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Rational Solutions to the Boussinesq Equation

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Abstract

Rational solutions to the Boussinesq equation are constructed as a quotient of two polynomials in x and t. For each positive integer N, the numerator is a polynomial of degree $N(N+1)-2$ in x and t, while the denominator is a polynomial of degree $N(N+1)$ in x and t. So we obtain a hierarchy of rational solutions depending on an integer N called the order of the solution. We construct explicit expressions of these rational solutions for $N=1$ to 4.

1. Introduction

We consider the Boussinesq equation (B) which can be written in the form

$$u_{tt}-u_{xx}+(u^2)_{xx}+\frac{1}{3}u_{xxxx}=0, \tag{1.1}$$

where the subscripts x and t denote partial derivatives.
This equation first appears first in 1871, in a paper written by Boussinesq [1, 2]. It is well known that the Boussinesq equation (1.1) is an equation solvable by inverse scattering [3, 4]. It gives the description of the propagation of long waves surfaces in shallow water. It appears in several physical applications as one-dimensional nonlinear lattice-waves [5], vibrations in a nonlinear string [6] and ion sound waves in plasma [7].
The first solutions were founded in 1977 by Hirota [8] by using Bäcklund transformations. Among the various works concerning this equation, we can mention the following studies. Ablowitz and Satsuma constructed non-singular rational solutions in 1978 by using the Hirota bilinear method [9]. Freemann and Nimmo expressed solutions in terms of wronskians in 1983 [10]. An algebra-geometrical method using trigonal curve was given by Matveev et al. in 1987 [11]. The same author constructed other types of solutions using Darboux transformation [12]. Bogdanov and Zakharov in 2002 constructed solutions by the $\overline{\partial}$ dressing method [13]. In $2008-2010$, Clarkson obtained solutions in terms of the generalized Okamoto, generalized Hermite or Yablonski Vorob'ev polynomials [14, 15].
Recently, in 2017, Clarkson et al. constructed new solutions as second derivatives of polynomials of degree $n(n+1)$ in x and t in [16].

In this paper, we study rational solutions of the Boussinesq equation. We present rational solutions as a quotient of two polynomials in x and t. These following solutions belong to an infinite hierarchy of rational solutions written in terms of polynomials for each positive integer N. The study here is limited to the simplest cases where $N=1,2,3,4$.

2. First order rational solutions

We consider the Boussinesq equation

$$u_{tt}-u_{xx}+(u^2)_{xx}+\frac{1}{3}u_{xxxx}=0,$$

We have the following result at order $N=1$:

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Theorem 2.1. *The function v defined by*

$$v(x,t) = \frac{-2}{(-x+t+a_1)^2},$$

is a solution to the Boussinesq equation (1.1) with a_1 an arbitrarily real parameter.

Proof It is straightforward.
□

The parameter a_1 is only a translation parameter; it is not crucial. In the following solutions, we will omit it.

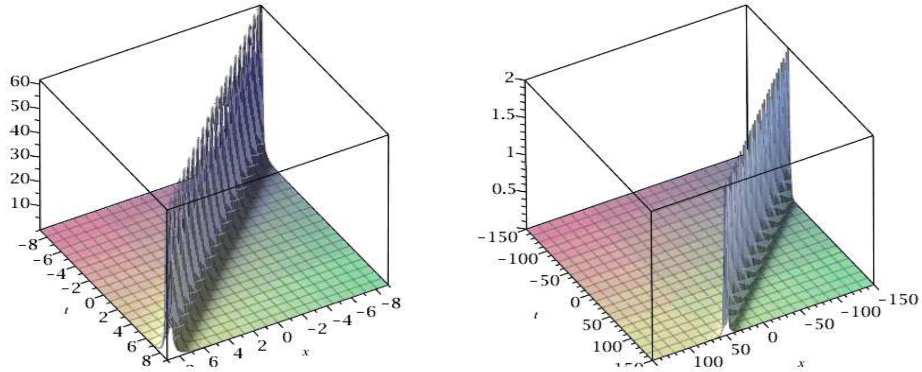

Figure 1. Solution of order 1 to (1.1), on the left $a_1 = 0$; on the right $a_1 = 100$.

