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Research Article

Explicit inverses of generalized Tribonacci circulant type matrices

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Abstract

In this paper, we consider generalized Tribonacci circulant type matrices, including the circulant and left circulant. Firstly, we discuss the invertibility of generalized Tribonacci circulant matrix and give the explicit determinant and inverse matrix based on constructing the transformation matrices. Afterwards, by utilizing the relation between circulant and left circulant, the invertibility of generalized Tribonacci left circulant matrix is also discussed. The determinant and inverse of generalized Tribonacci left circulant matrix are given respectively.

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1. Introduction and preliminaries

A matrix $\mathcal{C}_n \in M_n$ is called circulant matrix if it is of the form

$$\mathcal{C}_n = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{pmatrix}.$$

It is evidently determined by its first row, we denote $\mathcal{C}_n := Circ(c_0, c_1, \cdots, c_{n-1})$. And the $n \times n$ left circulant matrix $\mathcal{D}_n := LCirc(d_0, d_1, \cdots, d_{n-1})$ is defined as

$$\mathcal{D}_n = \begin{pmatrix} d_0 & d_1 & \cdots & d_{n-1} \\ d_1 & d_2 & \cdots & d_0 \\ \vdots & \vdots & \ddots & \vdots \\ d_{n-1} & d_0 & \cdots & d_{n-2} \end{pmatrix},$$

where each row is a cyclic shift of the row above to the left.

Circulant type matrices not only have many connections to problems in physics, geometry and numerical analysis, but also have important applications in various disciplines including signal and image processing [1], networks engineering [13, 19], solving ordinary and partial differential equations [6, 12]. In recent years, the investigation of the circulant

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type matrices attracts much attention. Bani-Domi and Kittaneh [2] gave two general norm equalities for circulant and skew circulant operator matrices. Karner et al. [14] presented the eigendecompositions and singular value decompositions of real circulant type matrices. In addition, the determinant of the circulant matrix $\mathcal{C}_n = Circ(c_0, c_1, \cdots, c_{n-1})$ is also given by the following formulas [7]:

$$\det \mathcal{C}_n = \prod_{j=0}^{n-1} g(\omega^j),$$

and if $\mathcal{C}_n$ is invertible, then

$$\mathcal{C}_n^{-1} = Circ(b_0, b_1, \cdots, b_{n-1}),$$

where $b_s = \frac{1}{n}\sum_{j=0}^{n-1} g(\omega^j)^{-1}\omega^{-js}(s = 0, 1, \cdots, n-1)$, $g(x) = \sum_{j=0}^{n-1} c_j x^j$ and $\omega = \exp(\frac{2\pi i}{n})$. Unfortunately, these formulas are not computationally feasible as their complexities increase drastically when n is getting large.

In order to overcome the deficiency above, some scholars gave the explicit determinants and inverses of circulant type matrices involving some famous numbers [3, 4, 8–11, 16, 18, 20, 21]. For example, Shen et al. [18] considered circulant matrices with Fibonacci and Lucas numbers and presented their explicit determinants and inverses by constructing the transformation matrices. Jiang et al. [10] discussed the invertibility of circulant type matrices involving the sum and product of Fibonacci and Lucas numbers and gave the determinants and the inverses of these matrices. Bozkurt and Tam [3] proposed the determinants and the inverses of circulant matrices involving Jacobsthal and Jacobsthal-Lucas numbers. Yazlik and Taskara [20] evaluated the determinant and the inverse of circulant matrix with generalized k-Horadam numbers. Liu and Jiang [16] discussed the invertibility of Tribonacci circulant type matrices and presented the explicit determinants and inverses of Tribonacci circulant and Tribonacci left circulant matrices. In addition, Jiang and Hong [9] considered Tribonacci skew circulant type matrices and gave their exact determinants and inverses by utilizing only the Tribonacci numbers.

