(\cdot)}(\mathbb{T})$ applying Lemma 7 and Lemma 10 we have

$$\begin{aligned} \|f-\sigma_n(f)\|_{p(\cdot),\omega} &\leq \|f-T_n^*\|_{p(\cdot),\omega} + \|T_n^*-\sigma_n(f)\|_{p(\cdot),\omega} \\ &= \|f-T_n^*\|_{p(\cdot),\omega} + \|\sigma_n(T_n^*-f)\|_{p(\cdot),\omega} \\ &= \mathcal{O}\left(\|f-T_n^*\|_{p(\cdot),\omega}\right) \\ &= \mathcal{O}\left(n^{-(1+r)}\right). \end{aligned} \tag{8}$$

By Lemma 11 for $\alpha=1$ and (8) we obtain

$$\begin{aligned} \|S_n(f)-\sigma_n(f)\|_{p(\cdot),\omega} &\leq \|S_n(f)-f\|_{p(\cdot),\omega} + \|f-\sigma_n(f)\|_{p(\cdot),\omega} \\ &= \mathcal{O}\left(n^{-(1+r)}\right). \end{aligned}$$

□

Lemma 13. *Let $(p_n)_{n=0}^\infty$ be a sequence of positive numbers. If $(p_n)_{n=0}^\infty \in AMIS$ and $(n+1)^{r+1} p_n = \mathcal{O}(P_n)$, then*

$$\sum_{m=1}^{n} m^{-(\alpha+r)} p_{n-m} = \mathcal{O}\left(n^{-(\alpha+r)} P_n\right)$$

for $r=0,1,2,...,$ and $0<\alpha<1$.

Proof. Let $r=0,1,2,...,$ and $0<\alpha<1$. In the case of $r=0$, Lemma 13 was proved in [12]. Similar way we can prove the other part of Lemma. Let k be integer part of $n/2$. If $(p_n)_{n=0}^\infty \in AMIS$ and $(n+1)^{r+1} p_n = \mathcal{O}(P_n)$, then

$$\begin{aligned} \sum_{m=1}^{n} m^{-(\alpha+r)} p_{n-m} &\leq K p_n \sum_{m=1}^{k} m^{-\alpha} + (k+1)^{-(\alpha+r)} \sum_{m=k+1}^{n} p_{n-m} \\ &= \mathcal{O}\left(P_n/n^{r+1}\right) \sum_{m=1}^{k} m^{-\alpha} + \mathcal{O}\left(n^{-(\alpha+r)}\right) P_n \end{aligned}$$

$$= \mathcal{O}\left(n^{-(\alpha+r)}\right) P_n.$$

□

3. Proofs of the main results

Proof of the Theorem 3. Let $0<\alpha<1$. Since

$$f(x)=\frac{1}{P_n}\sum_{m=0}^{n} p_{n-m} f(x)$$

we have

$$f(x)-N_n(f)(x)=\frac{1}{P_n}\sum_{m=0}^{n} p_{n-m}\left\{f(x)-S_m(f)(x)\right\}.$$

By Lemma 11, Lemma 13 and (4) we obtain

$$\begin{aligned}
\|f-N_n(f)\|_{p(\cdot),\omega} &\leq \frac{1}{P_n}\sum_{m=0}^{n} p_{n-m}\|f-S_m(f)\|_{p(\cdot),\omega} \\
&= \frac{1}{P_n}\sum_{m=1}^{n} p_{n-m}\mathcal{O}\left(m^{-(\alpha+r)}\right)+\frac{p_n}{P_n}\|f-S_0(f)\|_{p(\cdot),\omega} \\
&= \frac{1}{P_n}\mathcal{O}\left(n^{-(\alpha+r)}P_n\right)+\mathcal{O}\left((n+1)^{-(r+1)}\right) \\
&= \mathcal{O}\left(n^{-(\alpha+r)}\right).
\end{aligned}$$