In Figures 1., the singularity lines of respective equations $t = x$ and $t = x + a_1$ are clearly shown.

3. Second order rational solutions

The Boussinesq equation defined by (1.1) is always considered. We obtain the following solutions :

Theorem 3.1. *The function v defined by*

$$v(x,t) = -2\,\frac{n(x,t)}{d(x,t)^{(2)}}, \tag{3.1}$$

with

$$n(x,t) = 3x^4 + (-12t-4)x^3 + (18t^2+2+12t)x^2 + (-12t^2+8t-12t^3)x - 4t + 4t^3 - 10t^2 + 3t^4$$

and

$$d(x,t) = -x^3 + (3t+1)x^2 + (-3t^2-2t)x + t^3 + t^2 + 2t$$

is a rational solution to the Boussinesq equation (1.1), a quotient of two polynomials with the numerator of order 4 in x and t, the denominator of degree 6 in x and t.

Proof It is sufficient to replace the expression of the solution given by (3.1) and check that (1.1) is verified.
□

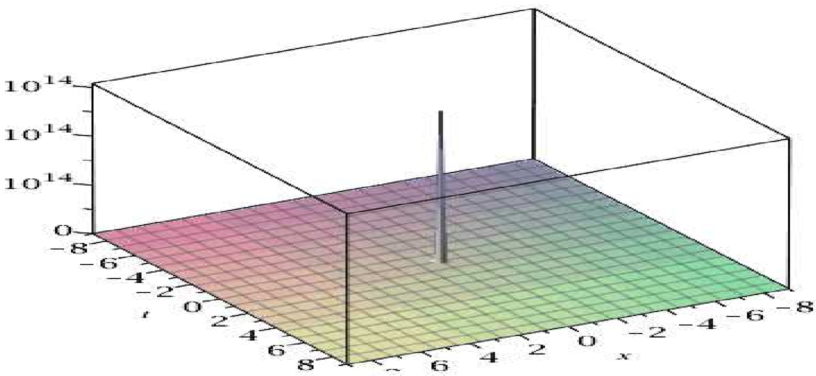

Figure 2. Solution of order 2 to (1.1).

This Figure 2. shows clearly the singularity in $(0;0)$.
The previous solution (3.1) can be rewritten as

$$-2\,\frac{3\,(t-x)^4 + 4\,(t-x)^3 - 4\,(t-x)^2 - 6t^2 + 6x^2 - 4t}{((t-x)^3 + (t-x)^2 + 2t)^2}.$$

So, with this expression, it is obvious to show that $(0;0)$ is a singularity as it can be seen in figure (2).

4. Rational solutions of order three

We obtain the following rational solutions to the Boussinesq equation defined by (1.1) :

Theorem 4.1. *The function v defined by*

$$v(x,t) = -2\,\frac{n(x,t)}{d(x,t)^{(2)}},$$

with
$n(x,t) = 6x^{10} + (-40 - 60t)x^9 + (270t^2 + 110 + 360t)x^8 + (-1440t^2 - 720t^3 - 160 - 880t)x^7 + (1260t^4 + 100 + 3080t^2 + 1120t + 3360t^3)x^6 + (-740t - 1512t^5 - 5040t^4 - 3360t^2 - 6160t^3)x^5 + (200t + 5040t^5 + 3100t^2 + 1260t^6 + 5600t^3 + 7700t^4)x^4 + (-6160t^5 - 720t^7 - 3360t^6 - 7000t^3 - 3200t^2 - 5600t^4)x^3 + (2000t^2 + 1440t^7 + 3080t^6 + 270t^8 + 8300t^4 + 8400t^3 + 3360t^5)x^2 + (-880t^7 - 5200t^3 - 8000t^4 - 60t^9 - 360t^8 - 4900t^5 - 1120t^6)x + 3200t^4 + 2600t^5 + 800t^3 + 160t^7 + 6t^{10} + 40t^9 + 110t^8 + 1140t^6$
and
$d(x,t) = x^6 + (-6t-4)x^5 + (15t^2 + 20t + 5)x^4 + (-20t^3 - 40t^2 - 30t)x^3 + (15t^4 + 40t^3 + 60t^2 + 20t)x^2 + (-6t^5 - 20t^4 - 50t^3 - 40t^2)x + t^6 + 4t^5 + 15t^4 + 20t^3 - 20t^2$
is a rational solution to the Boussinesq equation (1.1), quotient of two polynomials with numerator of order 10 *in x and t, denominator of degree* 12 *in x and t.*

