Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (3) (2019), 785 – 799
DOI : 10.15672/HJMS.2018.549

Research Article

Bihom-Nijienhuis operators and T^*-extensions of Bihom-Lie superalgebras

Juan Li, Liangyun Chen*, Bing Sun

School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

Abstract

The purpose of this article is to study Bihom-Nijienhuis operators and T^*-extensions of Bihom-Lie superalgebras. We show that the deformation generated by a Bihom-Nijienhuis operator is trivial. Moreover, we introduce the definition of T^*-extensions of Bihom-Lie superalgebras and show that T^*-extensions preserve many properties such as nilpotency, solvability and decomposition in some sense. In particular, we discuss the equivalence of T^*-extensions.

Mathematics Subject Classification (2010). 17A45, 17B30, 16S70

Keywords. Bihom-Lie superalgebras, Bihom-Nijienhuis operators, deformations, T^*-extensions

1. Introduction

The notion of Hom-Lie algebras was introduced by Hartwig, Larsson and Silvestrov to describe the structures on certain deformations of the Witt algebras and the Virasoro algebras [11]. Then, Hom-Lie algebras were generalized to Hom-Lie superalgebras by Ammar and Makhlouf [2, 4, 5]. Quadratic Hom-Lie algebras were studied first in [8]. In addition, (co)homology and deformations theory of Hom-algebras were studied in [1,3,13].

In 1997, Bordemann introduced the notion of T^*-extension of Lie algebras[6], which is a workable extensional technique since it is a one-step procedure: it is one of the main tools to prove that every symplectic quadratic Lie algebra is a special symplectic Manin algebra[7]. The Bihom-Nijienhuis operator and T^*-extension of Hom-Lie superalgebras was introduced by Liu in [12]. In addition, the representation and T^*-extension of Hom-Jordan-Lie algebras was introduced by Zhao in [15]

A Bihom-algebra is an algebra in such a way that the identities defining the structure are twisted by two homomorphisms α,β. This class of algebras was introduced from a categorical approach in [10] as an extension of the class of Hom-algebras. When the two linear maps are same automorphisms, Bihom-algebras will be return to Hom-algebras. These algebraic structures include Bihom-associative algebras, Bihom-Lie algebras and Bihom-bialgebras. The representation theory of Bihom-Lie algebras was introduced by Cheng in [9], in which, Bihom-cochain complexes, derivation, central extension, derivation

*Corresponding Author.
Email addresses: lij355@nenu.edu.cn (J. Li), chenly640@nenu.edu.cn (L. Chen), sunb427@nenu.edu.cn (B. Sun)
Received: 19.10.2017; Accepted: 12.01.2018

extension, trivial representation and adjoint representation of Bihom-Lie algebras were studied.

Recently, the definition of Bihom-Lie superalgebras were introduced in [14]. The notion of Bihom-Nijienhuis operators and T^*-extensions of Bihom-Lie superalgebras are not so well developed.

The paper is organized as follows. In Section 2 we give some definitions about Bihom-Lie superalgebras. In Section 3 we give the definition of Bihom-Nijienhuis operators of regular Bihom-Lie superalgebras. We show that the deformation generated by a Bihom-Nijienhuis operator is trivial. In Section 4 we study T^*-extensions of Bihom-Lie superalgebras. We show that T^*-extensions preserve many properties such as nilpotency, solvability and decomposition in some sense. In particular, we discuss the equivalence of T^*-extensions.

2. Preliminaries

Definition 2.1. [14] A *Bihom-Lie superalgebra* over a field $\mathbb{K}$ is a 4-tuple $(L,[\cdot,\cdot],\alpha,\beta)$, where L is a superspace, $\alpha\colon L\to L$ and $\beta\colon L\to L$ are even homomorphisms, $[\cdot,\cdot]\colon L\otimes L\to L$ is an even bilinear map, with notation $[\cdot,\cdot](a\otimes a')=[a,a']$, satisfying the following conditions, for all homogeneous elements $a,a',a''\in L$:

$$\alpha\circ\beta=\beta\circ\alpha, \tag{2.1}$$

$$[\beta(a),\alpha(a')]=-(-1)^{|a||a'|}[\beta(a'),\alpha(a)], \tag{2.2}$$

$$\circlearrowleft_{a,a',a''}(-1)^{|a||a''|}[\beta^2(a),[\beta(a'),\alpha(a'')]]=0. \tag{2.3}$$

Obviously, a Hom-Lie superalgebra $(L,[\cdot,\cdot],\alpha)$ is a particular case of a Bihom-Lie superalgebra, namely, $(L,[\cdot,\cdot],\alpha,\alpha)$. Conversely, a Bihom-Lie superalgebra $(L,[\cdot,\cdot],\alpha,\alpha)$ with isomorphism α is a Hom-Lie superalgebra $(L,[\cdot,\cdot],\alpha)$.

Definition 2.2.

(1) A Bihom-Lie superalgebra is called a *multiplicative* Bihom-Lie superalgebra if α and β are algebraic morphisms, i.e., for any $a',a''\in L$, we have

$$\alpha([a',a''])=[\alpha(a'),\alpha(a'')],\quad \beta([a',a''])=[\beta(a'),\beta(a'')]. \tag{2.4}$$

(2) A Bihom-Lie superalgebra $(L,[\cdot,\cdot]_L,\alpha,\beta)$ is regular if α and β are algebraic automorphisms.

(3) A sub-vector space $\eta\in L$ is a Bihom subalgebra of $(L,[\cdot,\cdot]_L,\alpha,\beta)$ if $\alpha(\eta)\in\eta$, $\beta(\eta)\in\eta$ and

$$[x,y]_L\in\eta,\quad \forall x,y\in\eta.$$

(4) A sub-vector space $\eta\in L$ is a Bihom ideal of $(L,[\cdot,\cdot]_L,\alpha,\beta)$ if $\alpha(\eta)\in\eta$, $\beta(\eta)\in\eta$ and

$$[x,y]_L\in\eta,\quad \forall x\in\eta, y\in L.$$

Definition 2.3. Let $(L,[\cdot,\cdot],\alpha,\beta)$ and $(L',[\cdot,\cdot]',\alpha',\beta')$ be two Bihom-Lie superalgebras. An even homomorphism $f:L\to L'$ is said to be a morphism of Bihom-Lie superalgebras if

$$f[u,v]=[f(u),f(v)]',\forall u,v\in L,$$

$$f\circ\alpha=\alpha'\circ f,$$

$$f\circ\beta=\beta'\circ f.$$

3. Bihom-Nijienhuis operators of Bihom-Lie superalgebras

Let $(L, [\cdot,\cdot], \alpha, \beta)$ be a regular Bihom-Lie superalgebra. We consider that L represents on itself via the bracket with respect to the morphisms α, β.

Definition 3.1. For any integer s, t, the $\alpha^s\beta^t$-adjoint representation of the regular Bihom-Lie superalgebra $(L, [\cdot,\cdot], \alpha, \beta)$, which we denote by $\mathrm{ad}_{s,t}$, is defined by

$$\mathrm{ad}_{s,t}(u)(v) = [\alpha^s\beta^t(u), v]_L, \forall u, v \in L.$$

Lemma 3.2. *With the above notations, we have*

$$ad_{s,t}(\alpha(u)) \circ \alpha = \alpha \circ ad_{s,t}(u),$$
$$ad_{s,t}(\beta(u)) \circ \beta = \beta \circ ad_{s,t}(u),$$
$$ad_{s,t}([\beta(u), v]) \circ \beta = ad_{s,t}(\alpha\beta(u)) \circ ad_{s,t}(v) - (-1)^{|u||v|} ad_{s,t}(\beta(v)) \circ ad_{s,t}(\alpha(u)).$$

Thus the definition of $\alpha^s\beta^t$-adjoint representation is well defined.

The set of k-cochains on L with values in M, which we denote by $C^k(L; M)$, is the set of k-linear homogenous maps from $L \times \cdots \times L$ (k-times) to M:

$$C^k(L; M) \triangleq \{f : \wedge^k L \to M \textit{ is a linear homogenous map}\},$$

where $f(u_1, \cdots, u_i, u_{i+1}, \cdots, u_n) = -(-1)^{|u_i||u_{i+1}|} f(u_1, \cdots, u_{i+1}, u_i, \cdots, u_n)$.

