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Effect of Inflation on Stochastic Optimal Investment Strategies for DC Pension under the Affine Interest Rate Model

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Abstract

In this paper, we seek to investigate the effect of inflation on the optimal investment strategies for DC Pension. Our model permits the plan member to make a defined contribution, as provided in the Nigerian Pension Reform Act of 2004. The plan member is free to invest in risk-free asset and two risky assets. A stochastic differential equation of the pension wealth that takes into account certainly agreed proportions of the plan member's salary, paid as a contribution towards the pension fund, is presented. The Hamilton-Jacobi-Bellman (H-J-B) equation, Legendre transformation, and dual theory are used to obtain the explicit solution of the optimal investment strategies for CRRA utility function. Our investigation reveals that the inflation has significant negative effect on optimal investment strategy, particularly, the CCRA is not constant with the investment strategy since the inflation parameters and coefficient of CRRA utility function have insignificant input on the investment strategy.

1. Introduction

There are two major designs of pension plan, namely, the defined benefit (DB) pension, and the defined contribution (DC) pension plan. As the names implies, in that of the DB, the benefits of the plan member are defined, and the sponsor bears the financial risk. Whereas, in the DC pension plan, the contributions are defined, the retirement benefits depends on the contributions and the investment returns, and the contributors (the plan members) bears the financial risk. Recently, the DC pension has taken dominance over the DB pension plan in the pension scheme, since DC pension plan is fully funded, which makes it easier for the plan managers (Pension Fund Administrators (PFAs') and the Pension Fund Custodians (PFCs') to invest equitably in the market, and also makes it easier for the plan members to receive their retirement benefit as and when due.

Investment strategies of the contributions, which in turn is a strong determinant of the investment returns vis-a-vis the benefits of the contributors at retirement must be given optimum attention. Recent publications in economic Journals and other reputable Mathematics and Science Journals have brought to light, a variety of methods of optimizing investment strategies and returns. For instance, some researchers have made various contributions in this direction, particularly, in DC Pension Plan. [1] did work on, "stochastic life styling: optimal dynamic asset allocation for defined contribution pension plans. In their work, various properties and characteristics of the optimal asset allocation strategy, both with and without the presence of non-hedge able salary risk were discussed. The significance of alternative optimal strategy by pension providers was established.

In order to deal with optimal investment strategy, the need for maximization of the expected utility of the terminal wealth became necessary. Example, the Constant Relative Risk Aversion (CRRA) utility function, and (or) the Constant Absolute Risk Aversion (CARA) utility function were used to maximize the terminal wealth. [1]-[4], and [5] used CRRA to maximize terminal wealth. However, [6] used the CRRA and the CARA to maximize terminal wealth.

[7] applied the well-known H-J-B equation, Legend transform, and dual theory to obtain the explicit solutions of CRRA and CARA utility function, for the maximization of the terminal wealth. In 2012, Nan-wei Han et al took a different direction. The investigated optimal asset allocation for DC pension plans under inflation. In their work, the retired individuals receive an annuity that is indexed by inflation and a

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downside protection on the amount of this annuity is considered. More so, in 2015, [1] considered an Inflationary market. In their work, the plan member made extra contribution to amortize the pension fund. The CRRA utility function was used to maximize the terminal wealth. This triggered our research. Ours is to investigate and view the extent of damage the inflation may have caused to enable us to introduce, not just an amortization fund, but an optimum amortization fund that would sufficiently dampen the effect of inflation. The approach used here is similar to that of [5]. The models we used is that of [8], though, we considered inflation of globally competing goods, and some real life assumptions are made to buttress this fact.

2. Preliminaries

We start with a complete and frictionless financial market that is continuously open over the fixed time interval $[0,T]$, for $T>0$, representing the retirement time of any plan member.
We assume that the market is composed of the risk-free asset (cash), the inflation-linked bond, and risky asset (the stock price subject to inflation). Let $(\Omega,\ F,P)$ be a complete probability space, where Ω is a real space and P is a probability measure, $\{W_s(t),\ W_I(t)\}$ are two standard orthogonal Brownian motions, $\{F_I(t),\ F_s(t)\}$ are right continuous filtrations whose information are generated by the two standard Brownian motions $\{W_s(t),\ W_I(t)\}$, whose sources of uncertainties are respectively to the inflation rate and the stock market. We assume also that at the early stage of the inflation, before government intervention policy, $\{W_R(t),\ W_I(t)\}$, $\{W_s(t),\ W_R(t)\}$ are two standard orthogonal Brownian motions, respectively.
Let $C(t)$ denote the price of the risk free asset at time t and it is modeled as follows

$$\frac{dC(t)}{C(t)}=r_R(t)dt, C(0)=1$$

$r(t)$ is the real interest rate process and is given by the stochastic differential equation (SDE)

$$dr_R(t)\ =\ (a\ -\ br_R\ (t))\ dt\ -\ \sigma_R dW_R(t),$$

$$\sigma_R=\ \sqrt{k_1 r_R(t)\ +\ k_2}\ ,\ \ t\geq 0,$$

where r_R is a real interest rate, $r_R(0)$, k_1, and k_2 are positive real numbers. If k_1 (resp., k_2) is equal to zero, we have a special case, as in [9], [10] dynamics. So under these dynamics, the term structure of the real interest rates is affine, which has been studied by [7], [4], [11] and [2].
Let $S(t)$ denote the price of the risky asset subject to inflation and its dynamics is given based on a continuous time stochastic process at $t\geq 0$ and the dynamics of the price process is described as follows

$$\frac{dS(t)}{S(t)}\ =(\ r_R(t)+\lambda_1\sigma_s{}^s+\lambda_2\sigma^I{}_s\theta_I)dt+\ \sigma^s{}_s dW_s+\sigma^I{}_s dW_I,\ \ S(0)=\ 1 \tag{2.1}$$

premium associated with the positive volatility constants $\sigma_s{}^s$ *and* $\sigma^I{}_s$, respectively, see [4]. θ_I represents the inflation price m with λ_1 *and* λ_2 represents the instantaneous market risk.
An inflation-linked bond with maturity T, whose price at time t is denoted by $B(t,I(t))$, $t\geq 0$, and its evolution is given by the SDE below (see [8])

$$\frac{dB(t,\ I(t)),\ )}{B(t,\ I(t)),}\ =(r_R(t)+\ \sigma_I\theta_I)\,dt+\sigma_I dW_I(t),\ \ B(T,\ I(T))=\ 1 \tag{2.2}$$

Let us denote the stochastic wage of the plan member, at time t, by $P(t)$ which is described by

$$\frac{dP(t)}{P(t)}\ =\ \mu_P(t)\ dt+\ \sigma^s{}_p dW_s(t)+\sigma^I{}_p dW_I(t),$$

where, $\mu_p(t)$ denotes the expected instantaneous rate of the wage, while $\sigma^s{}_p$ *and* $\sigma^I{}_p$ denote the two volatility scale factors of stock and inflation, respectively. Since the wage is stochastic, we let the instantaneous mean of the wage to be $\mu_P(t,\ r\ (t))=r(t)+\ u_p,$ where m_p is a real constant.

3. Methodology

3.1. Hamilton-Jacobi-Bellman (HJB) equation

Suppose, we represent $u=(u_B,u_S)$ as the strategy and we define the utility attained by the contributor from a given state y at time t as

$$G_u(t,r_R,y)=E_u\ V(X(T))\mid r_R(t)=r_R, Y(t)=y],\tag{3.1}$$

where t is the time, r_R is the real interest rate and y is the wealth. Our interest here is to find the optimal value function

$$G(t,r_R,y)=\sup_u G_u(t,r_R,y)$$

and the optimal strategy $u^*=(u_B{}^*,u_S{}^*)$ such that

$$G_{u^*}(t,r_R,y)=G(t,r_R,y).$$

3.2. Legendre transformation

The nonlinear partial differential equation obtained in (3.1) above is transformed into a linear partial differential equation, using the Legendre transform method and Dual theory.