Proof Replacing the expression of the solution given by (3.1), we check that the relation (1.1) is verified.
□

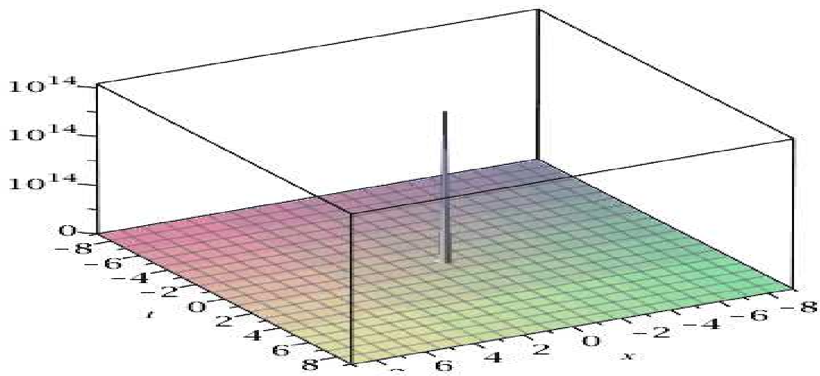

Figure 3. Solution of order 3 to (1.1).

The figure 3 clearly shows the singularity in (0;0).

5. Rational solutions of fourth order

The following solutions of order 4 to the Boussinesq equation defined by (1.1) are obtained :

Theorem 5.1. *The function v defined by*

$$v(x,t) = -2\,\frac{n(x,t)}{d(x,t)^{(2)}}, \tag{5.1}$$

with
$n(x,t) = 10x^{18} + (-180t - 180)x^{17} + (1460 + 3060t + 1530t^2)x^{16} + (-23600t - 8160t^3 - 6960 - 24480t^2)x^{15} + (30600t^4 + 21200 + 108000t + 122400t^3 + 178800t^2)x^{14} + (-781200t^2 - 842800t^3 - 428400t^4 - 321300t - 41300 - 85680t^5)x^{13} + (1113840t^5 + 2254000t^2 + 48300 + 2766400t^4 + 632800t + 3494400t^3 + 185640t^6)x^{12} + (-9703400t^3 - 4447800t^2 - 10810800t^4 - 318240t^7 - 805000t - 2227680t^6 - 29400 - 6704880t^5)x^{11} + (18972800t^3 + 28644000t^4 + 3500640t^7 + 24504480t^5 + 630000t + 12412400t^6 + 6013000t^2 + 437580t^8 + 7350)x^{10} + (-4375800t^8 - 17903600t^7 - 5467000t^2 - 26383000t^3 - 42042000t^6 - 54785500t^4 - 61345900t^5 - 294000t - 486200t^9)x^9 + (98313600t^6 + 20334600t^8 + 113097600t^5 + 24822000t^3 + 4375800t^9 + 55598400t^7 + 3228750t^2 + 73500t + 75778500t^4 + 437580t^{10})x^8 + (-318240t^{11} - 3500640t^{10} - 18246800t^9 - 57142800t^8 - 1176000t^2 - 150603600t^5 - 12544000t^3 - 67662000t^4 - 119790000t^7 - 171771600t^6)x^7 + (-882000t^3 + 45645600t^9 + 2227680t^{11} + 185640t^{12} + 119128800t^5 + 213150000t^6 + 111526800t^8 + 12892880t^{10} + 294000t^2 + 194409600t^7 + 19379500t^4)x^6 + (-78963500t^9 - 217182000t^7 - 85680t^{13} + 3920000t^3 - 1113840t^{12} - 7098000t^{11} - 140238000t^6 - 28108080t^{10} + 32928000t^4 - 164033100t^8 + 1528800t^5)x^5 + (13104000t^{11} + 41857200t^{10} + 158560500t^8 - 39690000t^4 - 980000t^3 + 30600t^{14} + 111132000t^7 + 101948000t^9 + 428400t^{13} + 2984800t^{12} - 115395000t^5 - 49808500t^6)x^4 + (-58107000t^8 - 45383800t^{10} + 19600000t^4 + 78400000t^7 - 122400t^{14} - 4477200t^{12} + 186984000t^6 - 16109800t^{11} + 113680000t^5 - 926800t^{13} - 81081000t^9 - 8160t^{15})x^3 + (-146510000t^6 - 52920000t^5 + 13708800t^{11} + 1530t^{16} + 1058400t^{13} - 59057250t^8 + 4256000t^{12} + 27617800t^{10} + 18942000t^9 + 200400t^{14} - 4900000t^4 + 24480t^{15} - 161994000t^7)x^2 + (89376000t^7 + 7840000t^5 - 690900t^{13} - 3389400t^{10} - 154800t^{14} - 180t^{17} + 50960000t^6 - 2519300t^{12} + 72912000t^8 - 26960t^{15} + 22778000t^9 - 3060t^{16} - 5635000t^{11})x - 16660000t^7 - 980000t^6 - 21070000t^8 - 13450500t^9 + 10t^{18} - 1960000t^5 + 180t^{17} + 1700t^{16} + 10560t^{15} + 52000t^{14} + 212800t^{13} + 521500t^{12} + 238000t^{11} - 3618650t^{10}$
and
$d(x,t) = x^{10} + (-10t - 10)x^9 + (45t^2 + 90t + 40)x^8 + (-120t^3 - 360t^2 - 350t - 70)x^7 + (210t^4 + 840t^3 + 1330t^2 + 700t + 35)x^6 +$