□

Proof of the Theorem 4. Let $f\in Lip_{\omega}^{p(\cdot)}(1,\omega)$ and $\sum_{k=1}^{n-1}|\Delta p_k|=\mathcal{O}\left(P_n/n^{r+1}\right)$. It is clear that

$$N_n(f)(x)=\frac{1}{P_n}\sum_{m=1}^{n} P_{n-m}u_m(f)(x).$$

By Abel transform

$$S_n(f)(x)-N_n(f)(x)=\frac{1}{P_n}\sum_{m=1}^{n}(P_n-P_{n-m})u_m(f)(x)$$

$$=\frac{1}{P_n}\sum_{m=1}^{n}\Delta_m\left(\frac{P_n-P_{n-m}}{m}\right)\left(\sum_{k=1}^{m}ku_k(f)(x)\right)+\frac{1}{n+1}\sum_{k=1}^{n}ku_k(f)(x),$$

hence

$$\|S_n(f)-N_n(f)\|_{p(\cdot),\omega} \leq \frac{1}{P_n}\sum_{m=1}^{n}\left|\Delta_m\left(\frac{P_n-P_{n-m}}{m}\right)\right|\left\|\sum_{k=1}^{m}ku_k(f)(x)\right\|_{p(\cdot),\omega}$$

$$+\frac{1}{n+1}\left\|\sum_{k=1}^{n} ku_k\left(f\right)\left(x\right)\right\|_{p(\cdot),\omega}.$$

Since

$$S_n\left(f\right)\left(x\right)-\sigma_n\left(f\right)\left(x\right)=\frac{1}{n+1}\sum_{k=1}^{n} ku_k\left(f\right)\left(x\right),$$

by Lemma 12 we get

$$\frac{1}{n+1}\left\|\sum_{k=1}^{n} ku_k\left(f\right)\left(x\right)\right\|_{p(\cdot),\omega}=\mathcal{O}\left(n^{-(1+r)}\right).$$

Hence

$$\left\|S_n\left(f\right)-N_n\left(f\right)\right\|_{p(\cdot),\omega}\leq\frac{1}{P_n}\sum_{m=1}^{n}\left|\Delta_m\left(\frac{P_n-P_{n-m}}{m}\right)\right|\mathcal{O}\left(m^{-r}\right)+\mathcal{O}\left(n^{-(1+r)}\right). \tag{9}$$

By a simple computations we have

$$\Delta_m\left(\frac{P_n-P_{n-m}}{m}\right)=\frac{1}{m\left(m+1\right)}\left(\sum_{k=n-m}^{n} p_k-\left(m+1\right)p_{n-m}\right)$$

and by induction one can easily obtain

$$\left|\sum_{k=n-m}^{n} p_k-\left(m+1\right)p_{n-m}\right|\leq\sum_{k=1}^{m} k\left|p_{n-k+1}-p_{n-k}\right|.$$

Thus,

$$\begin{aligned}\sum_{m=1}^{n}\left|\Delta_m\left(\frac{P_n-P_{n-m}}{m}\right)\right|m^{-r} &\leq \sum_{m=1}^{n}\left|\Delta_m\left(\frac{P_n-P_{n-m}}{m}\right)\right| \\ &\leq \sum_{m=1}^{n}\frac{1}{m\left(m+1\right)}\left(\sum_{k=1}^{m} k\left|p_{n-k+1}-p_{n-k}\right|\right) \\ &= \sum_{k=1}^{n} k\left|p_{n-k+1}-p_{n-k}\right|\left(\sum_{m=k}^{n}\frac{1}{m\left(m+1\right)}\right) \\ &\leq \sum_{k=1}^{n} k\left|p_{n-k+1}-p_{n-k}\right|\left(\sum_{m=k}^{\infty}\frac{1}{m\left(m+1\right)}\right) \\ &= \sum_{k=1}^{n}\left|p_{n-k+1}-p_{n-k}\right| \\ &= \sum_{k=1}^{n-1}\left|\Delta p_k\right|=\mathcal{O}\left(\frac{P_n}{n^{r+1}}\right).\end{aligned} \tag{10}$$