Theorem 3.1. *[12] Let $f: R^n \to R$ be a convex function for $z > 0$, define the Legendre transform*

$$L(z) = \max_{y} \{f(y) - zy\}, \tag{3.2}$$

where $L(z)$ is the Legendre dual of $f(y)$. Suppose, $f(y)$ is strictly convex, then the supremum (3.2) would be attained at one point, denoted by y_0 (i.e, the sup. exist). We write

$$L(z) = sup_y \{f(y) - zy\} = f(y_0) - zy_0$$

By Theorem 3.1 and the assumption of convexity of the value function $G(t, r_R, y)$, we define the Legendre transform

$$\hat{G}(t, r_R, z) = \sup_{y>0} G(t, r_R, y) - zy \mid 0 < y < \infty\} \;\; 0 < t < T. \tag{3.3}$$

Where $z > 0$ denotes the dual variable to y and $\hat{G}$ is the dual function of G.
The value of y where this optimum is attained is denoted by $h(t, r, z)$, so that

$$h(t, r_R, z) = \inf_{y>0} y \,\big|\, G(t, r_R, y) \geq zy + \hat{G}(t, r_R, z)\big\} \;\; 0 < t < T. \tag{3.4}$$

from (3.4), we see that the function h and $\widehat{G}$ are closely related, hence we write either of them as dual of G. To see this relationship,

$$\hat{G}(t, r_R, z) = G(t, r_R, h) - zh.$$

where

$$h(t, r_R, z) = y, \; G_y = z, \;\; and\ relating\ \hat{G}\ to\ h\ by\ h = -\hat{G}_z.$$

Replicating the idea in (3.3) and (3.4), above, we define the Legendre transform of the utility function $U(y)$, at terminal time, thus

$$\hat{U}(z) = sup_{x>0}\, U(x) - zx \mid 0 < x < \infty\},$$

where $z > 0$ denotes the dual variable to y, and $\hat{U}$ is the dual of U.
Similarly, the value of y where this optimum is attained is denoted by $G(z)$, such that

$$G(z) = sup_{x>0} \left\{w \mid U(x) \geq zx + \hat{U}(z)\right\}.$$

Consequently, we have

$$G(z) = (U')^{-1}(z)$$

where G is the inverse of the marginal utility U.
Since $h(T, r_R, y) = U(y)$, then at the terminal time, T, we can define

$$h(T, r_R, z) = \inf_{y>0} \left\{ y \middle| U(y) \geq zy + \hat{h}(T, r_R, z)\right\}$$

and

$$\hat{h}(T, r_R, z) = \sup_{y>0} \{U(y) - zy\}$$

so that

$$h(T, r_R, z) = (U')^{-1}(z). \tag{3.5}$$

4. Model formulation

Here, the contributions are continuously paid into the pension fund at the rate of $KP(t)$ where K is the mandatory rate of contribution. Let $W(t)$ denote the wealth of pension fund at time $t \in [0,T]$. $u_B(t)$ and $u_S(t)$ represent the proportion of the pension fund invested in the bond and the stock respectively. This implies that the proportion of the pension fund invested in the risk-free asset $u_C(t) = 1 - u_B(t) - u_S(t)$. The dynamics of the pension wealth is given by

$$dW(t) = u_C W(t)\frac{dC(t)}{C(t)} + u_B W(t)\frac{dB(t,I(t))}{B(t,I(t))} + u_S W(t)\frac{dS(t)}{S(t)} + KP(t)\,dt \tag{4.1}$$

Substituting f(1), (2.1) and (2.2) in (4.1) we have

$$dW(t) = W(t)\left[r_R(t) + \sigma_I\theta_I u_B + \left(\lambda_1\sigma^s{}_s + \lambda_2\sigma^I{}_s\theta_I\right)u_S\right]dt + KP(t)\,dt + W(t)\left(\sigma_I u_B + \sigma^I{}_s u_S\right)dW_I(t) + W(t)\sigma^s{}_s u_S dW_S(t) \tag{4.2}$$

Let the relative wealth $(t)t$be defined as follows

$$Y(t) = \frac{W(t)}{P(t)} \tag{4.3}$$

Applying product rule and Ito's formula to (4.3) and making use of (2.3) and (4.2) we arrive at the following equation

$$\begin{aligned} dY(t) &= Y(t)\left\{r(t) - \mu_p + (\sigma^s{}_p)^2 + (\sigma^I{}_p)^2 + \left[\left(\lambda_1\sigma^s{}_s + \lambda_2\sigma^I{}_s\theta_I\right) - \frac{1}{2}\sigma^I{}_s\sigma^I{}_p - \frac{1}{2}\sigma^s{}_s\sigma^s{}_p\right]u_S + \left(\sigma_I\theta_I - \frac{1}{2}\sigma_I\sigma^I{}_p\right)u_B\right\}dt \\ &+ Kdt + Y(t)\left(\sigma_I u_B + \sigma^I{}_s u_s - \sigma^I{}_p\right)dW_I + Y(t)\left(\sigma^s{}_s u_s - \sigma^s{}_p\right)dW_s \qquad Y(0) = W(0)/P(0) \end{aligned}$$

Simplifying,

$$dY(t) = X(c_1 + c_2 u_s + c_3 u_B)dt + Kdt + Y(t)\left(\sigma_I u_B + \sigma^I{}_s u_S - \sigma^I{}_p\right)dW_I(t) + Y(t)\left(\sigma^s{}_s u_S - \sigma^s{}_p\right)dW_S(t), \tag{4.4}$$

where

$$c_1 = r_R(t) - \mu_p + (\sigma^s{}_p)^2 + (\sigma^I{}_p)^2$$

$$c_2 = \left(\lambda_1\sigma^s{}_s + \lambda_2\sigma^I{}_s\theta_I\right) - \frac{1}{2}\sigma^I{}_s\sigma^I{}_p - \frac{1}{2}\sigma^s{}_s\sigma^s{}_p$$

$$c_3 = \sigma_I\theta_I - \frac{1}{2}\sigma_I\sigma^I{}_p$$

The Hamilton-Jacobi-Bellman (HJB) equation associated with (4.4) is

$$G_t + (a - br_R)G_r + \frac{1}{2}\sigma_{r_R}{}^2 G_{r_R r_R} + \sup_u\left\{\begin{array}{l} \{y(c_1 + u_s c_2 + u_B c_3)G_y + KG_y \\ +\frac{1}{2}y^2[\left(\left(\sigma_I u_B + \sigma^I{}_s u_s - \sigma^I{}_p\right)\right)^2 + \left(\sigma^s{}_s u_s - \sigma^s{}_p\right)^2]G_{yy} = 0 \end{array}\right\} \tag{4.5}$$

where G_t, G_y, $G_{r_R r_R}$, G_y and G_{yy} are partial derivatives of first and second orders with respect to time, real interest rate, and relative wealth. Differentiating (4.5) with respect to u_B and u_S, we obtain the first-order maximizing conditions for the optimal strategies $u_B{}^*$ *and* $u_S{}^*$, thus

$$c_3 G_y + y\sigma_I\left(\sigma_I u_B{}^* + \sigma^I{}_s u_S{}^* - \sigma^I{}_p\right)G_{yy} = 0 \tag{4.6}$$

$$c_2 G_y + x\sigma^I{}_s\left(\sigma_I u_B{}^* + \sigma^I{}_s u_S{}^* - \sigma^I{}_p\right)G_{yy} + y\sigma^s{}_s\left(\sigma^s{}_s u_S{}^* - \sigma^s{}_p\right)G_{yy} = 0 \tag{4.7}$$