$(-252t^5-1260t^4-2870t^3-2730t^2-700t)x^5+(210t^6+1260t^5+3850t^4+5600t^3+2975t^2+350t)x^4+(-120t^7-840t^6-3290t^5-6650t^4-5600t^3-1400t^2)x^3+(45t^8+360t^7+1750t^6+4620t^5+5425t^4+2100t^3+700t^2)x^2+(-10t^9-90t^8-530t^7-1750t^6-2660t^5-1400t^4-2800t^3)x+t^{10}+10t^9+70t^8+280t^7+525t^6+350t^5+2100t^4+1400t^3$

is a rational solution to the Boussinesq equation (1.1), quotient of two polynomials with numerator of order 18 *in x and t, denominator of degree* 20 *in x and t.*

Proof We have to check that the relation (1.1) is verified when we replace the expression of the solution given by (5.1).
□

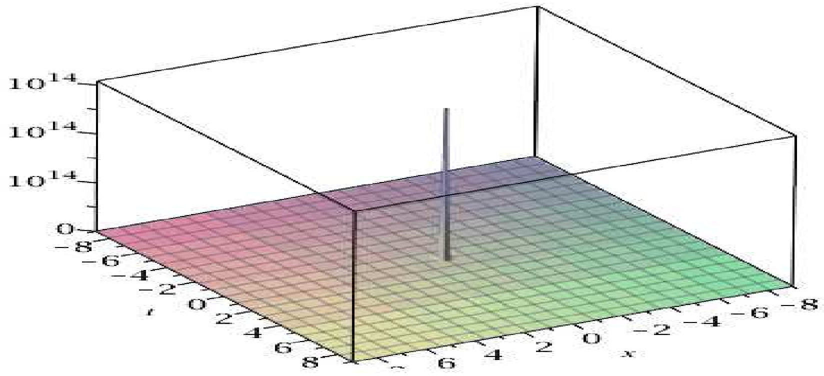

Figure 4. Solution of order 4 to (1.1).

As in the preceding cases, the figure 4 clearly shows the singularity in $(0;0)$.

6. Conclusion

Rational solutions to the Boussinesq equation of order 1, 2, 3, 4 have been constructed here. The following asymptotic behavior has been highlighted : $\lim_{t\to\infty} v(x,t)=0$, $\lim_{x\to\pm\infty} v(x,t)=0$.
It will relevant to construct rational solutions to the Boussinesq equation at order N and to give a representation of these solutions in terms of determinants. Namely, for every integer N, these solutions can be written as a quotient of determinants of order N, where the numerator is a polynomial of degree $N(N+1)-2$ in x, t, and the denominator is a polynomial of degree $N(N+1)$ in x, t.

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