For $n \geq 3$, the generalized Tribonacci sequence $\{T_n^{(a)}\}$ is defined by the following recurrence relations [15, 17]:

$$T_n^{(a)} = uT_{n-1}^{(a)} + vT_{n-2}^{(a)} + wT_{n-3}^{(a)}, \quad T_0^{(a)} = 0, \quad T_1^{(a)} = a, \quad T_2^{(a)} = au, \tag{1.1}$$

where a, u, v and w are arbitrary positive integers. When $a = u = v = w = 1$, the generalized Tribonacci sequence reduces to the Tribonacci sequence $\{T_n\}$ in [5]. Let λ_1, λ_2 and λ_3 be the roots of the characteristic equation $\lambda^3 - u\lambda^2 - v\lambda - w = 0$, then we have

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = u, \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = -v, \\ \lambda_1\lambda_2\lambda_3 = w. \end{cases}$$

and the Binet formula of the sequences $\{T_n^{(a)}\}$ have the form

$$T_n^{(a)} = b_1\lambda_1^n + b_2\lambda_2^n + b_3\lambda_3^n, \tag{1.2}$$

where $b_k = \lambda_k \prod_{j=1, j\neq k}^{3}(1/(\lambda_k - \lambda_j))$ $(k = 1, 2, 3)$.

In the present paper, we consider generalized Tribonacci circulant type matrices. Firstly, we define a generalized Tribonacci circulant matrix

$$\mathcal{R}_n = Circ(T_1^{(a)}, T_2^{(a)}, \cdots, T_n^{(a)})$$

and a generalized Tribonacci left circulant matrix

$$\mathcal{L}_n = LCirc(T_1^{(a)}, T_2^{(a)}, \cdots, T_n^{(a)}),$$

which are generalizations of Tribonacci circulant and Tribonacci left circulant matrices from [16], respectively. Afterwards, the invertibility of matrices $\mathcal{R}_n$ and $\mathcal{L}_n$ are discussed. Furthermore, the explicit determinants and inverses of these matrices are derived by constructing the transformation matrices, which are only related to the generalized Tribonacci numbers.

For the convenience of the discussion, throughout the paper we denote

$$p := -wT_n^{(a)},\ q := -vT_n^{(a)} - wT_{n-1}^{(a)},\ r := T_1^{(a)} - T_{n+1}^{(a)},$$

and adopt the following convention: for any sequence $\{a_n\}$, $\sum_{j=k}^{n} a_j = 0$ in the case $k > n$.

Lemma 1.1. [7] *Let $\mathcal{C}_n = Circ(c_1, c_2, \cdots, c_n)$ be an $n \times n$ circulant matrix. Then $\mathcal{C}_n$ is invertible if and only if $f(\omega^k) \neq 0 (k = 0, 1, 2, \cdots, n-1)$, where $f(x) = \sum_{j=1}^{n} c_j x^{j-1}$ and $\omega = \exp(\frac{2\pi i}{n})$.*

Lemma 1.2. [11] *Let*

$$\mathcal{K}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \end{pmatrix}$$

be an $n \times n$ matrix. Then it holds that $\mathcal{L}_n = \mathcal{K}_n \mathcal{R}_n$.

Lemma 1.3. [16] *The $n \times n$ tridiagonal matrix is given by*

$$\mathcal{A}_n = \begin{pmatrix} \tau_2 & \tau_1 & & & 0 \\ \tau_3 & \tau_2 & \tau_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \tau_3 & \tau_2 & \tau_1 \\ 0 & & & \tau_3 & \tau_2 \end{pmatrix},$$

then

$$\det \mathcal{A}_n = \begin{cases} \dfrac{\left(\frac{\tau_2+\sqrt{\tau_2^2-4\tau_1\tau_3}}{2}\right)^{n+1} - \left(\frac{\tau_2-\sqrt{\tau_2^2-4\tau_1\tau_3}}{2}\right)^{n+1}}{\sqrt{\tau_2^2-4\tau_1\tau_3}}, & \tau_2^2 \neq 4\tau_1\tau_3, \\ (n+1)(\frac{\tau_2}{2})^n, & \tau_2^2 = 4\tau_1\tau_3. \end{cases}$$

Lemma 1.4. [16] *Let*

$$\mathcal{B}_n = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} & a_n \\ \tau_2 & \tau_1 & & & & \\ \tau_3 & \tau_2 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \tau_2 & \tau_1 & \\ & & & \tau_3 & \tau_2 & \tau_1 \end{pmatrix}$$

be an $n \times n$ matrix. Then we have

$$\det \mathcal{B}_n = \sum_{k=1}^{n} (-1)^{k+1} \tau_1^{n-k} a_k \cdot \det \mathcal{A}_{k-1},$$

where

$$\det \mathcal{A}_{k-1} = \begin{cases} \dfrac{\left(\frac{\tau_2+\sqrt{\tau_2^2-4\tau_1\tau_3}}{2}\right)^{k} - \left(\frac{\tau_2-\sqrt{\tau_2^2-4\tau_1\tau_3}}{2}\right)^{k}}{\sqrt{\tau_2^2-4\tau_1\tau_3}}, & \tau_2^2 \neq 4\tau_1\tau_3, \\ k(\frac{\tau_2}{2})^{k-1}, & \tau_2^2 = 4\tau_1\tau_3. \end{cases}$$