The set of k-Bihom-cochains on L with coefficients in L, which we denote by $C^k_{\alpha,\beta}(L; L)$, is given by

$$C^k_{\alpha,\beta}(L; L) = \{f \in C^k(L; L) | \alpha \circ f = f \circ \alpha, \beta \circ f = f \circ \beta\}.$$

In particular, the set of 0-Bihom-cochains are given by:

$$C^0_{\alpha,\beta}(L; L) = \{u \in L | \alpha(u) = u, \beta(u) = u\}.$$

Associated to the $\alpha^s\beta^t$-adjoint representation, the corresponding operator

$$d_{s,t} : C^k_{\alpha,\beta}(L; L) \to C^{k+1}_{\alpha,\beta}(L; L)$$

is given by

$$\begin{aligned} & d_{s,t} f(u_1, \cdots, u_{k+1}) \\ = \ & \sum_{i=1}^{k+1} (-1)^i (-1)^{(|f|+|u_1|+\cdots+|u_{i-1}|)|u_i|} [\alpha^{s+1}\beta^{t+k-1}(u_i), f(u_1, \cdots, \widehat{u_i}, \cdots, u_{k+1})] \\ & + \sum_{i<j} (-1)^{i+j+1} (-1)^{(|u_1|+\cdots+|u_{i-1}|)|u_i|} (-1)^{(|u_1|+\cdots+|u_{j-1}|)|u_j|} (-1)^{|u_i||u_j|} \\ & f([\alpha^{-1}\beta(u_i), u_j], \beta(u_1), \cdots, \widehat{u_i}, \cdots, \widehat{u_j}, \cdots, \beta(u_{k+1})). \end{aligned}$$

For the $\alpha^s\beta^t$-adjoint representation $\mathrm{ad}_{s,t}$, we obtain the $\alpha^s\beta^t$-adjoint complex $(C^k_{\alpha,\beta}(L; L), d_{s,t})$ and the corresponding cohomology

$$H^k(L; ad_{s,t}) = Z^k(L; ad_{s,t})/B^k(L; ad_{s,t}).$$

Let $\psi \in C^2_{\alpha,\beta}(L; L)$ be a bilinear operator commuting with α and β, also $\psi(u, v) = -(-1)^{|u||v|}\psi(v, u)$. Consider a t-parametrized family of bilinear operations

$$[u, v]_t = [u, v] + t\psi(u, v). \tag{3.1}$$

Since ψ commutes with α, β, α, β are a morphisms with respect to the bracket $[\cdot,\cdot]_t$ for every t. If all the brackets $[\cdot,\cdot]_t$ endow $(L, [\cdot,\cdot]_t, \alpha, \beta)$ with regular Bihom-Lie superalgebra structures, we say that ψ generates a deformation of the regular Bihom-Lie superalgebra $(L, [\cdot,\cdot], \alpha, \beta)$. The skew-symmetry of $[\cdot,\cdot]_t$ means that

$$\begin{aligned} [\beta(v), \alpha(u)]_t &= [\beta(v), \alpha(u)] + t\psi(\beta(v), \alpha(u)) \\ \text{and } [\beta(u), \alpha(v)]_t &= [\beta(u), \alpha(v)] + t\psi(\beta(u), \alpha(v)). \end{aligned}$$

Then $[\beta(v),\alpha(u)]_t=-(-1)^{|u||v|}[\beta(u),\alpha(v)]_t$ if and only if

$$\psi(\beta(v),\alpha(u))=-(-1)^{|u||v|}\psi(\beta(u),\alpha(v)). \tag{3.2}$$

By computing the Bihom-Jacobi identity of $[\cdot,\cdot]_t$

$$\begin{aligned}
&\circlearrowleft_{u,v,w}(-1)^{|u||w|}[\beta^2(u),[\beta(v),\alpha(w)]_t]_t\\
=&\circlearrowleft_{u,v,w}(-1)^{|u||w|}[\beta^2(u),[\beta(v),\alpha(w)]_L+t\psi(\beta(v),\alpha(w))]_t\\
=&\circlearrowleft_{u,v,w}(-1)^{|u||w|}([\beta^2(u),[\beta(v),\alpha(w)]]_t+[\beta^2(u),t\psi(\beta(v),\alpha(w))]_t)\\
=&\circlearrowleft_{u,v,w}(-1)^{|u||w|}([\beta^2(u),[\beta(v),\alpha(w)]]+t\psi(\beta^2(u),[\beta(v),\alpha(w)])\\
&+[\beta^2(u),t\psi(\beta(v),\alpha(w))]+t\psi(\beta^2(u),t\psi(\beta(v),\alpha(w)))),
\end{aligned}$$

this is equivalent to the conditions

$$\circlearrowleft_{u,v,w}(-1)^{|u||w|}\psi(\beta^2(u),\psi(\beta(v),\alpha(w)))=0, \tag{3.3}$$

$$\circlearrowleft_{u,v,w}(-1)^{|u||w|}(\psi(\beta^2(u),[\beta(v),\alpha(w)])+[\beta^2(u),\psi(\beta(v),\alpha(w))])=0. \tag{3.4}$$

Obviously, (3.2) and (3.3) mean that ψ must itself define a Bihom-Lie superalgebra structure on L.

A deformation is said to be trivial if there is a linear operator $N\in C^1_{\alpha,\beta}(L;L)$ such that $T_t=\mathrm{id}+tN$ and

$$T_t[u,v]_t=[T_t(u),T_t(v)]. \tag{3.5}$$

Definition 3.3. A linear operator $N\in C^1_{\alpha,\beta}(L,L)$ is called a Bihom-Nijienhuis operator if we have

$$[Nu,Nv]=N[u,v]_N, \tag{3.6}$$

where the bracket $[\cdot,\cdot]_N$ is defined by

$$[u,v]_N\triangleq[Nu,v]+[u,Nv]-N[u,v]. \tag{3.7}$$

Theorem 3.4. *Let $N\in C^1_{\alpha,\beta}(L,L)_0$ be a Bihom-Nijienhuis operator. Then a deformation of the regular Bihom-Lie superalgebra $(L,[\cdot,\cdot],\alpha,\beta)$ can be obtained by putting*

$$\psi(u,v)=[u,v]_N. \tag{3.8}$$

Furthermore, this deformation is trivial.

Proof. To see that ψ generates a deformation, we need to check ψ satisfying (3.2), (3.3) and (3.4). First we can obtain

$$\begin{aligned}
\psi(\beta(u),\alpha(v))&=[\beta(u),\alpha(v)]_N\\
&=[N\beta(u),\alpha(v)]+[\beta(u),N\alpha(v)]-N[\beta(u),\alpha(v)]\\
&=-(-1)^{|u||v|}([N\beta(v),\alpha(u)]+[\beta(v),N\alpha(u)]-N[\beta(v),\alpha(u)])\\
&=-(-1)^{|u||v|}[\beta(v),\alpha(u)]_N\\
&=-(-1)^{|u||v|}\psi(\beta(v),\alpha(u)).
\end{aligned}$$

Next by (3.6),(3.7) and (3.8), we have

$$
\begin{aligned}
& \circlearrowleft_{u,v,w} (-1)^{|u||w|}\psi(\beta^2(u),\psi(\beta(v),\alpha(w))) \\
= \; & \circlearrowleft_{u,v,w} (-1)^{|u||w|}\psi(\beta^2(u),[N\beta(v),\alpha(w)]+[\beta(v),N\alpha(u)]-N[\beta(v),\alpha(w)]) \\
= \; & \circlearrowleft_{u,v,w} (-1)^{|u||w|}(\psi(\beta^2(u),[N\beta(v),\alpha(w)])+\psi(\beta^2(u),[\beta(v),N\alpha(u)]) \\
& -\psi(\beta^2(u),N[\beta(v),\alpha(w)])) \\
= \; & \circlearrowleft_{u,v,w} (-1)^{|u||w|}([N\beta^2(u),[N\beta(v),\alpha(w)]]+[N\beta^2(u),[\beta(v),N\alpha(w)]] \\
& -[N\beta^2(u),N[\beta(v),\alpha(w)]]-N[\beta^2(u),[N\beta(v),\alpha(w)]]-N[\beta^2(u),[\beta(v),N\alpha(w)]] \\
& +N[\beta^2(u),N[\beta(v),\alpha(w)]]+[\beta^2(u),[N\beta(v),N\alpha(w)]]) \\
= \; & (-1)^{|u||w|}(\underbrace{[N\beta^2(u),[N\beta(v),\alpha(w)]]}_{(1)}+\underbrace{[N\beta^2(u),[\beta(v),N\alpha(w)]]}_{(2)} \\
& -[N\beta^2(u),N[\beta(v),\alpha(w)]]\underbrace{-N[\beta^2(u),[N\beta(v),\alpha(w)]]}_{(3)}\underbrace{-N[\beta^2(u),[\beta(v),N\alpha(w)]]}_{(4)} \\
& +N[\beta^2(u),N[\beta(v),\alpha(w)]]+\underbrace{[\beta^2(u),[N\beta(v),N\alpha(w)]])}_{(5)} \\
& (-1)^{|u||v|}(\underbrace{[N\beta^2(v),[N\beta(w),\alpha(u)]]}_{(5')}+\underbrace{[N\beta^2(v),[\beta(w),N\alpha(u)]]}_{(1')} \\
& -[N\beta^2(v),N[\beta(w),\alpha(u)]]\underbrace{-N[\beta^2(v),[N\beta(w),\alpha(u)]]}_{(4')}\underbrace{-N[\beta^2(v),[\beta(w),N\alpha(u)]]}_{(6')} \\
& +N[\beta^2(v),N[\beta(w),\alpha(u)]]+\underbrace{[\beta^2(v),[N\beta(w),N\alpha(u)]])}_{(2')} \\
& (-1)^{|v||w|}(\underbrace{[N\beta^2(w),[N\beta(u),\alpha(v)]]}_{(2'')}+\underbrace{[N\beta^2(w),[\beta(u),N\alpha(v)]]}_{(5'')} \\
& -[N\beta^2(w),N[\beta(u),\alpha(v)]]\underbrace{-N[\beta^2(w),[N\beta(u),\alpha(v)]]}_{(6'')}\underbrace{-N[\beta^2(w),[\beta(u),N\alpha(v)]]}_{(3'')} \\
& +N[\beta^2(w),N[\beta(u),\alpha(v)]]+\underbrace{[\beta^2(w),[N\beta(u),N\alpha(v)]])}_{(1'')}.
\end{aligned}
$$