Solving (4.6) and (4.7) simultaneously we have

$$u_S{}^* = \frac{\sigma^I{}_s c_3 - c_2\sigma_I}{(\sigma^s{}_s)^2\sigma_I y}\frac{G_y}{G_{yy}} + \left(\frac{\sigma^s{}_p\sigma^I{}_s + \sigma^s{}_p\sigma^s{}_s - \sigma^I{}_p\sigma^I{}_s}{(\sigma^s{}_s)^2}\right) \tag{4.8}$$

$$u_B{}^* = \frac{\sigma^I{}_p}{\sigma_I} - \frac{\sigma^I{}_s(\sigma^s{}_p\sigma^I{}_s + \sigma^s{}_p\sigma^s{}_s - \sigma^I{}_p\sigma^I{}_s)}{(\sigma^s{}_s)^2\sigma_I} - \frac{\sigma^I{}_s(\sigma^I{}_s c_3 - c_2\sigma_I)}{(\sigma^s{}_s)^2 y}\frac{G_y}{G_{yy}} - \frac{c_3}{\sigma_I{}^2 y}\frac{G_y}{G_{yy}} \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.5), and assuming independent and identically distributed volatility scale of salary for stock and inflation (i.e., $\sigma^I{}_p = \sigma^s{}_p$), we have

$$\begin{aligned} &G_t + (a - br_R)G_{r_R} + \tfrac{1}{2}\sigma_{r_R}{}^2 G_{r_R r_R} + \left(K + y(\tfrac{1}{2}\rho_5 + \rho_1)\right)G_y \\ &+ \left(2\theta_I{}^2 + \tfrac{1}{2}(\sigma^I{}_p)^2 - \theta_I\sigma^I{}_p + \rho_2 + \rho_4\right)\frac{G_y{}^2}{G_{yy}} + \tfrac{1}{2}y^2\rho_3 = 0, \end{aligned} \tag{4.10}$$

$$\begin{aligned}\rho_1 = {} & \tfrac{3}{2}(\sigma^I{}_p)^2 + \lambda_1\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I + \lambda_2(\sigma^I{}_s)^2\theta_I{}^2\sigma_I - \tfrac12\lambda_1\sigma^s{}_s\sigma^I{}_s\sigma_I\sigma^I{}_p - \tfrac12\lambda_2(\sigma^I{}_s)^2\theta_I\sigma_I\sigma^I{}_p \\ & - \tfrac12(\sigma^I{}_s)^2\theta_I\sigma_I\sigma^I{}_p - \tfrac12\sigma^s{}_s\sigma^I{}_p\sigma^I{}_s\sigma_I\theta_I + \tfrac14(\sigma^I{}_s)^2(\sigma^I{}_p)^2\sigma_I + \tfrac14\sigma^s{}_s(\sigma^I{}_s)^2\sigma^I{}_s\sigma_I \\ & +\lambda_1\sigma^s{}_p + \tfrac{\lambda_2(\sigma^I{}_s)^2\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^2} + \tfrac{\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s} - \tfrac{\theta_I\lambda_2(\sigma^I{}_s)^2(\sigma^I{}_p)^2}{(\sigma^s{}_s)^2} - \tfrac{(\sigma^I{}_p)^2\sigma^I{}_s}{2\sigma^s{}_s} - u_p\end{aligned}$$

$$\begin{aligned}\rho_2 = {} & \tfrac{\lambda_1\sigma^I{}_s\sigma^I{}_p}{2\sigma^s{}_s} - \tfrac{2\lambda_1\lambda_2\sigma^I{}_s\theta_I}{\sigma^s{}_s} - \tfrac{\lambda_2{}^2(\sigma^I{}_s)^2\theta_I{}^2}{(\sigma^s{}_s)^2} + \tfrac{\lambda_2(\sigma^I{}_s)^2\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^2} + \tfrac{\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s} - \tfrac{3(\sigma^I{}_s)^2(\sigma^I{}_p)^2}{4\sigma^s{}_s{}^2} \\ & -\tfrac32\tfrac{\theta_I(\sigma^I{}_s)^2\sigma^I{}_p}{(\sigma^s{}_s)^2} - \tfrac{\theta_1\sigma^I{}_p\sigma^I{}_s}{2\sigma^s{}_s} - \tfrac{\theta_I{}^2(\sigma^I{}_s)^2}{(\sigma^s{}_s)^2} + \tfrac{\theta_I\sigma^I{}_s\lambda_1}{\sigma^s{}_s} + \tfrac{(\sigma^I{}_p)^2(\sigma^I{}_s)^2}{2(\sigma^s{}_s)^2} + \tfrac{(\sigma^s{}_p)^2\sigma^I{}_s}{4\sigma^s{}_s}\end{aligned}$$

$$\begin{aligned}\rho_3 = {} & (\sigma^I{}_s)^4\sigma_I\theta_I{}^2 - (\sigma^I{}_s)^4\sigma_I{}^2\theta_I\sigma^I{}_p + \tfrac{(\sigma^I{}_s)^4\sigma_I{}^2(\sigma^I{}_p)^2}{4} + (\sigma^s{}_s)^2(\sigma^I{}_s)^2\sigma_I{}^2\theta_I{}^2 - (\sigma^s{}_s)^2\sigma^I{}_s\sigma_I{}^2\theta_I\sigma^I{}_p \\ & +2\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p - 2\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p + \tfrac{(\sigma^s{}_s)^2(\sigma^I{}_s)^2\sigma_I{}^2(\sigma^I{}_p)^2}{4} - \tfrac{2(\sigma^I{}_p)^2(\sigma^I{}_s)^2}{(\sigma^s{}_s)^2}\end{aligned}$$

$$\begin{aligned}\rho_4 = {} & \tfrac{(\sigma^I{}_s)^4\theta_I{}^2}{(\sigma^s{}_s)^4} + \tfrac{(\sigma^I{}_s)^3\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^4} + \tfrac{(\sigma^I{}_s)^2(\sigma^I{}_p)^2}{4(\sigma^s{}_s)^4} - \tfrac{2\theta_I{}^2(\sigma^I{}_s)^2}{(\sigma^s{}_s)^2} + \tfrac{\sigma^I{}_s\sigma^I{}_p\theta_I}{(\sigma^s{}_s)^2} + \tfrac{\sigma^I{}_p(\sigma^I{}_s)^2\theta_I}{(\sigma^s{}_s)^2} - \tfrac{\sigma^I{}_s(\sigma^I{}_p)^2}{2(\sigma^s{}_s)^2} \\ & +\tfrac{2\lambda_1\lambda_2\sigma^I{}_s\theta_I}{\sigma^s{}_s} - \tfrac{\lambda_1\sigma^I{}_s\sigma^I{}_p}{\sigma^s{}_s} + \tfrac{\lambda_1{}^2(\sigma^I{}_s)^2\theta_I{}^2}{(\sigma^s{}_s)^2} - \tfrac{\lambda_2(\sigma^I{}_s)^2\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^2} - \tfrac{\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s} + \tfrac{(\sigma^I{}_s)^2(\sigma^I{}_p)^2}{4(\sigma^s{}_s)^2} + \tfrac{\sigma^I{}_s(\sigma^I{}_p)^2}{2\sigma^s{}_s}\end{aligned}$$

$$\begin{aligned}\rho_5 = {} & \frac{(\sigma^I{}_s)^3\sigma_I\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^2} - \frac{2(\sigma^I{}_s)^4\sigma_I\theta_I{}^2}{(\sigma^s{}_s)^2} + \frac{(\sigma^I{}_s)^4\sigma_I\sigma^I{}_p\theta_I}{(\sigma^s{}_s)^2} - \frac{(\sigma^I{}_s)^3\sigma_I(\sigma^I{}_p)^2}{2(\sigma^s{}_s)^2} + 2\theta_I{}^2\sigma_I(\sigma^I{}_s)^2 - \theta_I(\sigma^I{}_s)^2\sigma_I\sigma^I{}_p \\ + {} & \frac{(\sigma^I{}_p)^2(\sigma^I{}_s)^2\sigma_I}{2} - 2\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\lambda_1 - 2(\sigma^I{}_s)^2\sigma_I\theta_I{}^2\lambda_2 + \sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p + \sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p\lambda_1 + (\sigma^I{}_s)^2\sigma_I\sigma^I{}_p\lambda_2\theta_I \\ - {} & \frac{(\sigma^I{}_s)^2\sigma_I(\sigma^I{}_p)^2}{2} - \frac{\sigma^s{}_s\sigma^I{}_s\sigma_I(\sigma^I{}_p)^2}{2} - 2\sigma^s{}_s\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p + \frac{2\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s}\end{aligned}$$