By (9) and (10) we have

$$\left\| S_n(f) - N_n(f) \right\|_{p(\cdot),\omega} = \mathcal{O}\left(n^{-(1+r)}\right). \tag{11}$$

Combining Lemma 11 for $\alpha = 1$ and (11) we obtain

$$\begin{aligned} \left\| f - N_n(f) \right\|_{p(\cdot),\omega} &\leq \left\| f - S_n(f) \right\|_{p(\cdot),\omega} + \left\| S_n(f) - N_n(f) \right\|_{p(\cdot),\omega} \\ &= \mathcal{O}\left(n^{-(1+r)}\right). \end{aligned}$$

□

Proof of the Theorem 6. By Abel transform,

$$\begin{aligned} R_n(f)(x) &= \frac{1}{P_n} \sum_{m=0}^{n-1} \left\{ P_m \left(S_m(f)(x) - S_{m+1}(f)(x) \right) + P_n S_n(f)(x) \right\} \\ &= \frac{1}{P_n} \sum_{m=0}^{n-1} P_m \left(-u_{m+1}(f)(x) \right) + S_n(f)(x) \end{aligned}$$

and hence we have

$$R_n(f)(x) - S_n(f)(x) = -\frac{1}{P_n} \sum_{m=0}^{n-1} P_m u_{m+1}(f)(x). \tag{12}$$

Using Abel transform

$$\begin{aligned} \sum_{m=0}^{n-1} P_m u_{m+1}(f)(x) &= \sum_{m=0}^{n-1} \frac{P_m}{m+1} (m+1) u_{m+1}(f)(x) \\ &= \sum_{m=0}^{n-1} \Delta\left(\frac{P_m}{m+1}\right) \left(\sum_{k=0}^{m} (k+1) u_{k+1}(f)(x) \right) \\ &\quad + \frac{P_n}{n+1} \sum_{k=0}^{n-1} (k+1) u_{k+1}(f)(x). \end{aligned}$$

By Lemma 12 and (6) we get

$$\begin{aligned} \left\| \sum_{m=0}^{n-1} P_m u_{m+1}(f)(x) \right\|_{p(\cdot),\omega} &\leq \sum_{m=0}^{n-1} \left| \Delta\left(\frac{P_m}{m+1}\right) \right| \left\| \sum_{k=0}^{m} (k+1) u_{k+1}(f)(x) \right\|_{p(\cdot),\omega} \\ &\quad + \frac{P_n}{n+1} \left\| \sum_{k=0}^{n-1} (k+1) u_{k+1}(f)(x) \right\|_{p(\cdot),\omega} \\ &= \sum_{m=0}^{n-1} \left| \Delta\left(\frac{P_m}{m+1}\right) \right| (m+2) \left\| S_{m+1}(f) - \sigma_{m+1}(f) \right\|_{p(\cdot),\omega} \\ &\quad + P_n \left\| S_n(f) - \sigma_n(f) \right\|_{p(\cdot),\omega} \end{aligned}$$

$$= \sum_{m=0}^{n-1} \left| \Delta \left(\frac{P_m}{m+1} \right) \right| \mathcal{O}\left((m+1)^{-r} \right) + \mathcal{O}\left(\frac{P_n}{n^{r+1}} \right) = \mathcal{O}\left(\frac{P_n}{n^{r+1}} \right).$$

By (12) we have

$$\begin{aligned} \left\| R_n(f) - S_n(f) \right\|_{p(\cdot),\omega} &= \frac{1}{P_n} \left\| \sum_{m=0}^{n-1} P_m u_{m+1}(f)(x) \right\|_{p(\cdot),\omega} \\ &= \frac{1}{P_n} \mathcal{O}\left(\frac{P_n}{n^{r+1}} \right) = \mathcal{O}\left(n^{-(r+1)} \right). \end{aligned}$$

□

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Current address: Ahmet Testici: Department of Mathematics, Faculty of Art and Science, Balikesir University, 10145, Balikesir, Turkey

E-mail address: testiciahmet@hotmail.com

ORCID Address: https://orcid.org/0000-0002-1163-7037