Since N is a Bihom-Nijenhuis operator, we get

$$
\begin{aligned}
& -[N\beta^2(u),N[\beta(v),\alpha(w)]]+N[\beta^2(u),N[\beta(v),\alpha(w)]] \\
= \; & \underbrace{N^2[\beta^2(u),[\beta(v),\alpha(w)]]}_{(7)}\underbrace{-N[N\beta^2(u),[\beta(v),\alpha(w)]]}_{(6)},
\end{aligned}
$$

$$
\begin{aligned}
& -[N\beta^2(v),N[\beta(w),\alpha(u)]]+N[\beta^2(v),N[\beta(w),\alpha(u)]] \\
= \; & \underbrace{N^2[\beta^2(v),[\beta(w),\alpha(u)]]}_{(7')}\underbrace{-N[N\beta^2(v),[\beta(w),\alpha(u)]]}_{(3')}
\end{aligned}
$$

and

$$
\begin{aligned}
& -[N\beta^2(w),N[\beta(u),\alpha(v)]]+N[\beta^2(w),N[\beta(u),\alpha(v)]] \\
= \; & \underbrace{N^2[\beta^2(w),[\beta(u),\alpha(v)]]}_{(7'')}\underbrace{-N[N\beta^2(w),[\beta(u),\alpha(v)]]}_{(4'')}.
\end{aligned}
$$

We obtain $(i)+(i)^{'}+(i)^{''}=0$, for $i=1,\cdots,7$.

Furthermore, we have

$$\begin{aligned}
&\circlearrowleft_{u,v,w}(-1)^{|u||w|}(\psi(\beta^2(u),[\beta(v),\alpha(w)])+[\beta^2(u),\psi(\beta(v),\alpha(w))])\\
=&\circlearrowleft_{u,v,w}(-1)^{|u||w|}([\beta^2(u),[\beta(v),\alpha(w)]]_N+[\beta^2(u),[\beta(v),\alpha(w)]_N])\\
=&\circlearrowleft_{u,v,w}(-1)^{|u||w|}([N\beta^2(u),[\beta(v),\alpha(w)]]+[\beta^2(u),N[\beta(v),\alpha(w)]]-N[\beta^2(u),[\beta(v),\alpha(w)]]\\
&+[\beta^2(u),[N\beta(v),\alpha(w)]]+[\beta^2(u),[\beta(v),N\alpha(w)]]-[\beta^2(u),N[\beta(v),\alpha(w)]])\\
=&\circlearrowleft_{u,v,w}(-1)^{|u||w|}([N\beta^2(u),[\beta(v),\alpha(w)]]-N[\beta^2(u),[\beta(v),\alpha(w)]]\\
&+[\beta^2(u),[N\beta(v),\alpha(w)]]+[\beta^2(u),[\beta(v),N\alpha(w)]])\\
=&(-1)^{|u||w|}([N\beta^2(u),[\beta(v),\alpha(w)]]-N[\beta^2(u),[\beta(v),\alpha(w)]]\\
&+[\beta^2(u),[N\beta(v),\alpha(w)]]+[\beta^2(u),[\beta(v),N\alpha(w)]])\\
&(-1)^{|u||v|}([N\beta^2(v),[\beta(w),\alpha(u)]]-N[\beta^2(v),[\beta(w),\alpha(u)]]\\
&+[\beta^2(v),[N\beta(w),\alpha(u)]]+[\beta^2(v),[\beta(w),N\alpha(u)]])\\
&(-1)^{|v||w|}([N\beta^2(w),[\beta(u),\alpha(v)]]-N[\beta^2(w),[\beta(u),\alpha(v)]]\\
&+[\beta^2(w),[N\beta(u),\alpha(v)]]+[\beta^2(w),[\beta(u),N\alpha(v)]]).
\end{aligned}$$

This proves that ψ generates a deformation of the regular Bihom-Lie superalgebra $(L,[\cdot,\cdot],\alpha,\beta)$.

Let $T_t=\mathrm{id}+tN$, then

$$\begin{aligned}
T_t[u,v]_t &= (\mathrm{id}+tN)([u,v]+t\psi(u,v))\\
&= (\mathrm{id}+tN)([u,v]+t[u,v]_N)\\
&= [u,v]+t([u,v]_N+N[u,v])+t^2N[u,v]_N.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[T_t(u),T_t(v)] &= [u+tNu,v+tNv]\\
&= [u,v]+t([Nu,v]+[u,Nv])+t^2[Nu,Nv].
\end{aligned}$$

By the equations (3.6) and (3.7), we have

$$T_t[u,v]_t=[T_t(u),T_t(v)]_L,$$

which implies that the deformation is trivial. □

4. T^*-extensions of Bihom-Lie superalgebras

We provide in this section, for Bihom-Lie superalgebras, characterizations of T^*-extensions and observations about T^*-extensions of nilpotent and solvable Bihom-Lie superalgebras. This method was introduced by Martin Bordemann in [6].

Definition 4.1. Let $(L,[\cdot,\cdot],\alpha,\beta)$ be a Bihom-Lie superalgebra. A bilinear form f on L is said to be nondegenerate if

$$L^{\perp}=\{x\in L|f(x,y)=0,\forall y\in L\}=0;$$

superconsistent if

$$f(x,y)=0,\forall x\in L_{|x|},y\in L_{|y|},|x|+|y|\neq 0;$$

$\alpha\beta$-invariant if

$$f([\beta(x),\alpha(y)],\alpha(z))=f(\alpha(x),[\beta(y),\alpha(z)]),\forall x,y,z\in L;$$

supersymmetric if

$$f(x,y)=(-1)^{|x||y|}f(y,x).$$

A subspace I of L is called isotropic if $I\subseteq I^{\perp}$.

Throughout this section, we only consider superconsistent bilinear forms.

Definition 4.2. Let $(L,[\cdot,\cdot],\alpha,\beta)$ be a Bihom-Lie superalgebra over a field $\mathbb{K}$. If L admits a nondegenerate, $\alpha\beta$-invariant, and supersymmetric bilinear form f such that α, β are f-symmetric (i.e. $f(\alpha(x),y)=f(x,\alpha(y)), f(\beta(x),y)=f(x,\beta(y))$), then we call (L,f,α,β) a quadratic Bihom-Lie superalgebra. In particular, a quadratic $\mathbb{Z}_2$-graded vector space V is a $\mathbb{Z}_2$-graded vector space admitting a nondegenerate supersymmetric bilinear form.

Let $(L^{'},[\cdot,\cdot]',\alpha_1,\beta_1)$ be another Bihom-Lie superalgebra. Two quadratic Bihom-Lie superalgebras (L,f,α,β) and $(L^{'},f',\alpha_1,\beta_1)$ are said to be isometric if there exists a Bihom-Lie superalgebra isomorphism $\phi:L\to L^{'}$ such that $f(x,y)=f'(\phi(x),\phi(y)),\forall x,y\in L$.

Theorem 4.3. *Let $(L,[\cdot,\cdot],\alpha,\beta)$ be a Bihom-Lie sueralgebra, and $(M,\rho,\alpha_M,\beta_M)$ be a representation of of L. Let us consider M^* the dual space of M and $\tilde{\alpha},\tilde{\beta}:M^*\to M^*$ two even homomorphism defined by $\tilde{\alpha}(f)=f\circ\alpha,\tilde{\beta}(f)=f\circ\beta,\forall f\in L^*$. Then, the even linear map $\tilde{\rho}:L\to \mathrm{End}(M^*)$ defined by $\tilde{\rho}(x)(f)=-(-1)^{|f||x|}f\circ\rho(x),\forall f\in M^*,\forall x\in L$, is a representation of L on $(M^*,\tilde{\alpha},\tilde{\beta})$ if and only if*

$$\alpha\circ\rho(\alpha(x))=\rho(x)\circ\alpha;$$

$$\beta\circ\rho(\beta(x))=\rho(x)\circ\beta;$$

$$\rho(\alpha(x))\circ\rho(\beta(y))-(-1)^{|x||y|}\rho(y)\circ\rho(\alpha\beta(x))=\beta\circ\rho[\beta(x),y].$$

Proof. Let $f\in M^*$, $x,y\in L$. Firstly, we have

$$(\tilde{\rho}(\alpha(x))\circ\tilde{\alpha})(f)=-(-1)^{|f||x|}\tilde{\alpha}(f)\circ\rho(\alpha(x))=-(-1)^{|f||x|}f\circ\alpha\circ\rho(\alpha(x)),$$

$$\tilde{\alpha}(\tilde{\rho}(x))(f)=-(-1)^{|f||x|}\tilde{\alpha}(f\circ\rho(x))=-(-1)^{|f||x|}f\circ\rho(x)\circ\alpha.$$