with $G(T, r_R y,) = U(y)$.
Applying Legendre transform to (4.10), we have

$$\hat G_t + (a - b r_R(t))\hat G_{r_R} + \frac12\sigma_{r_R}{}^2\hat G_{r_R r_R} + [K + y(\frac12\rho_5 + \rho_1)]Z - \left(2\theta_I{}^2\frac12\left(\sigma^I{}_p\right)^2 - \theta_I\sigma^I{}_p + \rho_2 + \rho_4\right)Z^2\hat G_{zz} - \frac12 y^2\rho_3\frac{1}{\hat G_{zz}} = 0,$$

$$\begin{aligned}\rho_1 = {} & \tfrac{3}{2}(\sigma^I{}_p)^2 + \lambda_1\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I + \lambda_2(\sigma^I{}_s)^2\theta_I{}^2\sigma_I - \tfrac12\lambda_1\sigma^s{}_s\sigma^I{}_s\sigma_I\sigma^I{}_p - \tfrac12\lambda_2(\sigma^I{}_s)^2\theta_I\sigma_I\sigma^I{}_p \\ & - \tfrac12(\sigma^I{}_s)^2\theta_I\sigma_I\sigma^I{}_p - \tfrac12\sigma^s{}_s\sigma^I{}_p\sigma^I{}_s\sigma_I\theta_I + \tfrac14(\sigma^I{}_s)^2(\sigma^I{}_p)^2\sigma_I + \tfrac14\sigma^s{}_s(\sigma^I{}_s)^2\sigma^I{}_s\sigma_I + \lambda_1\sigma^s{}_p \\ & + \tfrac{\lambda_2(\sigma^I{}_s)^2\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^2} + \tfrac{\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s} - \tfrac{\theta_I\lambda_2(\sigma^I{}_s)^2(\sigma^I{}_p)^2}{(\sigma^s{}_s)^2} - \tfrac{(\sigma^I{}_p)^2\sigma^I{}_s}{2\sigma^s{}_s} - u_p\end{aligned}$$

$$\begin{aligned}\rho_2 = {} & \tfrac{\lambda_1\sigma^I{}_s\sigma^I{}_p}{2\sigma^s{}_s} - \tfrac{2\lambda_1\lambda_2\sigma^I{}_s\theta_I}{\sigma^s{}_s} - \tfrac{\lambda_2{}^2(\sigma^I{}_s)^2\theta_I{}^2}{(\sigma^s{}_s)^2} + \tfrac{\lambda_2(\sigma^I{}_s)^2\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^2} + \tfrac{\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s} - \tfrac{3(\sigma^I{}_s)^2(\sigma^I{}_p)^2}{4\sigma^s{}_s{}^2} \\ & -\tfrac32\tfrac{\theta_I(\sigma^I{}_s)^2\sigma^I{}_p}{(\sigma^s{}_s)^2} - \tfrac{\theta_1\sigma^I{}_p\sigma^I{}_s}{2\sigma^s{}_s} - \tfrac{e_I{}^2(\sigma^I{}_s)^2}{(\sigma^s{}_s)^2} + \tfrac{\theta_I\sigma^I{}_s\lambda_1}{\sigma^s{}_s} + \tfrac{(\sigma^I{}_p)^2(\sigma^I{}_s)^2}{2(\sigma^s{}_s)^2} + \tfrac{(\sigma^s{}_p)^2\sigma^I{}_s}{4\sigma^s{}_s}\end{aligned}$$

$$\begin{aligned}\rho_3 = {} & (\sigma^I{}_s)^4\sigma_I\theta_I{}^2 - (\sigma^I{}_s)^4\sigma_I{}^2\theta_I\sigma^I{}_p + \tfrac{(\sigma^I{}_s)^4\sigma_I{}^2(\sigma^I{}_p)^2}{4} + (\sigma^s{}_s)^2(\sigma^I{}_s)^2\sigma_I{}^2\theta_I{}^2 - (\sigma^s{}_s)^2\sigma^I{}_s\sigma_I{}^2\theta_I\sigma^I{}_p \\ & +2\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p - 2\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p + \tfrac{(\sigma^s{}_s)^2(\sigma^I{}_s)^2\sigma_I{}^2(\sigma^I{}_p)^2}{4} - \tfrac{2(\sigma^I{}_p)^2(\sigma^I{}_s)^2}{(\sigma^s{}_s)^2}\end{aligned}$$

$$\begin{aligned}\rho_4 = {} & \tfrac{(\sigma^I{}_s)^4\theta_I{}^2}{(\sigma^s{}_s)^4} + \tfrac{(\sigma^I{}_s)^3\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^4} + \tfrac{(\sigma^I{}_s)^2(\sigma^I{}_p)^2}{4(\sigma^s{}_s)^4} - \tfrac{2\theta_I{}^2(\sigma^I{}_s)^2}{(\sigma^s{}_s)^2} + \tfrac{\sigma^I{}_s\sigma^I{}_p\theta_I}{(\sigma^s{}_s)^2} + \tfrac{\sigma^I{}_p(\sigma^I{}_s)^2\theta_I}{(\sigma^s{}_s)^2} - \tfrac{\sigma^I{}_s(\sigma^I{}_p)^2}{2(\sigma^s{}_s)^2} \\ & +\tfrac{2\lambda_1\lambda_2\sigma^I{}_s\theta_I}{\sigma^s{}_s} - \tfrac{\lambda_1\sigma^I{}_s\sigma^I{}_p}{\sigma^s{}_s} + \tfrac{\lambda_1{}^2(\sigma^I{}_s)^2\theta_I{}^2}{(\sigma^s{}_s)^2} - \tfrac{\lambda_2(\sigma^I{}_s)^2\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^2} - \tfrac{\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s} + \tfrac{(\sigma^I{}_s)^2(\sigma^I{}_p)^2}{4(\sigma^s{}_s)^2} + \tfrac{\sigma^I{}_s(\sigma^I{}_p)^2}{2\sigma^s{}_s}\end{aligned}$$

$$\begin{aligned}\rho_5 = {} & \frac{(\sigma^I{}_s)^3\sigma_I\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^2} - \frac{2(\sigma^I{}_s)^4\sigma_I\theta_I{}^2}{(\sigma^s{}_s)^2} + \frac{(\sigma^I{}_s)^4\sigma_I\sigma^I{}_p\theta_I}{(\sigma^s{}_s)^2} - \frac{(\sigma^I{}_s)^3\sigma_I(\sigma^I{}_p)^2}{2(\sigma^s{}_s)^2} + 2\theta_I{}^2\sigma_I(\sigma^I{}_s)^2 - \theta_I(\sigma^I{}_s)^2\sigma_I\sigma^I{}_p \\ + {} & \frac{(\sigma^I{}_p)^2(\sigma^I{}_s)^2\sigma_I}{2} - 2\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\lambda_1 - 2(\sigma^I{}_s)^2\sigma_I\theta_I{}^2\lambda_2 + \sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p + \sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p e_1 + (\sigma^I{}_s)^2\sigma_I\sigma^I{}_p\lambda_2\theta_I \\ - {} & \frac{(\sigma^I{}_s)^2\sigma_I(\sigma^I{}_p)^2}{2} - \frac{\sigma^s{}_s\sigma^I{}_s\sigma_I(\sigma^I{}_p)^2}{2} - 2\sigma^s{}_s\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p + \frac{2\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s}\end{aligned}$$