2. Determinant and inverse of generalized Tribonacci circulant matrix

In this section, let $\mathcal{R}_n = Circ(T_1^{(a)}, T_2^{(a)}, \cdots, T_n^{(a)})$ be a generalized Tribonacci circulant matrix. Afterwards, we give an explicit formula for the determinant of the matrix $\mathcal{R}_n$. In addition, we also discuss the invertibility of $\mathcal{R}_n$ and compute the inverse matrix of $\mathcal{R}_n$.

Theorem 2.1. *Let $\mathcal{R}_n = Circ(T_1^{(a)}, T_2^{(a)}, \cdots, T_n^{(a)})$ be a generalized Tribonacci circulant matrix. Then we have*

$$\begin{aligned} \det \mathcal{R}_n &= a\Big[\Big(T_1^{(a)} - uT_n^{(a)} + \sum_{k=1}^{n-2}(T_{k+2}^{(a)} - uT_{k+1}^{(a)})\delta^{n-k-1}\Big)\kappa_1 \\ &\quad - \Big(-vT_n^{(a)} + (T_1^{(a)} - T_{n+1}^{(a)})\delta + w\sum_{k=1}^{n-2} T_k^{(a)}\delta^{n-k-1}\Big)\kappa_2\Big], \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \delta &= \frac{-q+\sqrt{q^2-4pr}}{2r}, \quad \kappa_1 = r^{n-2} + w\sum_{k=1}^{n-2}(-1)^{k+1}r^{n-k-2}T_{n-k-1}^{(a)} \cdot \det \mathcal{A}_{k-1}, \\ \kappa_2 &= \sum_{k=1}^{n-2}(-1)^{k+1}r^{n-k-2}(T_{n-k+1}^{(a)} - uT_{n-k}^{(a)}) \cdot \det \mathcal{A}_{k-1}, \end{aligned}$$

and

$$\det \mathcal{A}_{k-1} = \begin{cases} \dfrac{\left(\frac{q+\sqrt{q^2-4pr}}{2}\right)^k - \left(\frac{q-\sqrt{q^2-4pr}}{2}\right)^k}{\sqrt{q^2-4pr}}, & q^2 \neq 4pr, \\ k(\frac{q}{2})^{k-1}, & q^2 = 4pr. \end{cases}$$

Proof. Obviously, $\det \mathcal{R}_1 = a, \det \mathcal{R}_2 = a^2(1-u^2)$ and $\det \mathcal{R}_3 = a^3(1-u^3-2uv)[1-u(u^2+v)] + a^3v^2(u^2+v)$ satisfy the formula (2.1). In the case $n > 3$, let

$$\Gamma_1 = \begin{pmatrix} 1 & & & & & & & 0 \\ -u & & & & & & & 1 \\ -v & & & & & & 1 & -u \\ -w & & & & & 1 & -u & -v \\ 0 & & & & 1 & -u & -v & -w \\ \vdots & & & \iddots & \iddots & \iddots & \iddots & \\ 0 & & 1 & -u & -v & -w & & \\ 0 & 1 & -u & -v & -w & & & \end{pmatrix}$$

and

$$\Pi_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \delta^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & \delta^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \delta & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

be two $n \times n$ matrices. Then we have

$$\Gamma_1 \mathcal{R}_n \Pi_1 = \begin{pmatrix} T_1^{(a)} & \sigma_1 & T_{n-1}^{(a)} & T_{n-2}^{(a)} & \cdots & T_4^{(a)} & T_3^{(a)} & T_2^{(a)} \\ 0 & \sigma_2 & \phi_3 & \phi_4 & \cdots & \phi_{n-2} & \phi_{n-1} & \phi_n \\ 0 & \sigma_3 & \phi & wT_{n-3}^{(a)} & \cdots & wT_3^{(a)} & wT_2^{(a)} & wT_1^{(a)} \\ 0 & 0 & q & r & & & & \\ 0 & 0 & p & q & \ddots & & & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \\ 0 & 0 & & & \ddots & q & r & \\ 0 & 0 & & & & p & q & r \end{pmatrix},$$

where

$$\phi_k = T_{n-k+3}^{(a)} - uT_{n-k+2}^{(a)} \ (k = 3, 4, \cdots, n), \tag{2.2}$$