Similarly,

$$(\tilde{\rho}(\beta(x))\circ\tilde{\beta})(f)=-(-1)^{|f||x|}\tilde{\beta}(f)\circ\rho(\beta(x))=-(-1)^{|f||x|}f\circ\beta\circ\rho(\beta(x)),$$

$$\tilde{\beta}(\tilde{\rho}(x))(f)=-(-1)^{|f||x|}\tilde{\beta}(f\circ\rho(x))=-(-1)^{|f||x|}f\circ\rho(x)\circ\beta.$$

Therefore, we have

$$(\tilde{\rho}([\beta(x),y])\circ\tilde{\beta})(f)=-(-1)^{|f|(|x|+|y|)}\tilde{\beta}(f)\circ\rho[\beta(x),y]=-(-1)^{|f|(|x|+|y|)}f\circ\beta\circ\rho[\beta(x),y]$$

and

$$\begin{aligned}
&(\tilde{\rho}(\alpha\beta(x))\circ\tilde{\rho}(y)-(-1)^{|x||y|}\tilde{\rho}(\beta(y))\circ\tilde{\rho}(\alpha(x)))(f)\\
=\;&-(-1)^{|x|(|f|+|y|)}\tilde{\rho}(y)(f)\circ\rho(\alpha\beta(x))+(-1)^{|f||y|}\tilde{\rho}(\alpha(x))(f)\circ\rho(\beta(y))\\
=\;&(-1)^{|f|(|x|+|y|)+|x||y|}f\circ\rho(y)\circ\rho(\alpha\beta(x))-(-1)^{|f|(|x|+|y|)}f\circ\rho(\alpha(x))\circ\rho(\beta(y))\\
=\;&-(-1)^{|f|(|x|+|y|)}f\circ(\rho(\alpha(x))\circ\rho(\beta(y))-(-1)^{|x||y|}\rho(y)\circ\rho(\alpha\beta(x))).
\end{aligned}$$

Then $\tilde{\rho}$ is a representation of L on $(M^*,\tilde{\alpha},\tilde{\beta})$. □

Corollary 4.4. *Let* ad *be the adjoint representation of a Bihom-Lie superalgebra $(L,[\cdot,\cdot],\alpha,\beta)$. Let us consider the even linear map $\pi:L\to\mathrm{End}(L^*)$ defined by*

$$\pi(x)(f)(y)=-(-1)^{|f||x|}f\circ\mathrm{ad}(x)(y),\forall x,y\in L.$$

Then π is a representation of L on $(L^*,\tilde{\alpha},\tilde{\beta})$ if and only if

$$\alpha\circ\mathrm{ad}\alpha(x)=\mathrm{ad}x\circ\alpha,\tag{4.1}$$

$$\beta\circ\mathrm{ad}\beta(x)=\mathrm{ad}x\circ\beta,\tag{4.2}$$

$$\mathrm{ad}(\alpha(x))\circ\mathrm{ad}\beta(y)-(-1)^{|x||y|}\mathrm{ad}y\circ\mathrm{ad}(\alpha\beta(x))=\beta\circ\mathrm{ad}[\beta(x),y].\tag{4.3}$$

We call the representation π the coadjoint representation of L.

Lemma 4.5. *Under the above notations, let $(L, [\cdot,\cdot], \alpha, \beta)$ be a Bihom-Lie superalgebra, and $\omega : L \times L \to L^*$ be an even bilinear map. Assume that the coadjoint representation exists and α, β are bijective. The $\mathbb{Z}_2$-graded vector space $L \oplus L^*$, provided with the following bracket and two even linear maps defined respectively by*

$$[x+f, y+g]_{L\oplus L^*} = [x,y] + \omega(x,y) + \pi(x)g - (-1)^{|x||y|}\pi(\alpha^{-1}\beta(y)\tilde{\alpha}\tilde{\beta}^{-1}(f), \tag{4.4}$$

$$\alpha'(x+f) = \alpha(x) + f\circ\alpha, \tag{4.5}$$

$$\beta'(x+f) = \beta(x) + f\circ\beta. \tag{4.6}$$

Then $(L\oplus L^*, [\cdot,\cdot]_{L\oplus L^*}, \alpha', \beta')$ is a Bihom-Lie superalgebra if and only if

$$w(\beta(x), \alpha(y)) = -(-1)^{|x||y|} w(\beta(y), \alpha(x)), \tag{4.7}$$

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|}\big(w(\beta^2(x), [\beta(y), \alpha(z)]) + \pi(\beta^2(x)) w(\beta(y), \alpha(z))\big) = 0. \tag{4.8}$$

Proof. For any elements $x+f, y+g, z+h \in L\oplus L^*$. First we have

$$\begin{aligned}\alpha'\circ\beta'(x+f) &= \alpha'(\beta(x)+f\circ\beta) = \alpha\beta(x) + f\circ\beta\circ\alpha \\ &= \beta\alpha(x) + f\circ\alpha\circ\beta = \beta'\circ\alpha'(x+f).\end{aligned}$$

Next we show that

$$\begin{aligned}&[\beta'(x+f), \alpha'(y+g)]_{L\oplus L^*} \\ =\ & [\beta(x)+f\circ\beta, \alpha(y)+g\circ\alpha]_{L\oplus L^*} \\ =\ & [\beta(x),\alpha(y)] + w(\beta(x),\alpha(y)) + \pi(\beta(x))(g\circ\alpha) - (-1)^{|x||y|}\pi(\alpha^{-1}\beta(\alpha(y)))\tilde{\alpha}\tilde{\beta}^{-1}(f\circ\beta) \\ =\ & [\beta(x),\alpha(y)] + w(\beta(x),\alpha(y)) + \pi(\beta(x))(g\circ\alpha) - (-1)^{|x||y|}\pi(\beta(y))(f\circ\alpha).\end{aligned}$$

In the same way, we have

$$\begin{aligned}[\beta'(y+g), \alpha'(x+f)]_{L\oplus L^*} &= [\beta(y),\alpha(x)] + w(\beta(y),\alpha(x)) + \pi(\beta(y))(f\circ\alpha) \\ &\quad -(-1)^{|x||y|}\pi(\beta(x))(g\circ\alpha).\end{aligned}$$

Then, we obtain $[\beta'(x+f), \alpha'(y+g)]_{L\oplus L^*} = -(-1)^{|x||y|}[\beta'(y+g), \alpha'(x+f)]_{L\oplus L^*}$ if and only if

$$w(\beta(x),\alpha(y)) = -(-1)^{|x||y|} w(\beta(y),\alpha(x)).$$

Therefore,

$$\begin{aligned}&(-1)^{|x||z|}[\beta'^2(x+f), [\beta'(y+g), \alpha'(z+h)]_{L\oplus L^*}]_{L\oplus L^*} \\ =\ & (-1)^{|x||z|}[\beta^2(x)+f\circ\beta^2, [\beta(y)+g\circ\beta, \alpha(z)+h\circ\alpha]_{L\oplus L^*}]_{L\oplus L^*} \\ =\ & (-1)^{|x||z|}[\beta^2(x)+f\circ\beta^2, [\beta(y),\alpha(z)] + w(\beta(y),\alpha(z)) \\ & +\pi(\beta(y))(h\circ\alpha) - (-1)^{|y||z|}\pi(\alpha^{-1}\beta(\alpha(z)))(\tilde{\alpha}\tilde{\beta}^{-1}(g\circ\beta))]_{L\oplus L^*} \\ =\ & (-1)^{|x||z|}[\beta^2(x)+f\circ\beta^2, [\beta(y),\alpha(z)] + w(\beta(y),\alpha(z)) \\ & +\pi(\beta(y))(h\circ\alpha) - (-1)^{|y||z|}\pi(\beta(z))(g\circ\alpha)]_{L\oplus L^*} \\ =\ & (-1)^{|x||z|}[\beta^2(x), [\beta(y),\alpha(z)]] + (-1)^{|x||z|}w(\beta^2(x), [\beta(y),\alpha(z)]) \\ & +(-1)^{|x||z|}\pi(\beta^2(x))w(\beta(y),\alpha(z)) + (-1)^{|x||z|}\pi(\beta^2(x))\pi(\beta(y))(h\circ\alpha) \\ & -(-1)^{|z|(|x|+|y|)}\pi(\beta^2(x))\pi(\beta(z))(g\circ\alpha) - (-1)^{|x||y|}\pi(\alpha^{-1}\beta[\beta(y),\alpha(z)])(f\circ\beta\circ\alpha).\end{aligned}$$

And

$$\begin{aligned}&(-1)^{|x||y|}[\beta'^2(y+g), [\beta'(z+h), \alpha'(x+f)]_{L\oplus L^*}]_{L\oplus L^*} \\ =\ & (-1)^{|x||y|}[\beta^2(y), [\beta(z),\alpha(x)]] + (-1)^{|x||y|}w(\beta^2(y), [\beta(z),\alpha(x)]) \\ & +(-1)^{|x||y|}\pi(\beta^2(y))w(\beta(z),\alpha(x)) + (-1)^{|x||y|}\pi(\beta^2(y))\pi(\beta(z))(f\circ\alpha) \\ & -(-1)^{|x|(|z|+|y|)}\pi(\beta^2(y))\pi(\beta(x))(h\circ\alpha) - (-1)^{|z||y|}\pi(\alpha^{-1}\beta[\beta(z),\alpha(x)])(g\circ\beta\circ\alpha),\end{aligned}$$

$$
\begin{aligned}
& (-1)^{|z||y|}[\beta'^2(z+h),[\beta'(x+f),\alpha'(y+g)]_{L\oplus L^*}]_{L\oplus L^*} \\
= \; & (-1)^{|z||y|}[\beta^2(z),[\beta(x),\alpha(y)]] + (-1)^{|z||y|}w(\beta^2(z),[\beta(x),\alpha(y)]) \\
& +(-1)^{|z||y|}\pi(\beta^2(z))w(\beta(x),\alpha(y)) + (-1)^{|z||y|}\pi(\beta^2(z))\pi(\beta(x))(g\circ\alpha) \\
& -(-1)^{|y|(|z|+|x|)}\pi(\beta^2(z))\pi(\beta(y))(f\circ\alpha) - (-1)^{|x||z|}\pi(\alpha^{-1}\beta[\beta(x),\alpha(y)])(h\circ\beta\circ\alpha).
\end{aligned}
$$