Differentiating equation (4.9) for $\hat G$ with respect to z we obtain a linear PDE in terms of h and its derivatives and using $y = h = -\hat G_z$, we have

$$h_t+(a-br)h_{r_R}+\frac{1}{2}{\sigma_{r_R}}^2h_{r_Rr_R}-Zh_2\left(\frac{1}{2}\rho_5+\rho_1\right)-\left[k+h\left(\frac{1}{2}\rho_5+\rho_1\right)\right]-$$
$$\left(2{\theta_I}^2\frac{1}{2}\left({\sigma^I}_p\right)^2-\theta_I{\sigma^I}_p+\rho_2+\rho_4\right)\left(h_z2z+z^2h_{zz}\right)+\frac{1}{2}\rho_3\left(2h+\frac{h^2hh_{zz}}{{h_z}^2}\right)=0 \tag{4.11}$$

where

$$\rho_1=\tfrac{3}{2}({\sigma^I}_p)^2+\lambda_1{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I+\lambda_2({\sigma^I}_s)^2{\theta_I}^2\sigma_I-\tfrac{1}{2}\lambda_1{\sigma^s}_s{\sigma^I}_s\sigma_I{\sigma^I}_p-\tfrac{1}{2}\lambda_2({\sigma^I}_s)^2\theta_I\sigma_I{\sigma^I}_p$$
$$-\tfrac{1}{2}({\sigma^I}_s)^2\theta_I\sigma_I{\sigma^I}_p-\tfrac{1}{2}{\sigma^s}_s{\sigma^I}_p{\sigma^I}_s\sigma_I\theta_I+\tfrac{1}{4}({\sigma^I}_s)^2({\sigma^I}_p)^2\sigma_I+\tfrac{1}{4}{\sigma^s}_s({\sigma^I}_s)^2{\sigma^I}_s\sigma_I$$
$$+\lambda_1{\sigma^s}_p+\tfrac{\lambda_2({\sigma^I}_s)^2\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}+\tfrac{\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s}-\tfrac{\theta_I\lambda_2({\sigma^I}_s)^2({\sigma^I}_p)^2}{({\sigma^s}_s)^2}-\tfrac{({\sigma^I}_p)^2{\sigma^I}_s}{2{\sigma^s}_s}-u_p$$

$$\rho_2=\tfrac{\lambda_1{\sigma^I}_s{\sigma^I}_p}{2{\sigma^s}_s}-\tfrac{2\lambda_1\lambda_2{\sigma^I}_s\theta_I}{{\sigma^s}_s}-\tfrac{{\lambda_2}^2({\sigma^I}_s)^2{\theta_I}^2}{({\sigma^s}_s)^2}+\tfrac{\lambda_2({\sigma^I}_s)^2\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}+\tfrac{\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s}-\tfrac{3({\sigma^I}_s)^2({\sigma^I}_p)^2}{4{{\sigma^s}_s}^2}$$
$$-\tfrac{3}{2}\tfrac{\theta_I({\sigma^I}_s)^2{\sigma^I}_p}{({\sigma^s}_s)^2}-\tfrac{\theta_I{\sigma^I}_p{\sigma^I}_s}{2{\sigma^s}_s}-\tfrac{{\theta_I}^2({\sigma^I}_s)^2}{({\sigma^s}_s)^2}+\tfrac{\theta_I{\sigma^I}_s\lambda_1}{{\sigma^s}_s}+\tfrac{({\sigma^I}_p)^2({\sigma^I}_s)^2}{2({\sigma^s}_s)^2}+\tfrac{({\sigma^s}_p)^2{\sigma^I}_s}{4{\sigma^s}_s}$$

$$\rho_3=({\sigma^I}_s)^4\sigma_I{\theta_I}^2-({\sigma^I}_s)^4{\sigma_I}^2\theta_I{\sigma^I}_p+\tfrac{({\sigma^I}_s)^4{\sigma_I}^2({\sigma^I}_p)^2}{4}+({\sigma^s}_s)^2({\sigma^I}_s)^2{\sigma_I}^2{\theta_I}^2-({\sigma^s}_s)^2{\sigma^I}_s{\sigma_I}^2\theta_I{\sigma^I}_p$$
$$+2{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p-2{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p+\tfrac{({\sigma^s}_s)^2({\sigma^I}_s)^2{\sigma_I}^2({\sigma^I}_p)^2}{4}-\tfrac{2({\sigma^I}_p)^2({\sigma^I}_s)^2}{({\sigma^s}_s)^2}$$

$$\rho_4=\tfrac{({\sigma^I}_s)^4{\theta_I}^2}{({\sigma^s}_s)^4}+\tfrac{({\sigma^I}_s)^3\theta_I{\sigma^I}_p}{({\sigma^s}_s)^4}+\tfrac{({\sigma^I}_s)^2({\sigma^I}_p)^2}{4({\sigma^s}_s)^4}-\tfrac{2{\theta_I}^2({\sigma^I}_s)^2}{({\sigma^s}_s)^2}+\tfrac{{\sigma^I}_s{\sigma^I}_p\theta_I}{({\sigma^s}_s)^2}+\tfrac{{\sigma^I}_p({\sigma^I}_s)^2\theta_I}{({\sigma^s}_s)^2}-\tfrac{{\sigma^I}_s({\sigma^I}_p)^2}{2({\sigma^s}_s)^2}$$
$$+\tfrac{2\lambda_1\lambda_2{\sigma^I}_s\theta_I}{{\sigma^s}_s}-\tfrac{\lambda_1{\sigma^I}_s{\sigma^I}_p}{{\sigma^s}_s}+\tfrac{{\lambda_1}^2({\sigma^I}_s)^2{\theta_I}^2}{({\sigma^s}_s)^2}-\tfrac{\lambda_2({\sigma^I}_s)^2\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}-\tfrac{\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s}+\tfrac{({\sigma^I}_s)^2({\sigma^I}_p)^2}{4({\sigma^s}_s)^2}+\tfrac{{\sigma^I}_s({\sigma^I}_p)^2}{2{\sigma^s}_s}$$

$$\rho_5=\tfrac{({\sigma^I}_s)^3\sigma_I\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}-\tfrac{2({\sigma^I}_s)^4\sigma_I{\theta_I}^2}{({\sigma^s}_s)^2}+\tfrac{({\sigma^I}_s)^4\sigma_I{\sigma^I}_p\theta_I}{({\sigma^s}_s)^2}-\tfrac{({\sigma^I}_s)^3\sigma_I({\sigma^I}_p)^2}{2({\sigma^s}_s)^2}+2{\theta_I}^2\sigma_I({\sigma^I}_s)^2-\theta_I({\sigma^I}_s)^2\sigma_I{\sigma^I}_p+\tfrac{({\sigma^I}_p)^2({\sigma^I}_s)^2\sigma_I}{2}-2{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I\lambda_1$$
$$-2({\sigma^I}_s)^2\sigma_I{\theta_I}^2\lambda_2+{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p+{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p\lambda_1+({\sigma^I}_s)^2\sigma_I{\sigma^I}_p\lambda_2\theta_I-\tfrac{({\sigma^I}_s)^2\sigma_I({\sigma^I}_p)^2}{2}$$
$$-\tfrac{{\sigma^s}_s{\sigma^I}_s\sigma_I({\sigma^I}_p)^2}{2}-2{\sigma^s}_s\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p+\tfrac{2\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s},\ {\sigma^I}_p={\sigma^s}_p$$