$$\phi = T_1^{(a)} - uT_n^{(a)} - vT_{n-1}^{(a)}, \quad \sigma_1 = \sum_{k=1}^{n-1} T_{k+1}^{(a)} \delta^{n-k-1}, \tag{2.3}$$

$$\sigma_2 = T_1^{(a)} - uT_n^{(a)} + \sum_{k=1}^{n-2} (T_{k+2}^{(a)} - uT_{k+1}^{(a)}) \delta^{n-k-1}, \tag{2.4}$$

$$\sigma_3 = -vT_n^{(a)} + (T_1^{(a)} - T_{n+1}^{(a)})\delta + w \sum_{k=1}^{n-2} T_k^{(a)} \delta^{n-k-1}. \tag{2.5}$$

Let

$$\Omega = \begin{pmatrix} \phi & wT_{n-3}^{(a)} & \cdots & wT_3^{(a)} & wT_2^{(a)} & wT_1^{(a)} \\ q & r & & & & \\ p & q & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & q & r & \\ & & & p & q & r \end{pmatrix}$$

and

$$\Psi = \begin{pmatrix} \phi_3 & \phi_4 & \cdots & \phi_{n-2} & \phi_{n-1} & \phi_n \\ q & r & & & & \\ p & q & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & q & r & \\ & & & p & q & r \end{pmatrix}$$

be two $(n-2) \times (n-2)$ matrices. According to Lemma 1.4, we obtain

$$\begin{aligned} \det \Omega &= r^{n-2} + w \sum_{k=1}^{n-2} (-1)^{k+1} r^{n-k-2} T_{n-k-1}^{(a)} \cdot \det \mathcal{A}_{k-1} = \kappa_1, \\ \det \Psi &= \sum_{k=1}^{n-2} (-1)^{k+1} r^{n-k-2} (T_{n-k+1}^{(a)} - uT_{n-k}^{(a)}) \cdot \det \mathcal{A}_{k-1} = \kappa_2. \end{aligned}$$

Hence

$$\begin{aligned}\det\Gamma_1\det\mathcal{R}_n\det\Pi_1 &= T_1^{(a)}\cdot(\sigma_2\det\Omega-\sigma_3\det\Psi)\\ &= a\Big[\Big(T_1^{(a)}-uT_n^{(a)}+\sum_{k=1}^{n-2}(T_{k+2}^{(a)}-uT_{k+1}^{(a)})\delta^{n-k-1}\Big)\kappa_1\\ &\quad-\Big(-vT_n^{(a)}+(T_1^{(a)}-T_{n+1}^{(a)})\delta+w\sum_{k=1}^{n-2}T_k^{(a)}\delta^{n-k-1}\Big)\kappa_2\Big],\end{aligned}$$

while $\det\Gamma_1=\det\Pi_1=(-1)^{\frac{(n-1)(n-2)}{2}}$. Thus, the proof is completed. □

Theorem 2.2. *Let $\mathcal{R}_n=Circ(T_1^{(a)},T_2^{(a)},\cdots,T_n^{(a)})(n>2)$ be a generalized Tribonacci circulant matrix. If $\lambda_j^n\neq 1(j=1,2,3)$, $q\neq p+r$ and $\sqrt{4pr-q^2}/q\neq\pm\tan\frac{2k\pi}{n}$ for any integer $k\in(\frac{n}{4},\frac{3n}{4})$, then $\mathcal{R}_n$ is an invertible matrix.*

Proof. Since $\lambda_j^n\neq 1(j=1,2,3)$, $1-\lambda_1\omega^k\neq 0, 1-\lambda_2\omega^k\neq 0$ and $1-\lambda_3\omega^k\neq 0$ for any $k\in\{1,2,\cdots,n-1\}$, then applying the Binet formula (1.2), we obtain

$$\begin{aligned}f(\omega^k) &= \sum_{j=1}^n T_j^{(a)}(\omega^k)^{j-1}=\sum_{j=1}^n(b_1\lambda_1^j+b_2\lambda_2^j+b_3\lambda_3^j)(\omega^k)^{j-1}\\ &= \frac{b_1\lambda_1(1-\lambda_1^n)}{1-\lambda_1\omega^k}+\frac{b_2\lambda_2(1-\lambda_2^n)}{1-\lambda_2\omega^k}+\frac{b_3\lambda_3(1-\lambda_3^n)}{1-\lambda_3\omega^k}\\ &= \frac{T_1^{(a)}-T_{n+1}^{(a)}+(-vT_n^{(a)}-wT_{n-1}^{(a)})\omega^k-wT_n^{(a)}\omega^{2k}}{1-u\omega^k-v\omega^{2k}-w\omega^{3k}}.\end{aligned}$$