Since π is the coadjoint representation of L, we have

$$
\begin{aligned}
& -(-1)^{|x||z|}\pi(\alpha^{-1}\beta[\beta(x),\alpha(y)])(h\circ\beta\circ\alpha) \\
= \; & -(-1)^{|x||z|}\pi([\beta(\alpha^{-1}\beta(x)),\beta(y)])\circ\tilde{\beta}(h\circ\alpha) \\
= \; & -(-1)^{|x||z|}\pi(\alpha\beta(\alpha^{-1}\beta(x)))\pi(\beta(y))(h\circ\alpha) \\
& +(-1)^{|x||z|+|x||y|}\pi(\beta(\beta(y)))\pi(\alpha(\alpha^{-1}\beta(x)))(h\circ\alpha) \\
= \; & -(-1)^{|x||z|}\pi(\beta^2(x))\pi(\beta(y))(h\circ\alpha) + (-1)^{|x|(|z|+|y|)}\pi(\beta^2(y)))\pi(\beta(x))(h\circ\alpha).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& -(-1)^{|x||y|}\pi(\alpha^{-1}\beta[\beta(y),\alpha(z)])(f\circ\beta\circ\alpha) \\
= \; & -(-1)^{|x||y|}\pi(\beta^2(y))\pi(\beta(z))(f\circ\alpha) + (-1)^{|y|(|z|+|x|)}\pi(\beta^2(z)))\pi(\beta(y))(f\circ\alpha)
\end{aligned}
$$

and

$$
\begin{aligned}
& -(-1)^{|z||y|}\pi(\alpha^{-1}\beta[\beta(z),\alpha(x)])(g\circ\beta\circ\alpha) \\
= \; & -(-1)^{|z||y|}\pi(\beta^2(z))\pi(\beta(x))(g\circ\alpha) + (-1)^{|z|(|y|+|x|)}\pi(\beta^2(x)))\pi(\beta(z))(g\circ\alpha).
\end{aligned}
$$

Consequently,

$$
\circlearrowleft_{x+f,y+g,z+h} (-1)^{|x||z|}[\beta'^2(x+f),[\beta'(y+g),\alpha'(z+h)]_{L\oplus L^*}]_{L\oplus L^*} = 0
$$

if and only if

$$
\circlearrowleft_{x,y,z} (-1)^{|x||z|}(w(\beta^2(x),[\beta(y),\alpha(z)]) + \pi(\beta^2(x))w(\beta(y),\alpha(z))) = 0.
$$

Hence the lemma follows. □

Clearly, L^* is an abelian Bihom-ideal of $(L\oplus L^*,[\cdot,\cdot]_{L\oplus L^*},\alpha',\beta')$ and L is isomorphic to the factor Bihom-Lie superalgebra $(L\oplus L^*)/L^*$. Moreover, consider the following supersymmetric bilinear form q_L on $L\oplus L^*$ for all $x+f, y+g \in L\oplus L^*$,

$$
q_L(x+f,y+g) = f(y) + (-1)^{|x||y|}g(x).
$$

Then we have the following lemma.

Lemma 4.6. *Let L, L^*, ω and q_L be as above. Then the 4-uplet $(L\oplus L^*, q_L, \alpha', \beta')$ is a quadratic Bihom-Lie superalgebra if and only if ω is supercyclic in the following sense:*

$$
\omega(\beta(x),\alpha(y))(\alpha(z)) = (-1)^{|x|(|z|+|y|)}\omega(\beta(y),\alpha(z))(\alpha(x)), \text{ for all } x,y,z\in L. \tag{4.9}
$$

Proof. If $x+f$ is orthogonal to all elements $y+g$ of $L\oplus L^*$, then $f(y)=0$ and $g(x)=0$, which implies that $x=0$ and $f=0$. So the supersymmetric bilinear form q_L is nondegenerate.

Now suppose that $x+f, y+g, z+h \in L\oplus L^*$, we have

$$
\begin{aligned}
q_L(\alpha'(x+f),y+g) &= q_L(\alpha(x)+f\circ\alpha, y+g) \\
&= f\circ\alpha(y) + (-1)^{|x||y|}g(\alpha(x)) \\
&= f(\alpha(x)) + (-1)^{|x||y|}g\circ\alpha(x) \\
&= q_L(x+f,\alpha'(y+g)),
\end{aligned}
$$

Then $\alpha^{'}$ is q_L-symmetric. In the same way, $\beta^{'}$ is q_L-symmetric. On the one hand,

$$\begin{aligned}
& q_L([\beta^{'}(x+f), \alpha^{'}(y+g)]_{L\oplus L^*}, \alpha^{'}(z+h)) \\
= & q_L([\beta(x)+f\circ\beta, \alpha(y)+g\circ\alpha]_{L\oplus L^*}, \alpha(z)+h\circ\alpha) \\
= & q_L([\beta(x),\alpha(y)]+\omega(\beta(x),\alpha(y))+\pi(\beta(x))g\circ\alpha \\
& -(-1)^{|x||y|}\pi(\alpha^{-1}\beta\alpha(y))\tilde{\alpha}\tilde{\beta}^{-1}(f\circ\beta), \alpha(z)+h\circ\alpha) \\
= & q_L([\beta(x),\alpha(y)]+\omega(\beta(x),\alpha(y))+\pi(\beta(x))g\circ\alpha \\
& -(-1)^{|x||y|}\pi(\beta(y))(f\circ\alpha), \alpha(z)+h\circ\alpha) \\
= & \omega(\beta(x),\alpha(y))(\alpha(z))+\pi(\beta(x))(g\circ\alpha)(\alpha(z))-(-1)^{|x||y|}\pi(\beta(y))(f\circ\alpha)(\alpha(z)) \\
& +(-1)^{|z|(|x|+|y|)}h\circ\alpha([\beta(x),\alpha(y)]) \\
= & \omega(\beta(x),\alpha(y))(\alpha(z))-(-1)^{|x||y|}g\circ\alpha([\beta(x),\alpha(z)])+f\circ\alpha([\beta(y),\alpha(z)]) \\
& +(-1)^{|z|(|x|+|y|)}h\circ\alpha([\beta(x),\alpha(y)]) \\
= & \omega(\beta(x),\alpha(y))(\alpha(z))+(-1)^{|x|(|z|+|y|)}g\circ\alpha([\beta(z),\alpha(x)])+f\circ\alpha([\beta(y),\alpha(z)]) \\
& -(-1)^{|z|(|x|+|y|)+|x||y|}h\circ\alpha([\beta(y),\alpha(x)]).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& q_L(\alpha^{'}(x+f), [\beta^{'}(y+g), \alpha^{'}(z+h)]_{L\oplus L^*}) \\
= & q_L(\alpha(x)+f\circ\alpha, [\beta(y)+g\circ\beta, \alpha(z)+h\circ\alpha]_{L\oplus L^*}) \\
= & q_L(\alpha(x)+f\circ\alpha, [\beta(y),\alpha(z)]+\omega(\beta(y),\alpha(z))+\pi(\beta(y))h\circ\alpha \\
& -(-1)^{|z||y|}\pi(\alpha^{-1}\beta\alpha(z))\tilde{\alpha}\tilde{\beta}^{-1}(g\circ\beta)) \\
= & q_L(\alpha(x)+f\circ\alpha, [\beta(y),\alpha(z)]+\omega(\beta(y),\alpha(z))+\pi(\beta(y))h\circ\alpha \\
& -(-1)^{|z||y|}\pi(\beta(z))(g\circ\alpha)) \\
= & f\circ\alpha([\beta(y),\alpha(z)])+(-1)^{|x|(|z|+|y|)}\omega(\beta(y),\alpha(z))(\alpha(x)) \\
& +(-1)^{|x|(|z|+|y|)}\pi(\beta(y))h\circ\alpha(\alpha(x))-(-1)^{|x|(|z|+|y|)+|z||y|}\pi(\beta(z))(g\circ\alpha))(\alpha(x)) \\
= & (-1)^{|x|(|z|+|y|)}\omega(\beta(y),\alpha(z))(\alpha(x))+(-1)^{|x|(|z|+|y|)}g\circ\alpha([\beta(z),\alpha(x)]) \\
& +f\circ\alpha([\beta(y).\alpha(z)])-(-1)^{|z|(|x|+|y|)+|x||y|}h\circ\alpha([\beta(y),\alpha(x)]).
\end{aligned}$$

Hence the lemma follows. □

Now, for a supercyclic ω , which satisfies (4.7) and (4.8), we shall call the quadratic Bihom-Lie superalgebra $(L\oplus L^*, q_L, \alpha^{'}, \beta^{'})$ the T^*-extension of L (by ω) and denote the Bihom-Lie superalgebra $(L\oplus L^*, [\cdot,\cdot]_{L\oplus L^*}, \alpha^{'}, \beta^{'})$ by $T^*_\omega L$.