$$u_C=1-u_B-u_S$$

$${u_S}^*=\frac{{\sigma^I}_p}{{\sigma^s}_s}-\left(\frac{-\lambda_1{\sigma^s}_s-\lambda_2{\sigma^I}_s\theta_2+{\sigma^I}_s{\sigma^I}_p+{\sigma^I}_s(\theta_I-\frac{{\sigma^I}_p}{2})}{y({\sigma^s}_s)^2}\right)zh_z \tag{4.12}$$

$${u_B}^*=\frac{{\sigma^I}_p}{\sigma_I}-\frac{{\sigma^I}_p{\sigma^I}_s}{{\sigma^s}_s\sigma_I}+\frac{[({\sigma^I}_s)^2\theta_I-{\sigma^I}_s\sigma_I{\sigma^I}_p-{\sigma^I}_s\lambda_1{\sigma^s}_s-({\sigma^I}_s)^2\lambda_2\theta_I+\frac{({\sigma^I}_s)^2{\sigma^I}_p}{2}+\frac{{\sigma^s}_s{\sigma^I}_s{\sigma^I}_p}{2}]}{h({\sigma^s}_s)^2}zh_z+\frac{\theta_Izh_z}{h\sigma_I}, \tag{4.13}$$

$${\sigma^I}_p={\sigma^s}_p$$

We will now solve (4.11) for h and substitute into (4.12) and (4.13) to obtain the optimal investment strategies.

5. Explicit Solution of the optimal investment strategies for The CRRA Utility Function

Assume the investor takes a power utility function

$$U(x)=\frac{y^p}{p},\quad p<1,\ \ p\neq 0 \tag{5.1}$$

The relative risk aversion of an investor with utility described in (5.1) is constant and (5.1) is a CRRA utility.
From (3.5) we have $h(T,r,z)=(V')^{-1}(z)$ and from (5.1), we have

$$h(T,r_R,z)=z^{\frac{1}{p-1}}$$

We assume a solution to (4.11) with the following form

$$h(t,r,z)=g(t,r)\left[z^{\frac{1}{p-1}}\right]+v(t),\qquad v(T)=0,\ \ g(T,s)=1.$$

Then

$$h_t=g_tz^{\frac{1}{p-1}}+v',\ \ h_z=-\frac{g}{1-p}z^{\left(\frac{1}{p-1}-1\right)},\ \ h_{r_Rz}=-\frac{g_{r_R}}{1-p}z^{\left(\frac{1}{p-1}-1\right)} \tag{5.2}$$

$$h_{zz}=\frac{(2-p)g}{(1-p)^2}z^{(\frac{1}{p-1}-1)},\ h_{r_R}=g_{r_R}z^{\frac{1}{p-1}},\ h_{r_Rr_R}=g_{r_Rr_R}z^{\frac{1}{p-1}}.$$

Substituting (5.2) into (4.11), we have

$$\left\{\begin{array}{l} g_t+(a-br_R)g_{r_R}-\frac{g_{r_Rr_R}\sigma_{r_R}{}^2}{2}+\frac{g(\frac{\rho_5}{2}+\rho_1)}{1-p}-\frac{g\rho_5}{2}-g\rho_1+\frac{2g(2\theta_I{}^2\frac{1}{2}(\sigma^I{}_p)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{1-p} \\ -\frac{(2-p)g(2\theta_I{}^2\frac{1}{2}(\sigma^I{}_p)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{(1-p)^2} \end{array}\right\}z^{\frac{1}{p-1}}+v^I(t)-k-\frac{1}{2}v\rho_5-v\rho_1=0$$

$$+y^I(t)-\rho_1y(t)-\rho_0=0 \tag{5.3}$$

Splitting (5.3), we have

$$v^I(t)-(\frac{1}{2}\rho_5+\rho_1)v(t)-k=0$$

$$g_t+(a-br_R)g_{r_R}-\frac{g_{r_Rr_R}\sigma_{r_R}{}^2}{2}+\frac{g(\frac{\rho_5}{2}+\rho_1)}{1-p}-\frac{g\rho_5}{2}-g\rho_1+\frac{2g(2\theta_I{}^2\frac{1}{2}(\sigma^I{}_p)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{1-p}-\frac{(2-p)g(2\theta_I{}^2\frac{1}{2}(\sigma^I{}_p)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{(1-p)^2}=0 \tag{5.4}$$

Considering the boundary condition,

$$v(T)=0,$$

yields the solution

$$v(t)=-\frac{k}{\rho_*}(1-e^{-\rho_*(T-t)}),$$

where $\rho_3=0,\ \rho_*=\frac{1}{2}\rho_5+\rho_1$
Next, obtain the solution of (5.4), by assuming, a solution of the form

$$g(t,r_R)=M(t)\,e^{N(t)r_R}M(T)=1,\ N(T)=0$$

$$g_{r_R}=M(t)N(t)e^{N(t)r_R},\ g_{r_Rr_R}=M(t)N^2(t)e^{N(t)r_R}\ and\ g_t=r_RM(t)N^I(t)\ e^{N(t)r_R}+M^I(t)\ e^{N(t)r_R} \tag{5.5}$$

Substituting (5.5) into (5.4), we have

$$\begin{aligned} & N_tr_R+\frac{M_t}{M}+Na-Nbr_R+\frac{1}{2}N^2k_1r_R+\frac{1}{2}N^2k_2+\frac{\rho_5}{2(1-p)}+\frac{\rho_1}{1-p}-\frac{1}{2}\rho_5-\frac{1}{2}\rho_1 \\ + & \frac{2(2\theta_I{}^2\frac{1}{2}(\sigma^I{}_p)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{1-p}-\frac{(2-p)(2\theta_I{}^2\frac{1}{2}(\sigma^I{}_p)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{(1-p)^2}=0,\rho_3 \end{aligned}$$

Splitting (5.6), we have

$$\begin{array}{l} \frac{M_t}{M}+Na+\frac{1}{2}N^2k_1+\frac{\rho_5}{2(1-p)}+\frac{\rho_1}{1-p}-\frac{1}{2}\rho_5-\frac{1}{2}\rho_1 \\ +\frac{2(2\theta_I{}^2\frac{1}{2}(\sigma^I{}_p)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{1-p}-\frac{(2-p)(2\theta_I{}^2\frac{1}{2}(\sigma^I{}_p)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{(1-p)^2}=0 \end{array} \tag{5.6}$$

$$N_t-Nb+\frac{1}{2}N^2k_1=0 \tag{5.7}$$

Solving (5.6) and (5.7), we obtain

$$N(t)=\frac{2b[t-T]}{k_1}$$

$$M(t)=c_1e^{\left\{a\pm(a^2-2k_2H)^{\frac{1}{2}}k_2{}^{-1}t\right\}},\ c_1=e^c,$$

$$H=\frac{\rho_5}{2(1-p)}+\frac{\rho_1}{1-p}-\frac{1}{2}\rho_5-\frac{1}{2}\rho_1+\frac{2(2\theta_I{}^2\frac{1}{2}(\sigma^I{}_p)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{1-p}-\frac{(2-p)(2\theta_I{}^2\frac{1}{2}(\sigma^I{}_p)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{(1-p)^2}M(T)=1$$

where

$$d_1=\frac{4b}{2k_1}$$

$$d_2=0$$

$$g(r_R,t)=\frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}exp\frac{2b(t-T)}{k_1}r_R$$

Therefore, the solution of (4.11) becomes

$$h(t,r_R,z)=\frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}z^{\frac{1}{p-1}}-\frac{k}{\rho_*}(1-e^{-\rho_*(T-t)}),$$

where $\rho_3=0,\ \rho_*=\frac{1}{2}\rho_5+\rho_1$

Theorem 5.1. *Let the optimal investment strategies for cash, bond and stock be given as follows* $u_C{}^*=1-u_B{}^*-u_S{}^*$*.Then* $N(t)=\frac{2b[t-T]}{k_1}$ *with* $d_1=\frac{4b}{2k_1}$ *and* $d_2=0$.