By a way of contradiction, if there exists $l\in\{1,2,\cdots,n-1\}$ such that $f(\omega^l)=0$, then we have $T_1^{(a)}-T_{n+1}^{(a)}+(-vT_n^{(a)}-wT_{n-1}^{(a)})\omega^l-wT_n^{(a)}\omega^{2l}=0$, hence

$$\begin{aligned}\omega^l &= \frac{vT_n^{(a)}+wT_{n-1}^{(a)}\pm\sqrt{(-vT_n^{(a)}-wT_{n-1}^{(a)})^2+4wT_n^{(a)}(T_1^{(a)}-T_{n+1}^{(a)})}}{-2wT_n^{(a)}}\\ &= \frac{-q\pm\sqrt{q^2-4pr}}{2p}.\end{aligned}$$

From $q\neq p+r$, we can deduce that $q^2-4pr<0$, this implies that ω^l is an imaginary number. While

$$\omega^l=\exp\left(\frac{2l\pi i}{n}\right)=\cos\frac{2l\pi}{n}+i\sin\frac{2l\pi}{n},$$

so we have

$$\begin{cases}\cos(\frac{2l\pi}{n})=-\frac{q}{2p}<0,\\ \sin(\frac{2l\pi}{n})=\pm\frac{\sqrt{4pr-q^2}}{2p}.\end{cases}$$

It follows that $\sqrt{4pr-q^2}/q=\pm\tan\frac{2l\pi}{n}$ is valid for integer $l\in(\frac{n}{4},\frac{3n}{4})$, which is a contradiction. Hence, we obtain $f(\omega^k)\neq 0$ for any $k\in\{1,2,\cdots,n-1\}$, while it is obvious that $f(1)=\sum_{j=1}^n T_j^{(a)}\neq 0$. Thus, according to Lemma 1.1, the proof is completed. □

Lemma 2.3. *Let $\mathcal{G} = [g_{j,k}]$ be an $(n-3)\times(n-3)$ matrix given by*

$$g_{j,k} = \begin{cases} T_1^{(a)} - T_{n+1}^{(a)}, & \textit{if } j = k, \\ -vT_n^{(a)} - wT_{n-1}^{(a)}, & \textit{if } j = k+1, \\ -wT_n^{(a)}, & \textit{if } j = k+2, \\ 0, & \textit{otherwise.} \end{cases}$$

Then the inverse $\mathcal{G}^{-1} = [g'_{j,k}]$ of the matrix $\mathcal{G}$ is equal to

$$g'_{j,k} = \begin{cases} \frac{1}{r}(\frac{\beta^{j-k+1}-\alpha^{j-k+1}}{\beta-\alpha}), & \textit{if } j \geq k, \\ 0, & \textit{if } j < k. \end{cases}$$

where $\alpha = -\frac{q+\sqrt{q^2-4pr}}{2r}$, $\beta = -\frac{q-\sqrt{q^2-4pr}}{2r}$.

Proof. Let $c_{j,k} = \sum_{l=1}^{n-3} g_{j,l}g'_{l,k}$. It is clear that $c_{j,k} = 0$ for $j < k$. In the case $j = k$, we have

$$c_{j,j} = g_{j,j}g'_{j,j} = (T_1^{(a)} - T_{n+1}^{(a)}) \cdot \frac{1}{r} = (T_1^{(a)} - T_{n+1}^{(a)}) \cdot \frac{1}{T_1^{(a)} - T_{n+1}^{(a)}} = 1.$$

For $j \geq k+1$, we obtain

$$\begin{aligned} c_{j,k} &= \sum_{l=1}^{n-3} g_{j,l}g'_{l,k} = g_{j,j-2}g'_{j-2,k} + g_{j,j-1}g'_{j-1,k} + g_{j,j}g'_{j,k} \\ &= -wT_n^{(a)}\frac{1}{r}\Big(\frac{\beta^{j-k-1}-\alpha^{j-k-1}}{\beta-\alpha}\Big) + (-vT_n^{(a)} - wT_{n-1}^{(a)})\frac{1}{r}\Big(\frac{\beta^{j-k}-\alpha^{j-k}}{\beta-\alpha}\Big) \\ &\quad +(T_1^{(a)} - T_{n+1}^{(a)})\frac{1}{r}\Big(\frac{\beta^{j-k+1}-\alpha^{j-k+1}}{\beta-\alpha}\Big) \\ &= 0. \end{aligned}$$