Definition 4.7. Let L be a Bihom-Lie superalgebra over a field $\mathbb{K}$. We inductively define a derived series

$$(L^{(n)})_{n\geq 0}: L^{(0)}=L,\ L^{(n+1)}=[L^{(n)}, L^{(n)}],$$

and a central descending series

$$(L^n)_{n\geq 0}: L^0=L,\ L^{n+1}=[L^n, L].$$

L is called solvable and nilpotent(of length k) if and only if there is a (smallest) integer k such that $L^{(k)}=0$ and $L^k=0$, respectively.

In the following theorem we discuss some properties of $T^*_\omega L$.

Theorem 4.8. *Let $(L, [\cdot,\cdot], \alpha, \beta)$ be a regular Bihom-Lie superalgebra over a field $\mathbb{K}$.*

(1) *If L is solvable (nilpotent) of length k, then the T^*-extension $T^*_\omega L$ is solvable (nilpotent) of length r, where $k\leq r\leq k+1$ $(k\leq r\leq 2k-1)$.*
(2) *If L is decomposed into a direct sum of two Bihom-ideals of L, so is the trivial T^*-extension $T^*_0 L$.*

Proof. (1) Firstly we suppose that L is solvable of length k. Since $(T^*_\omega L)^{(n)}/L^* \cong L^{(n)}$ and $L^{(k)} = 0$, we have $(T^*_\omega L)^{(k)} \subseteq L^*$, which implies $(T^*_\omega L)^{(k+1)} = 0$ because L^* is abelian, and it follows that $T^*_\omega L$ is solvable of length k or $k+1$.

Suppose now that L is nilpotent of length k. Since $(T^*_\omega L)^n/L^* \cong L^n$ and $L^k = 0$, we have $(T^*_\omega L)^k \subseteq L^*$. Let $g \in (T^*_\omega L)^k \subseteq L^*, b \in L$, $x_1 + f_1, \cdots, x_{k-1} + f_{k-1} \in T^*_\omega L$, $1 \leq i \leq k-1$, we have

$$\begin{aligned}
& [[\cdots[g, x_1 + f_1]_{L\oplus L^*}, \cdots]_{L\oplus L^*}, x_{k-1} + f_{k-1}]_{L\oplus L^*}(b) \\
= \ & (-1)^{|x_1||g|+|x_2|(|x_1|+|g|)+\cdots+|x_{k-1}|(|x_1|+\cdots+|x_{k-1}|+|g|)} \\
& g\mathrm{ad}(x_1)\mathrm{ad}(\beta^{-1}\alpha(x_2))\cdots\mathrm{ad}(x_{k-1})\beta^{-(k-1)}\alpha^{k-1}(b) \\
= \ & (-1)^{|x_1||g|+|x_2|(|x_1|+|g|)+\cdots+|x_{k-1}|(|x_1|+\cdots+|x_{k-1}|+|g|)} \\
& g([x_1, [\beta^{-1}\alpha(x_2), [\cdots, [\beta^{-(k-2)}\alpha^{k-2}(x_{k-1}), \beta^{-(k-1)}\alpha^{k-1}(b)]\cdots]]]) \\
\in \ & g(L^k) = 0.
\end{aligned}$$

This proves that $(T^*_\omega L)^{2k-1} = 0$. Hence $T^*_w L$ is nilpotent of length at least k and at most $2k-1$.

(2) Suppose that $0 \neq L = I \oplus J$, where I and J are two nonzero Bihom-ideals of $(L[\cdot,\cdot], \alpha, \beta)$. Let I^* (resp. J^*) denote the subspace of all linear forms in L^* vanishing on J (resp. I). Clearly, I^* (resp. J^*) can canonically be identified with the dual space of I (resp. J) and $L^* \cong I^* \oplus J^*$.

Since $[I^*, L]_{L\oplus L^*}(J) = I^*([L, \beta^{-1}\alpha(J)]) \subseteq I^*([L, J]) \subseteq I^*(J) = 0$ and $[I, L^*]_{L\oplus L^*}(J) = L^*([I, J]) \subseteq L^*(I \cap J) = 0$, we have $[I^*, L]_{L\oplus L^*} \subseteq I^*$ and $[I, L^*]_{L\oplus L^*} \subseteq I^*$. Then

$$\begin{aligned}
[T^*_0 I, T^*_0 L]_{L\oplus L^*} &= [I \oplus I^*, L \oplus L^*]_{L\oplus L^*} \\
&= [I, L] + [I, L^*]_{L\oplus L^*} + [I^*, L]_{L\oplus L^*} + [I^*, L^*]_{L\oplus L^*} \subseteq I \oplus I^* = T^*_0 I.
\end{aligned}$$

$T^*_0 I$ is a Bihom-ideal of L and so is $T^*_0 J$ in the same way. Hence $T^*_0 L$ can be decomposed into the direct sum $T^*_0 I \oplus T^*_0 J$ of two nonzero Bihom-ideals of $T^*_0 L$. □

In the proof of a criterion for recognizing T^*-extensions of a Bihom-Lie superalgebra, we will need the following result.

Lemma 4.9. *Let (L, q_L, α, β) be a quadratic regular Bihom-Lie superalgebra of even dimension n over a field $\mathbb{K}$ and I be an isotropic $n/2$-dimensional subspace of L. If I is a Bihom-ideal of $(L, [\cdot,\cdot], \alpha, \beta)$, then $[\beta(I), \alpha(I)] = 0$.*

Proof. Since $\mathrm{dim}I + \mathrm{dim}I^\perp = n/2 + \mathrm{dim}I^\perp = n$ and $I \subseteq I^\perp$, we have $I = I^\perp$. If I is a ideal of $(L, [\cdot,\cdot], \alpha, \beta)$, then $q_L(\alpha(L), [\beta(I), \alpha(I^\perp)]) = q_L([\beta(L), \alpha(I)], \alpha(I^\perp)) \subseteq q_L([\beta(L), I], \alpha(I^\perp)) \subseteq q_L(I, I^\perp) = 0$, which implies $[\beta(I), \alpha(I)] = [\beta(I), \alpha(I^\perp)] \subseteq \alpha(L)^\perp = 0$. □

Theorem 4.10. *Let (L, q_L, α, β) be a quadratic regular Bihom-Lie superalgebra of even dimension n over a field $\mathbb{K}$ of characteristic not equal to two. Then (L, q_L, α, β) is isometric to a T^*-extension $(T^*_\omega B, q_B, \alpha', \beta')$ if and only if n is even and $(L, [\cdot,\cdot], \alpha, \beta)$ contains an isotropic Bihom-ideal I of dimension $n/2$. In particular, $B \cong L/I$, with B^* satisfying $\alpha(B^*) \subseteq B^*$ and $\beta(B^*) \subseteq B^*$.*

Proof. ($\Longrightarrow$) Since $\mathrm{dim}B = \mathrm{dim}B^*$, $\mathrm{dim}T^*_\omega B$ is even. Moreover, it is clear that B^* is a Bihom-ideal of half the dimension of $T^*_\omega B$ and by the definition of q_B, we have $q_B(B^*, B^*) = 0$, i.e., $B^* \subseteq (B^*)^\perp$ and so B^* is isotropic.

($\Longleftarrow$) Suppose that I is an $n/2$-dimensional isotropic Bihom-ideal of L. By Lemma 4.9, $[\beta(I), \alpha(I)] = 0$. Let $B = L/I$ and $p : L \to B$ be the canonical projection. Clearly, $|p(x)| = |x|, \forall x \in L_{|x|}$. Since $\mathrm{ch}\mathbb{K} \neq 2$, we can choose an isotropic complement subspace B_0 to I in L, i.e., $L = B_0 \dotplus I$ and $B_0 \subseteq B_0^\perp$. Then $B_0^\perp = B_0$ since $\mathrm{dim}B_0 = n/2$.

Denote by p_0 (resp. p_1) the projection $L \to B_0$ (resp. $L \to I$) and let q_L^* denote the homogeneous linear map $I \to B^* : i \mapsto q_L^*(i)$, where $q_L^*(i)(p(x)) := q_L(i, x)$, it is clear $|q_L^*(x)| = |x|, \forall x \in L_{|x|}$. We claim that q_L^* is a linear isomorphism. In fact, if $p(x) = p(y)$, then $x - y \in I$, hence $q_L(i, x - y) \in q_L(I, I) = 0$ and so $q_L(i, x) = q_L(i, y)$, which implies q_L^* is well-defined and it is easily seen that q_L^* is linear. If $q_L^*(i) = q_L^*(j)$, then $q_L^*(i)(p(x)) = q_L^*(j)(p(x)), \forall x \in L$, i.e., $q_L(i, x) = q_L(j, x)$, which implies $i - j \in L^\perp = 0$, hence q_L^* is injective. Note that $\dim I = \dim B^*$, then q_L^* is surjective.