Proof. Let

$$\begin{aligned} u_S{}^* &= \frac{\sigma^I{}_p}{\sigma^s{}_s}-\left(\frac{-\lambda_1\sigma^s{}_s-\lambda_2\sigma^I{}_s\theta_2+\sigma^I{}_s\sigma^I{}_p+\sigma^I{}_s(\theta_I-\frac{\sigma^I{}_p}{2})}{\sigma^s{}_s}\right)\frac{1}{p-1} \\ &\times \frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}\left(\frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}-\frac{\rho_*z^{\frac{1}{p-1}}}{k(1-e^{-\rho_*(T-t)})}\right) \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} u_B{}^* &= \frac{\sigma^I{}_p}{\sigma_I}-\frac{\sigma^I{}_p\sigma^I{}_s}{\sigma^s{}_s\sigma_I}+\frac{\left[\left(\sigma^I{}_s\right)^2\theta_I-\sigma^I{}_s\sigma_I\sigma^I{}_p-\sigma^I{}_s\lambda_1\sigma^s{}_s-\left(\sigma^I{}_s\right)^2\lambda_2\theta_I+\frac{\left(\sigma^I{}_s\right)^2\sigma^I{}_p}{2}+\frac{\sigma^s{}_s\sigma^I{}_s\sigma^I{}_p}{2}\right]}{\left(\sigma^s{}_s\right)^2} \\ &\times \frac{1}{p-1}\frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}\left(\frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}-\frac{\rho_*z^{\frac{1}{p-1}}}{k(1-e^{-\rho_*(T-t)})}\right) \\ &+ \frac{\theta_I}{p-1}\frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}\left(\frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}-\frac{\rho_*z^{\frac{1}{p-1}}}{\sigma_Ik(1-e^{-\rho_*(T-t)})}\right., \end{aligned} \tag{5.9}$$

$$\rho_3=0,\ \rho_*=\frac{1}{2}\rho_5+\rho_1,\ \sigma^I{}_p=\sigma^s{}_p$$

Then

$$H=\frac{\rho_5}{2(1-p)}+\frac{\rho_1}{1-p}-\frac{1}{2}\rho_5-\frac{1}{2}\rho_1+\frac{2(2\theta_I{}^2\frac{1}{2}\left(\sigma^I{}_p\right)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{1-p}-\frac{(2-p)(2\theta_I{}^2\frac{1}{2}\left(\sigma^I{}_p\right)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{(1-p)^2} \tag{5.10}$$

$$\begin{aligned} \rho_1 &= \frac{3}{2}\left(\sigma^I{}_p\right)^2+\lambda_1\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I+\lambda_2\left(\sigma^I{}_s\right)^2\theta_I{}^2\sigma_I-\frac{1}{2}\lambda_1\sigma^s{}_s\sigma^I{}_s\sigma_I\sigma^I{}_p-\frac{1}{2}\lambda_2\left(\sigma^I{}_s\right)^2\theta_I\sigma_I\sigma^I{}_p-\frac{1}{2}\left(\sigma^I{}_s\right)^2\theta_I\sigma_I\sigma^I{}_p \\ &- \frac{1}{2}\sigma^s{}_s\sigma^I{}_p\sigma^I{}_s\sigma_I\theta\ldots_I+\frac{1}{4}\left(\sigma^I{}_s\right)^2\left(\sigma^I{}_p\right)^2\sigma_I+\frac{1}{4}\sigma^s{}_s\left(\sigma^I{}_s\right)^2\sigma^I{}_s\sigma_I+\lambda_1\sigma^s{}_p+\frac{\lambda_2\left(\sigma^I{}_s\right)^2\theta_I\sigma^I{}_p}{\left(\sigma^s{}_s\right)^2} \\ &+ \frac{\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s}-\frac{\theta_I\lambda_2\left(\sigma^I{}_s\right)^2\left(\sigma^I{}_p\right)^2}{\left(\sigma^s{}_s\right)^2}-\frac{\left(\sigma^I{}_p\right)^2\sigma^I{}_s}{2\sigma^s{}_s}-u_p. \end{aligned}$$

And

$$\begin{aligned} \rho_2 &= \frac{\lambda_1\sigma^I{}_s\sigma^I{}_p}{2\sigma^s{}_s}-\frac{2\lambda_1\lambda_2\sigma^I{}_s\theta_I}{\sigma^s{}_s}-\frac{\lambda_2{}^2\left(\sigma^I{}_s\right)^2\theta_I{}^2}{\left(\sigma^s{}_s\right)^2}+\frac{\lambda_2\left(\sigma^I{}_s\right)^2\theta_I\sigma^I{}_p}{\left(\sigma^s{}_s\right)^2}+\frac{\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s}-\frac{3\left(\sigma^I{}_s\right)^2\left(\sigma^I{}_p\right)^2}{4\sigma^s{}_s{}^2} \\ &- \frac{3}{2}\frac{\theta_I\left(\sigma^I{}_s\right)^2\sigma^I{}_p}{\left(\sigma^s{}_s\right)^2}-\frac{\theta_1\sigma^I{}_p\sigma^I{}_s}{2\sigma^s{}_s}-\frac{\theta_I{}^2\left(\sigma^I{}_s\right)^2}{\left(\sigma^s{}_s\right)^2}+\frac{\theta_I\sigma^I{}_s\lambda_1}{\sigma^s{}_s}+\frac{\left(\sigma^I{}_p\right)^2\left(\sigma^I{}_s\right)^2}{2\left(\sigma^s{}_s\right)^2}+\frac{\left(\sigma^s{}_p\right)^2\sigma^I{}_s}{4\sigma^s{}_s}. \end{aligned}$$

$$\begin{aligned} \rho_3 &= \left(\sigma^I{}_s\right)^4\sigma_I\theta_I{}^2-\left(\sigma^I{}_s\right)^4\sigma_I{}^2\theta_I\sigma^I{}_p+\frac{\left(\sigma^I{}_s\right)^4\sigma_I{}^2\left(\sigma^I{}_p\right)^2}{4}+(\sigma^s{}_s)^2\left(\sigma^I{}_s\right)^2\sigma_I{}^2\theta_I{}^2-(\sigma^s{}_s)^2\sigma^I{}_s\sigma_I{}^2\theta_I\sigma^I{}_p \\ &+ 2\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p-2\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p+\frac{(\sigma^s{}_s)^2\left(\sigma^I{}_s\right)^2\sigma_I{}^2\left(\sigma^I{}_p\right)^2}{4}-\frac{2\left(\sigma^I{}_p\right)^2\left(\sigma^I{}_s\right)^2}{\left(\sigma^s{}_s\right)^2}, \end{aligned}$$

$$\begin{aligned}\rho_4 \quad = \quad & \frac{(\sigma^I{}_s)^4\theta_I{}^2}{(\sigma^s{}_s)^4}+\frac{(\sigma^I{}_s)^3\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^4}+\frac{(\sigma^I{}_s)^2(\sigma^I{}_p)^2}{4(\sigma^s{}_s)^4}-\frac{2\theta_I{}^2(\sigma^I{}_s)^2}{(\sigma^s{}_s)^2}+\frac{\sigma^I{}_s\sigma^I{}_p\theta_I}{(\sigma^s{}_s)^2}+\frac{\sigma^I{}_p(\sigma^I{}_s)^2\theta_I}{(\sigma^s{}_s)^2}-\frac{\sigma^I{}_s(\sigma^I{}_p)^2}{2(\sigma^s{}_s)^2}+\frac{2\lambda_1\lambda_2\sigma^I{}_s\theta_I}{\sigma^s{}_s}\\ - \quad & \frac{\lambda_1\sigma^I{}_s\sigma^I{}_p}{\sigma^s{}_s}+\frac{\lambda_1{}^2(\sigma^I{}_s)^2\theta_I{}^2}{(\sigma^s{}_s)^2}-\frac{\lambda_2(\sigma^I{}_s)^2\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^2}-\frac{\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s}+\frac{(\sigma^I{}_s)^2(\sigma^I{}_p)^2}{4(\sigma^s{}_s)^2}+\frac{\sigma^I{}_s(\sigma^I{}_p)^2}{2\sigma^s{}_s}\end{aligned}$$