Hence, we verify $\mathcal{G}\mathcal{G}^{-1} = I_{n-3}$, where I_{n-3} is the $(n-3)\times(n-3)$ identity matrix. Similarly, we can verify $\mathcal{G}^{-1}\mathcal{G} = I_{n-3}$. Thus, the proof is completed. $\square$

Theorem 2.4. *Let $\mathcal{R}_n = Circ(T_1^{(a)}, T_2^{(a)}, \cdots, T_n^{(a)})(n > 4)$ be an invertible generalized Tribonacci circulant matrix. If $\sigma_2 \neq 0$, then we have*

$$\begin{aligned} \mathcal{R}_n^{-1} &= Circ\Big(x'_2 - (u+\frac{\sigma_3}{\sigma_2})x'_3 - vx'_4 - wx'_5,\ -ux'_2 + (-v+u\frac{\sigma_3}{\sigma_2})x'_3 - wx'_4, \\ &\quad x'_n,\ x'_{n-1} - ux'_n,\ x'_{n-2} - ux'_{n-1} - vx'_n, \cdots, x'_3 - ux'_4 - vx'_5 - wx'_6\Big), \end{aligned}$$

where

$$
\begin{aligned}
x_2^{'} &= \frac{1}{\sigma_2},\ x_3^{'} = \frac{-r\phi_3+\sigma}{\ell r\sigma_2},\\
x_k^{'} &= \frac{r\phi_3-\sigma}{\ell r^2\sigma_2}\sum_{j=1}^{n-k+1}\rho_{j+k-1}\Big(\frac{\beta^j-\alpha^j}{\beta-\alpha}\Big)\\
&\quad -\frac{1}{r\sigma_2}\sum_{j=1}^{n-k+1}\phi_{j+k-1}\Big(\frac{\beta^j-\alpha^j}{\beta-\alpha}\Big)\quad (k=4,5,\cdots,n),\\
\sigma &= \sum_{j=1}^{n-3}\phi_{j+3}\Big[q\Big(\frac{\beta^j-\alpha^j}{\beta-\alpha}\Big)+p\Big(\frac{\beta^{j-1}-\alpha^{j-1}}{\beta-\alpha}\Big)\Big],\\
\ell &= \rho_3-\frac{q}{r}\sum_{j=1}^{n-3}\rho_{j+3}\Big(\frac{\beta^j-\alpha^j}{\beta-\alpha}\Big)-\frac{p}{r}\sum_{j=1}^{n-4}\rho_{j+4}\Big(\frac{\beta^j-\alpha^j}{\beta-\alpha}\Big),\\
\rho_3 &= \phi-\phi_3\frac{\sigma_3}{\sigma_2},\ \rho_j = wT^{(a)}_{n-j+1}-\phi_j\frac{\sigma_3}{\sigma_2}\quad (j=4,5,\cdots,n),
\end{aligned}
$$

and $\sigma_2,\ \sigma_3,\ \phi,\ \phi_j(j=3,4,\cdots,n)$ *are given by (2.2), (2.3), (2.4) and (2.5).*

Proof. Let

$$
\Gamma_2=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & -\frac{\sigma_3}{\sigma_2} & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}
$$

and

$$
\Pi_2=\begin{pmatrix} 1 & -\sigma_1 & -T^{(a)}_{n-1}+\frac{\sigma_1\phi_3}{\sigma_2} & \cdots & -T^{(a)}_2+\frac{\sigma_1\phi_n}{\sigma_2}\\ 0 & a & \frac{-a\phi_3}{\sigma_2} & \cdots & \frac{-a\phi_n}{\sigma_2}\\ 0 & 0 & a & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & a \end{pmatrix}
$$

be two $n\times n$ matrices. Then we have

$$\Gamma_2\Gamma_1\mathcal{R}_n\Pi_1\Pi_2 = a(\mathcal{D}\oplus\mathcal{F}),$$

where $\mathcal{D}\oplus\mathcal{F}$ is the direct sum of $\mathcal{D}$ and $\mathcal{F}$, $\mathcal{D}=diag(1,\sigma_2)$ is a diagonal matrix and

$$
\mathcal{F}=\begin{pmatrix} \rho_3 & \rho_4 & \rho_5 & \cdots & \rho_{n-2} & \rho_{n-1} & \rho_n\\ q & r & & & & & \\ p & q & r & & & & \\ & p & q & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & q & r & \\ & & & & p & q & r \end{pmatrix}
$$

is an $(n-2)\times(n-2)$ matrix. If we denote $\Gamma=\Gamma_2\Gamma_1, \Pi=\Pi_1\Pi_2$, then we obtain

$$\mathcal{R}_n^{-1} = a^{-1}\Pi(\mathcal{D}^{-1}\oplus\mathcal{F}^{-1})\Gamma.$$