In addition, q_L^* has the following property:

$$\begin{aligned}
& q_L^*([\beta(x), \alpha(i)])(p(\alpha(y))) \\
= \ & q_L([\beta(x), \alpha(i)], \alpha(y)) \\
= \ & -(-1)^{|x||i|} q_L([\beta(i), \alpha(x)], \alpha(y)) \\
= \ & -(-1)^{|x||i|} q_L(\alpha(i), [\beta(x), \alpha(y)]) \\
= \ & -(-1)^{|x||i|} q_L^*(\alpha(i)) p([\beta(x), \alpha(y)]) \\
= \ & -(-1)^{|x||i|} q_L^*(\alpha(i))[p(\beta(x)), p(\alpha(y))] \\
= \ & -(-1)^{|x||i|} q_L^*(\alpha(i))(\mathrm{ad} p(\beta(x))(p(\alpha(y)))) \\
= \ & \pi(p(\beta(x))) q_L^*(\alpha(i)))(p(\alpha(y)) \\
= \ & [p(\beta(x)), q_L^*(\alpha(i))]_{L \oplus L^*}(p(\alpha(y))),
\end{aligned}$$

where $x, y \in L$, $i \in I$. A similar computation shows that

$$q_L^*([\beta(x), \alpha(i)]) = [p(\beta(x)), q_L^*(\alpha(i))]_{L \oplus L^*}, \quad q_L^*([\beta(i), \alpha(x)]) = [q_L^*(\beta(i)), p(\beta(x))]_{L \oplus L^*}.$$

Define a homogeneous bilinear map

$$\begin{aligned}
\omega : \quad & B \times B & \longrightarrow \quad & B^* \\
& (p(b_0), p(b_0')) & \longmapsto \quad & q_L^*(p_1([b_0, b_0'])),
\end{aligned}$$

where $b_0, b_0' \in B_0$. Then $|w| = 0$ and w is well-defined since the restriction of the projection p to B_0 is a linear isomorphism.

Let φ be the linear map $L \to B \oplus B^*$ defined by $\varphi(b_0 + i) = p(b_0) + q_L^*(i), \forall b_0 + i \in B_0 \dot{+} I = L$. Since the restriction of p to B_0 and q_L^* are linear isomorphisms, φ is also a linear isomorphism. Note that

$$\begin{aligned}
& \varphi([\beta(b_0 + i), \alpha(b_0' + i')]) \\
= \ & \varphi([\beta(b_0), \alpha(b_0')] + [\beta(b_0), \alpha(i')] + [\beta(i), \alpha(b_0')]) \\
= \ & \varphi(p_0([\beta(b_0), \alpha(b_0')]) + p_1([\beta(b_0), \alpha(b_0')]) + [\beta(b_0), \alpha(i')] + [\beta(i), \alpha(b_0')]) \\
= \ & p(p_0([\beta(b_0), \alpha(b_0')])) + q_L^*(p_1([\beta(b_0), \alpha(b_0')]) + [\beta(b_0), \alpha(i')] + [\beta(i), \alpha(b_0')]) \\
= \ & [p(\beta(b_0)), p(\alpha(b_0'))] + \omega(p(\beta(b_0)), p(\alpha(b_0'))) + [p(\beta(b_0)), q_L^*(\alpha(i'))] \\
& + [q_L^*(\beta(i)), p(\alpha(b_0'))] \\
= \ & [p(\beta(b_0)), p(\alpha(b_0'))] + \omega(p(\beta(b_0)), p(\alpha(b_0'))) + \pi(p(\beta(b_0))(q_L^*(\alpha(i'))) \\
& -(-)^{|b_0||b_0'|} \pi(p(\beta(b_0'))(q_L^*(\alpha(i))) \\
= \ & [p(\beta(b_0)) + q_L^*(\beta(i)), p(\alpha(b_0')) + q_L^*(\alpha(i'))]_{B \oplus B^*} \\
= \ & [\varphi\beta((b_0 + i)), \varphi\alpha((b_0' + i'))]_{L \oplus L^*}.
\end{aligned}$$

Then φ is an isomorphism of $\mathbb{Z}$-graded superalgebras, and $(B \oplus B^*, [\cdot,\cdot]_{B\oplus B^*}, \overline{\alpha}, \overline{\beta})$ is a Bihom-Lie superalgebra. Furthermore, we have

$$\begin{aligned} q_B(\varphi(b_0+i), \varphi(b_0'+i')) &= q_B(p(b_0)+q_L^*(i), p(b_0')+q_L^*(i')) \\ &= q_L^*(i)(p(b_0')) + (-)^{|b_0||b_0'|} q_L^*(i')(p(b_0)) \\ &= q_L(i, b_0') + (-)^{|b_0||b_0'|} q_L(i', b_0) \\ &= q_L(b_0+i, b_0'+i'), \end{aligned}$$

then φ is isometric. The relation

$$\begin{aligned} & q_B([\beta'(\varphi((x)), \alpha'(\varphi(y))], \alpha'(\varphi(z))) \\ = \; & q_B([\varphi(\beta(x)), \varphi(\alpha(y))], \varphi(\alpha(z))) = q_B(\varphi([\beta(x), \alpha(y)]), \varphi(\alpha(z))) \\ = \; & q_L([\beta(x), \alpha(y)], \alpha(z)) = q_L(\alpha(x), [\beta(y), \alpha(z)]) \\ = \; & q_B(\varphi(\alpha(x)), [\varphi(\beta(y)), \varphi(\alpha(z))]) = q_B(\alpha'(\varphi(x)), [\beta'(\varphi(y)), \alpha'(\varphi(z))]), \end{aligned}$$

which implies that q_B is a nondegenerate, $\alpha\beta$-invariant and supersymmetric bilinear form, and so $(B \oplus B^*, q_B, \alpha', \beta')$ is a quadratic Bihom-Lie superalgebra. In this way, we get a T^*-extension $T^*_\omega B$ of B and consequently, (L, q_L, α, β) and $(T^*_\omega B, q_B, \alpha', \beta')$ are isometric as required. □

Let$(L, [\cdot,\cdot], \alpha, \beta)$ be a Bihom-Lie superalgebra over a field $\mathbb{K}$, and let $\omega_1 : L \times L \to L^*$ and $\omega_2 : L \times L \to L^*$ be two different bilinear maps satisfying (4.7), (4.8), (4.9) and $|\omega_1| = |\omega_2| = 0$. The T^*-extensions $T^*_{\omega_1} L$ and $T^*_{w_2} L$ of L are said to be equivalent if there exists an isomorphism of Bihom-Lie superalgebras $\phi : T^*_{\omega_1} L \to T^*_{\omega_2} L$ which is the identity on the Bihom-ideal L^* and which induces the identity on the factor Bihom-Lie superalgebra $T^*_{\omega_1} L/L^* \cong L \cong T^*_{\omega_2} L/L^*$. The two T^*-extensions $T^*_{\omega_1} L$ and $T^*_{\omega_2} L$ are said to be isometrically equivalent if they are equivalent and ϕ is an isometry.

Proposition 4.11. *Let $(L, [\cdot,\cdot], \alpha, \beta)$, where α, β are bijective, be a Bihom-Lie superalgebra over a field $\mathbb{K}$ of characteristic not equal to 2, and ω_1, ω_2 be two bilinear maps $L \times L \to L^*$ satisfying* (4.7), (4.8), (4.9) *and $|\omega_1| = |\omega_2| = 0$. Then we have*

(i) *$T^*_{\omega_1} L$ is equivalent to $T^*_{\omega_2} L$ if and only if there is $z \in C^1(L, L^*)_0$ such that*

$$\omega_1(x,y) - \omega_2(x,y) = \pi(x)z(y) - (-1)^{|x||y|}\pi(\alpha^{-1}\beta(y)\tilde{\alpha}\tilde{\beta}^{-1}z(x) - z([x,y]), \forall x, y \in L. \quad (4.10)$$

If this is the case, then the supersymmetric part z_s of z, defined by $z_s(x)(y) := \frac{1}{2}(z(x)(y) + (-1)^{|x||y|}z(y)(x))$, for all $x, y \in L$, induces a supersymmetric $\alpha\beta$-invariant bilinear form on L.

(ii) *$T^*_{\omega_1} L$ is isometrically equivalent to $T^*_{\omega_2} L$ if and only if there is $z \in C^1(L, L^*)_0$ such that* (4.10) *holds for all $x, y \in L$ and the supersymmetric part z_s of z vanishes.*

Proof. (i) $T^*_{\omega_1} L$ is equivalent to $T^*_{\omega_2} L$ if and only if there is an isomorphism of Bihom-Lie superalgebras $\Phi : T^*_{\omega_1} L \to T^*_{\omega_2} L$ satisfying $\Phi|_{L^*} = 1_{L^*}$ and $x - \Phi(x) \in L^*, \forall x \in L$.