and

$$\begin{aligned}\rho_5 = & \tfrac{(\sigma^I{}_s)^3\sigma_I\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^2}-\tfrac{2(\sigma^I{}_s)^4\sigma_I\theta_I{}^2}{(\sigma^s{}_s)^2}+\tfrac{(\sigma^I{}_s)^4\sigma_I\sigma^I{}_p\theta_I}{(\sigma^s{}_s)^2}-\tfrac{(\sigma^I{}_s)^3\sigma_I(\sigma^I{}_p)^2}{2(\sigma^s{}_s)^2}+2\theta_I{}^2\sigma_I(\sigma^I{}_s)^2\\ & -\theta_I(\sigma^I{}_s)^2\sigma_I\sigma^I{}_p+\tfrac{(\sigma^I{}_p)^2(\sigma^I{}_s)^2\sigma_I}{2}-2\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\lambda_1-2(\sigma^I{}_s)^2\sigma_I\theta_I{}^2\lambda_2+\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p\\ & +\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p\lambda_1+(\sigma^I{}_s)^2\sigma_I\sigma^I{}_p\lambda_2\theta_I-\tfrac{(\sigma^I{}_s)^2\sigma_I(\sigma^I{}_p)^2}{2}-\tfrac{\sigma^s{}_s\sigma^I{}_s\sigma_I(\sigma^I{}_p)^2}{2}-2\sigma^s{}_s\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p+\tfrac{2\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s}.\end{aligned} \tag{5.11}$$

It therefore follows that

$$N(t)=\frac{2b[t-T]}{k_1}$$

$$d_1=\frac{4b}{2k_1}$$

$$d_2=0.$$

□

Remark 5.1
If we let $\sigma^I{}_p=\sigma^I{}_p=\theta_I=0$, the optimal strategies (5.8) and (5.9) would be of the form of the [7]
Recall from [7], the coefficients $d_1,\ d_2$ degenerates to $\frac{4b}{2k_1}$ and zero, in the absence of the coefficient of the CRRA (i.e, as $p\to 0$), however, in this work, even in the presence of the coefficient of CRRA the coefficients $d_1,\ d_2$ are already degenerate. We therefore, conclude that, under the inflationary market, the CRRA utility function has little or no effect on the investment strategy.
The associated optimal investment strategy for a logarithmic utility function, as $p\to 0$ is given by

$$u_S{}^* = \quad \frac{\sigma^I{}_p}{\sigma^s{}_s}+\left(\frac{-\lambda_1\sigma^s{}_s-\lambda_2\sigma^I{}_s\theta_2+\sigma^I{}_s\sigma^I{}_p+\sigma^I{}_s(\theta_I-\frac{\sigma^I{}_p}{2})}{\sigma^s{}_s}\right)\frac{e^{\left\{a\pm\left(a^2-2k_2\rho_1\right)^{\frac12}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm(a^2-2k_2\rho_1)^{\frac12}k_2{}^{-1}T\right\}}}\left(\frac{e^{\left\{a\pm\left(a^2-2k_2\rho_1\right)^{\frac12}k_2{}^{-1}T\right\}}}{e^{\left\{a\pm(a^2-2k_2\rho_1)^{\frac12}k_2{}^{-1}t\right\}}}-\frac{\rho_*}{zk(1-e^{-\rho_*(T-t)})}\right)$$

$$\begin{aligned}u_B{}^* \quad = \quad & \frac{\sigma^I{}_p}{\sigma_I}-\frac{\sigma^I{}_p\sigma^I{}_s}{\sigma^s{}_s\sigma_I}-\frac{\left[(\sigma^I{}_s)^2\theta_I-\sigma^I{}_s\sigma_I\sigma^I{}_p-\sigma^I{}_s\lambda_1\sigma^s{}_s-(\sigma^I{}_s)^2\lambda_2\theta_I+\frac{(\sigma^I{}_s)^2\sigma^I{}_p}{2}+\frac{\sigma^s{}_s\sigma^I{}_s\sigma^I{}_p}{2}\right]}{(\sigma^s{}_s)^2}\\ \times \quad & \frac{e^{\left\{a\pm\left(a^2-2k_2\rho_1\right)^{\frac12}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm(a^2-2k_2\rho_1)^{\frac12}k_2{}^{-1}T\right\}}}\left(\frac{e^{\left\{a\pm\left(a^2-2k_2\rho_1\right)^{\frac12}k_2{}^{-1}T\right\}}}{e^{\left\{a\pm(a^2-2k_2\rho_1)^{\frac12}k_2{}^{-1}t\right\}}}-\frac{\rho_*}{zk(1-e^{-\rho_*(T-t)})}\right)\\ + \quad & \frac{\theta_I}{p-1}\,\frac{e^{\left\{a\pm\left(a^2-2k_2\rho_1\right)^{\frac12}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm(a^2-2k_2\rho_1)^{\frac12}k_2{}^{-1}T\right\}}}\left(\frac{e^{\left\{a\pm\left(a^2-2k_2\rho_1\right)^{\frac12}k_2{}^{-1}T\right\}}}{e^{\left\{a\pm(a^2-2k_2\rho_1)^{\frac12}k_2{}^{-1}t\right\}}}-\frac{\rho_*}{z\sigma_I k\left(1-e^{-\rho_*(T-t)}\right)}\right),\end{aligned}$$

$$\rho_3=0,\ \rho_*=\frac{1}{2}\rho_5+\rho_1,\ \sigma^I{}_p=\sigma^s{}_p,$$

$$H=\frac{\rho_5}{2(1-p)}+\frac{\rho_1}{1-p}-\frac{1}{2}\rho_5-\frac{1}{2}\rho_1+\frac{2(2\theta_I{}^2\frac{1}{2}\left(\sigma^I{}_p\right)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{1-p}-\frac{(2-p)(2\theta_I{}^2\frac{1}{2}\left(\sigma^I{}_p\right)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{(1-p)^2}.$$

6. Discussion and conclusion

6.1. Discussion

From Proposition 5.1, we deduced that in the absence of inflation, proportions of the pension wealth invested in stock and bond would be at least at minimal returns, and the optimal investment strategy, with CRRA utility function, would be constant. From (5.10) and (5.11), we observe that the optimal investment process is lumped with a lot of inflation radicals. More so, from remark 5.1, we discovered that the CRRA utility function does not have much effect on inflation and its effect on wealth investment. From the analysis, we see that the returns on investment of the pension wealth will reduce drastically, therefore, the contributor require the extra measure to dampen the effect of inflation on the investment strategy. From this analysis, we deduce also that the more the returns on optimal investment degenerates, which is as a result of inflation-affected optimal investment strategy, the more the price of stock becomes non-increasing, then the need for more wealth investment in both stock and bond becomes necessary, in order to recover for the lost times, and pull down the price of stock, hence the need for an amortization fund by the plan member becomes necessary.

6.2. Conclusion

The optimal investment strategy for a prospective investor in a DC pension scheme, under the inflationary market, with stochastic salary, under the affine interest rate model has been studied. Relevant to this work, the CRRA utility function was used and we obtained the optimal investment strategies for cash, bond and stock using the Legendre transform and dual theory. More so, the effects of inflation parameters and the coefficient of CRRA utility function were analyzed, with insignificant input on the investment strategy. We conclude, therefore, inflation has significant negative effect on optimal investment strategy, particularly, the CRRA utility function is not constant with the investment strategy.

6.3. Recommendation

From the result obtained in this work, we recommend the investigation of the effect of extra contribution on optimal investment strategy, in DC pension scheme, under inflationary market.

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