Let $\mathcal{P}=(\rho_4,\rho_5,\cdots,\rho_n)$ be a row vector and $\mathcal{Q}=(q,p,0,\cdots,0)^T$ be a column vector. Then $\mathcal{F}$ can be rewritten as the following block form

$$
\mathcal{F}=\begin{pmatrix} \rho_3 & \mathcal{P}\\ \mathcal{Q} & \mathcal{G} \end{pmatrix}.
$$

Hence

$$\mathcal{F}^{-1}=\begin{pmatrix} \frac{1}{\gamma} & -\frac{1}{\gamma}\mathcal{P}\mathcal{G}^{-1} \\ -\frac{1}{\gamma}\mathcal{G}^{-1}\mathcal{Q} & \mathcal{G}^{-1}+\frac{1}{\gamma}\mathcal{G}^{-1}\mathcal{Q}\mathcal{P}\mathcal{G}^{-1} \end{pmatrix},$$

where $\gamma=\rho_3-\mathcal{P}\mathcal{G}^{-1}\mathcal{Q}$. According to Lemma 2.3, we obtain

$$\gamma=\rho_3-\frac{q}{r}\sum_{j=1}^{n-3}\rho_{j+3}\Big(\frac{\beta^j-\alpha^j}{\beta-\alpha}\Big)-\frac{p}{r}\sum_{j=1}^{n-4}\rho_{j+4}\Big(\frac{\beta^j-\alpha^j}{\beta-\alpha}\Big)=\ell.$$

Since the last row elements of the matrix Π are $0, a, -\frac{a\phi_3}{\sigma_2}, -\frac{a\phi_4}{\sigma_2}, \cdots, -\frac{a\phi_n}{\sigma_2}$, and $\mathcal{D}^{-1}=diag(1,\sigma_2^{-1})$, the last row of $a^{-1}\Pi(\mathcal{D}^{-1}\oplus\mathcal{F}^{-1})$ is given by the following equations:

$$\begin{aligned}
y_1 &= 0,\ y_2=\frac{1}{\sigma_2}=x_2',\ y_3=\frac{-r\phi_3+\sigma}{\ell r\sigma_2}=x_3',\\
y_4 &= \frac{r\phi_3-\sigma}{\ell r^2\sigma_2}\sum_{j=1}^{n-3}\rho_{j+3}\Big(\frac{\beta^j-\alpha^j}{\beta-\alpha}\Big)-\frac{1}{r\sigma_2}\sum_{j=1}^{n-3}\phi_{j+3}\Big(\frac{\beta^j-\alpha^j}{\beta-\alpha}\Big)=x_4',\\
&\vdots\\
y_{n-1} &= \frac{r\phi_3-\sigma}{\ell r^2\sigma_2}\sum_{j=1}^{2}\rho_{n+j-2}\Big(\frac{\beta^j-\alpha^j}{\beta-\alpha}\Big)-\frac{1}{r\sigma_2}\sum_{j=1}^{2}\phi_{n+j-2}\Big(\frac{\beta^j-\alpha^j}{\beta-\alpha}\Big)=x_{n-1}',\\
y_n &= \frac{r\phi_3-\sigma}{\ell r^2\sigma_2}\rho_n-\frac{1}{r\sigma_2}\phi_n=x_n'.
\end{aligned}$$