Suppose that $\Phi : T^*_{\omega_1} L \to T^*_{\omega_2} L$ is an isomorphism of Bihom-Lie superalgebra and define a linear map $z : L \to L^*$ by $z(x) := \Phi(x) - x$, then $z \in C^1(L, L^*)_0$ and for all $x + f, y + g \in T^*_{\omega_1} L$, we have

$$\begin{aligned} & \Phi([x+f, y+g]_{L\oplus L^*}) \\ = \; & \Phi([x,y] + \omega_1(x,y) + \pi(x)g - (-1)^{|x||y|}\pi(\alpha^{-1}\beta(y)\tilde{\alpha}\tilde{\beta}^{-1}(f)) \\ = \; & [x,y] + z([x,y]) + \omega_1(x,y) + \pi(x)g - (-1)^{|x||y|}\pi(\alpha^{-1}\beta(y)\tilde{\alpha}\tilde{\beta}^{-1}(f). \end{aligned}$$

On the other hand,

$$\begin{aligned}
& [\Phi(x+f), \Phi(y+g)]_{L\oplus L^*} \\
= \ & [x+z(x)+f, y+z(y)+g]_{L\oplus L^*} \\
= \ & [x,y]+\omega_2(x,y)+\pi(x)g+\pi(x)z(y)-(-1)^{|x||y|}\pi(\alpha^{-1}\beta(y)\tilde{\alpha}\tilde{\beta}^{-1}z(x) \\
& -(-1)^{|x||y|}\pi(\alpha^{-1}\beta(y)\tilde{\alpha}\tilde{\beta}^{-1}(f).
\end{aligned}$$

Since Φ is an isomorphism, (4.10) holds.

Conversely, if there exists $z \in C^1(L, L^*)_0$ satisfying (4.10), then we can define $\Phi : T^*_{\omega_1}L \to T^*_{\omega_2}L$ by $\Phi(x+f) := x+z(x)+f$. It is easy to prove that Φ is an isomorphism of Bihom-Lie superalgebras such that $\Phi|_{L^*} = \mathrm{id}_{L^*}$ and $x - \Phi(L) \in L^*, \forall x \in L$, i.e. $T^*_{\omega_1}L$ is equivalent to $T^*_{\omega_2}L$.

Consider the supersymmetric bilinear form $q_L : L \times L \to \mathbb{K}, (x,y) \mapsto z_s(x)(y)$ induced by z_s. Note that

$$\begin{aligned}
& \omega_1(\beta(x),\alpha(y))(\alpha(m)) - \omega_2(\beta(x),\alpha(y))(\alpha(m)) \\
= \ & \pi(\beta(x))z(\alpha(y))(\alpha(m)) - (-1)^{|x||y|}\pi(\alpha^{-1}\beta(\alpha(y)))\tilde{\alpha}\tilde{\beta}^{-1}z(\beta(x))(\alpha(m)) \\
& -z([\beta(x),\alpha(y)])(\alpha(m)) \\
= \ & \pi(\beta(x))z(\alpha(y))(\alpha(m)) - (-1)^{|x||y|}\pi(\alpha(y))z(\alpha(x))(\alpha(m)) - z([\beta(x),\alpha(y)])(\alpha(m)) \\
= \ & -(-1)^{|x||y|}z(\alpha(y))([\beta(x),\alpha(m)]) + z(\alpha(x))([\beta(y),\alpha(m)]) - z([\beta(x),\alpha(y)])(\alpha(m))
\end{aligned}$$

and

$$\begin{aligned}
& (-1)^{|x|(|y|+|m|)}(\omega_1(\beta(y),\alpha(m))(\alpha(x)) - \omega_2(\beta(y),\alpha(m))(\alpha(x))) \\
= \ & (-1)^{|x|(|y|+|m|)}(\pi(\beta(y))z(\alpha(m))(\alpha(x)) - (-1)^{|m||y|}\pi(\beta(m))z(\alpha(y))(\alpha(x)) \\
& -z([\beta(y),\alpha(m)])(\alpha(x))) \\
= \ & (-1)^{|x|(|y|+|m|)}\big(-(-1)^{|m||y|}z(\alpha(m))([\beta(y),\alpha(x)]) + z(\alpha(y))([\beta(m),\alpha(x)]) \\
& -z([\beta(y),\alpha(m)])(\alpha(x))) \\
= \ & (-1)^{|m|(|y|+|x|)}z(\alpha(m))([\beta(x),\alpha(y)]) - (-1)^{|x||y|}z(\alpha(y))([\beta(x),\alpha(m)]) \\
& -(-1)^{|x|(|y|+|m|)}z([\beta(y),\alpha(m)])(\alpha(x)).
\end{aligned}$$

Since both ω_1 and ω_2 satisfy (4.9), the right hand sides of above two equations are equal. Hence,

$$\begin{aligned}
& -(-1)^{|x||y|}z(\alpha(y))([\beta(x),\alpha(m)]) + z(\alpha(x))([\beta(y),\alpha(m)]) - z([\beta(x),\alpha(y)])(\alpha(m)) \\
= \ & (-1)^{|m|(|y|+|x|)}z(\alpha(m))([\beta(x),\alpha(y)]) - (-1)^{|x||y|}z(\alpha(y))([\beta(x),\alpha(m)]) \\
& -(-1)^{|x|(|y|+|m|)}z([\beta(y),\alpha(m)])(\alpha(x)).
\end{aligned}$$

That is

$$\begin{aligned}
& z(\alpha(x))([\beta(y),\alpha(m)]) + (-1)^{|x|(|y|+|m|)}z([\beta(y),\alpha(m)])(\alpha(x)) \\
= \ & z([\beta(x),\alpha(y)])(\alpha(m)) + (-1)^{|m|(|y|+|x|)}z(\alpha(m))([\beta(x),\alpha(y)]) - .
\end{aligned}$$

Since $\mathrm{ch}\mathbb{K} \neq 2$, $q_L(\alpha(x),[\beta(y),\alpha(m)]) = q_L([\beta(x),\alpha(y)],\alpha(m))$, which proves the $\alpha\beta$-invariance of the supersymmetric bilinear form q_L induced by z_s.

(ii) Let the isomorphism Φ be defined as in (i). Then for all $x+f, y+g \in L \oplus L^*$, we have

$$\begin{aligned}
& q_B(\Phi(x+f),\Phi(y+g)) = q_B(x+z(x)+f, y+z(y)+g) \\
= \ & z(x)(y) + f(y) + (-1)^{|x||y|}z(y)(x) + (-1)^{|x||y|}g(x) \\
= \ & z(x)(y) + (-1)^{|x||y|}z(y)(x) + f(y) + (-1)^{|x||y|}g(x) \\
= \ & 2z_s(x)(y) + q_B(x+f, y+g).
\end{aligned}$$

Thus, Φ is an isometry if and only if $z_s = 0$. □

Acknowledgment. This paper is supported by NNSF of China (No. 11471090 and No.11771069), and NSF of Jilin province (No. 20170101048JC).

References

[1] K. Abdaoui, F. Ammar and A. Makhlouf, *Constructions and cohomology of Hom-Lie color algebras*, Comm. Algebra **43** (11), 4581-4612, 2015.

[2] F. Ammar, I. Ayadi, S. Mabrouk and A. Makhlouf, *Quadratic color Hom-Lie algebras*, arXiv:1204.5155.

[3] F. Ammar Z. Ejbehi and A. Makhlouf, *Cohomology and deformations of Hom-algebras*, J. Lie Theory **21** (4), 813-836, 2011.

[4] F. Ammar and A. Makhlouf, *Hom-Lie superalgebras and Hom-Lie admissible superalgebras*, J. Algebra **324** (7), 1513-1528, 2010.

[5] F. Ammar, A. Makhlouf and N. Saadoui, *Cohomology of Hom-Lie superalgebras and q-deforemed Witt superalgebra*, Czechoslovak Math. J. **63(138)** (3), 721-761, 2013.

[6] M. Bordemann, *Nondegenerate invariant bilinear forms on nonassociative algebras*, Acta Math. Univ. Comenian. (N.S.) **66** (2), 151-201, 1997.

[7] I. Bajo, S. Benayadi and A. Medina, *Symplectic structures on quadratic Lie algebras*, J. Algebra **316** (1), 174-188, 2007.

[8] S. Benayadi and A. Makhlouf, *Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms*, J. Geom. Phys. **76**, 38-60, 2014.

[9] Y. Cheng and H. Qi, *Representations of Bihom-Lie algebras*, arXiv:1610.04302.

[10] G. Graziani, A. Makhlouf, C. Menini and F. Panaite, *BiHom-associative algebras, BiHom-Lie algebras and BiHom-bialgebras*, SIGMA Symmetry Integrability Geom. Methods Appl. **11(86)** (1), 1-34, 2015.

[11] J. Hartwig, D. Larsson and S. Silvestrov, *Deformations of Lie algebras using σ-derivations*, J. Algebra **295** (2), 314-361, 2006.

[12] Y. Liu, L. Chen and Y. Ma, *Hom-Nijienhuis operators and T^*-extensions of hom-Lie superalgebras*, Linear Algebra Appl. **439** (7), 2131-2144, 2013.

[13] Y. Sheng, *Representations of hom-Lie algebras*, Algebr. Represent. Theory **15** (6), 1081-1098, 2012.

[14] S. Wang and S. Guo, *BiHom-Lie superalgebra structures*, arXiv:1610.02290.

[15] J. Zhao, L. Chen and L. Ma, *Representations and T^*-extensions of hom-Jordan-Lie algebras*, Comm. Algebra **44** (7), 2786-2812, 2016.