While

$$\Gamma=\Gamma_2\Gamma_1=\begin{pmatrix}
1 & & & & & & & & 0\\
-u & & & & & & & & 1\\
-v+u\frac{\sigma_3}{\sigma_2} & & & & & & & 1 & -u-\frac{\sigma_3}{\sigma_2}\\
-w & & & & & & 1 & -u & -v\\
0 & & & & & 1 & -u & -v & -w\\
\vdots & & & & \iddots & \iddots & \iddots & \iddots & \\
0 & & 1 & -u & -v & -w & & & \\
0 & 1 & -u & -v & -w & & & &
\end{pmatrix},$$

hence, if we denote $\mathcal{R}_n^{-1}:=Circ(x_1,x_2,\cdots,x_n)$, then its last row elements are of the forms:

$$\begin{aligned}
x_2 &= -ux_2'+(-v+u\frac{\sigma_3}{\sigma_2})x_3'-wx_4',\\
x_3 &= x_n',\\
x_4 &= x_{n-1}'-ux_n',\\
x_5 &= x_{n-2}'-ux_{n-1}'-vx_n',\\
x_6 &= x_{n-3}'-ux_{n-2}'-vx_{n-1}'-wx_n',\\
&\vdots\\
x_n &= x_3'-ux_4'-vx_5'-wx_6',\\
x_1 &= x_2'-(u+\frac{\sigma_3}{\sigma_2})x_3'-vx_4'-wx_5'.
\end{aligned}$$

Thus, the proof is completed. □

3. Determinant and inverse of generalized Tribonacci left circulant matrix

In this section, let $\mathcal{L}_n = LCirc(T_1^{(a)}, T_2^{(a)}, \cdots, T_n^{(a)})$ be a generalized Tribonacci left circulant matrix. By using the obtained conclusions, we give a determinant formula of the matrix $\mathcal{L}_n$. Furthermore, we discuss the invertibility of the matrix $\mathcal{L}_n$ and derive the inverse of $\mathcal{L}_n$.

According to Lemma 1.2, Theorem 2.1,Theorem 2.2 and Theorem 2.4, we can obtain the following theorems.

Theorem 3.1. *Let $\mathcal{L}_n = LCirc(T_1^{(a)}, T_2^{(a)}, \cdots, T_n^{(a)})$ be a generalized Tribonacci left circulant matrix. Then we have*

$$\begin{aligned}\det \mathcal{L}_n &= (-1)^{\frac{(n-1)(n-2)}{2}} a\Bigg[\Bigg(T_1^{(a)} - uT_n^{(a)} + \sum_{k=1}^{n-2}(T_{k+2}^{(a)} - uT_{k+1}^{(a)})\delta^{n-k-1}\Bigg)\kappa_1 \\ &\quad - \Bigg(- vT_n^{(a)} + (T_1^{(a)} - T_{n+1}^{(a)})\delta + w\sum_{k=1}^{n-2} T_k^{(a)}\delta^{n-k-1}\Bigg)\kappa_2\Bigg],\end{aligned}$$

where δ, κ_1 and κ_2 are given by Theorem 2.1.

Theorem 3.2. *Let $\mathcal{L}_n = LCirc(T_1^{(a)}, T_2^{(a)}, \cdots, T_n^{(a)})(n > 2)$ be a generalized Tribonacci left circulant matrix. If $\lambda_j^n \neq 1 (j = 1, 2, 3)$, $q \neq p + r$ and $\sqrt{4pr - q^2}/q \neq \pm \tan \frac{2k\pi}{n}$ for any integer $k \in (\frac{n}{4}, \frac{3n}{4})$, then $\mathcal{L}_n$ is an invertible matrix.*

Theorem 3.3. *Let $\mathcal{L}_n = LCirc(T_1^{(a)}, T_2^{(a)}, \cdots, T_n^{(a)})(n > 4)$ be an invertible generalized Tribonacci left circulant matrix. If $\sigma_2 \neq 0$, then we have*

$$\begin{aligned}\mathcal{L}_n^{-1} &= LCirc\Bigg(x_2' - (u + \frac{\sigma_3}{\sigma_2})x_3' - vx_4' - wx_5',\ x_3' - ux_4' - vx_5' - wx_6', \cdots, \\ &\quad x_{n-3}' - ux_{n-2}' - vx_{n-1}' - wx_n',\ x_{n-2}' - ux_{n-1}' - vx_n',\ x_{n-1}' - ux_n', \\ &\quad x_n',\ -ux_2' + (-v + u\frac{\sigma_3}{\sigma_2})x_3' - wx_4'\Bigg),\end{aligned}$$

where σ_2, σ_3 and $x_j' (j = 2, 3, \cdots, n)$ are given by Theorem 2.4.

Proof. Since $\mathcal{L}_n^{-1} = \mathcal{R}_n^{-1}\mathcal{K}_n^{-1} = \mathcal{R}_n^{-1}\mathcal{K}_n$, the proof is trivial by Theorem 2.4. □

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