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OPTIMAL INEQUALITIES FOR MULTIPLY WARPED PRODUCT SUBMANIFOLDS

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Abstract. In an earlier paper [3] the first author proved that, for any isometric immersion of a warped product $N_1 \times_f N_2$ into a Riemannian m-manifold of constant sectional curvature c, the warping function f satisfies the optimal general inequality:

$$\frac{\Delta f}{f} \leq \frac{(n_1+n_2)^2}{4n_2} H^2 + n_1 c,$$

where $n_i = \dim N_i$, $i = 1, 2$, H^2 is the squared mean curvature, and Δ is the Laplacian operator of N_1. Moreover, he proved in [2] that for a CR-warped product $N_T \times_f N_\perp$ in a Kaehler manifold, the second fundamental form h satisfies $||h||^2 \geq 2n_2||\nabla(\ln f)||^2$, where $n_2 = \dim N_\perp$. In this article we extend these inequalities to multiply warped product manifolds in an arbitrary Riemannian or Kaehlerian manifold. We also provide some examples to illustrate that our results are sharp. Moreover, several applications are also obtained.

1. Introduction

Let $N_1, \cdots, N_k$ be Riemannian manifolds and let $N = N_1 \times \cdots \times N_k$ be the Cartesian product of $N_1, \ldots, N_k$. For each i, denote by $\pi_i : N \to N_i$ the canonical projection of N onto N_i. When there is no confusion, we identify N_i with a horizontal lift of N_i in N via π_i.

If $f_2, \cdots, f_k : N_1 \to \mathbf{R}_+$ are positive-valued functions, then

$$\langle X, Y \rangle := \langle \pi_{1*}X, \pi_{1*}Y \rangle + \sum_{i=2}^{k} (f_i \circ \pi_1)^2 \langle \pi_{i*}X, \pi_{i*}Y \rangle$$

defines a Riemannian metric g on N, called a multiply warped product metric. The product manifold N endowed with this metric is denoted by $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$.

For a multiply warped product manifold $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$, let $\mathcal{D}_i$ denote the distributions obtained from the vectors tangent to N_i (or more precisely, vectors

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tangent to the horizontal lifts of N_i). Assume that

$$\phi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to \tilde{M}$$

is an isometric immersion of a multiply warped product $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into a Riemannian manifold $\tilde{M}$. Denote by h the second fundamental form of ϕ. Then the immersion ϕ is called *mixed totally geodesic* if $h(\mathcal{D}_i, \mathcal{D}_j) = \{0\}$ holds for distinct $i, j \in \{1, \ldots, k\}$.

Let $\psi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to \tilde{M}$ be an isometric immersion of a multiply warped product $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into an arbitrary Riemannian manifold $\tilde{M}$. Denote by $\operatorname{trace} h_i$ the trace of h restricted to N_i, that is

$$\operatorname{trace} h_i = \sum_{\alpha=1}^{n_i} h(e_\alpha, e_\alpha)$$

for some orthonormal frame fields $e_1, \ldots, e_{n_i}$ of $\mathcal{D}_i$.

The first author proved in [3] the following general optimal result:

Theorem A. *Let $\phi : N_1 \times_f N_2 \to R^m(c)$ be an isometric immersion of a warped product into a Riemannian m-manifold of constant sectional curvature c. Then we have*

$$\frac{\Delta f}{f} \leq \frac{(n_1 + n_2)^2}{4n_2} H^2 + n_1 c, \tag{1.1}$$

where $n_i = \dim N_i$, $i = 1, 2$, H^2 is the squared mean curvature of ϕ, and Δ is the Laplacian operator of N_1.

The equality sign of (1.1) holds identically if and only if $\phi : N_1 \times_f N_2 \to R^m(c)$ is a mixed totally geodesic immersion satisfying $\operatorname{trace} h_1 = \operatorname{trace} h_2$.

One purpose of this article is to extend inequality (1.1) to the following general inequality for arbitrary isometric immersions of multiply warped product manifolds into arbitrary Riemannian manifolds.

Theorem 1.1. *Let $\phi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to \tilde{M}^m$ be an isometric immersion of a multiply warped product $N := N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into an arbitrary Riemannian m-manifold. Then we have*

$$\sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j} \leq \frac{n^2}{4} H^2 + n_1(n - n_1) \max \tilde{K}, \quad n = \sum_{j=1}^{k} n_j, \tag{1.2}$$

where $\max \tilde{K}(p)$ denotes the maximum of the sectional curvature function of $\tilde{M}^m$ restricted to 2-plane sections of the tangent space T_pN of N at $p = (p_1, \ldots, p_k)$.

The equality sign of (1.2) holds identically if and only if the following two statements hold:

(1) *ϕ is a mixed totally geodesic immersion satisfying* $\operatorname{trace} h_1 = \cdots = \operatorname{trace} h_k$;

(2) *at each point $p \in N$, the sectional curvature function $\tilde{K}$ of $\tilde{M}^m$ satisfies $\tilde{K}(u, v) = \max \tilde{K}(p)$ for each unit vector u in $T_{p_1}(N_1)$ and each unit vector v in $T_{(p_2, \cdots, p_k)}(N_2 \times \cdots \times N_k)$.*

As an immediate consequence of Theorem 1.1, we have the following.

Corollary 1.1. *Let $\phi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to R^m(c)$ be an isometric immersion of a multiply warped product $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into a Riemannian m-manifold $R^m(c)$ of constant curvature c. Then we have*

$$\sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j} \leq \frac{n^2}{4} H^2 + n_1(n - n_1)c, \quad n = \sum_{j=1}^{k} n_j. \tag{1.3}$$

The equality sign of (1.3) *holds identically if and only if ϕ is a mixed totally geodesic immersion satisfying* $\text{trace}\, h_1 = \cdots = \text{trace}\, h_k$.

A submanifold M of a Kaehler manifold $\tilde{M}$ is called totally real if the almost complex structure J of $\tilde{M}$ carries each tangent space of M into its corresponding normal space. A submanifold N of a Kaehler manifold $\tilde{M}$ is called a CR-warped product if N is the warped $N_T \times_f N_2$ of a holomorphic submanifold N_T and a totally real submanifold N_2 of $\tilde{M}$ (see [2] for details).

The first author proved in [2] that for any CR-warped product in a Kaehler manifold $\tilde{M}$ the second fundamental form h of $N_T \times_f N_2$ in $\tilde{M}$ and the warping function f satisfy the following optimal inequality:

$$||h||^2 \geq 2n_2 ||\nabla(\ln f)||^2, \tag{1.4}$$

where $\nabla(\ln f)$ is the gradient of $\ln f$.

In the following, a multiply warped product $N_T \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ in a Kaehler manifold $\tilde{M}$ is called a *multiply CR-warped product* if N_T is a holomorphic submanifold and $N_\perp := {}_{f_2}N_2 \times \cdots \times_{f_k} N_k$ is a totally real submanifold of $\tilde{M}$.

The second purpose of this article is to extend (1.4) to the following.

Theorem 1.2. *Let $N = N_T \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ be a multiply CR-warped product in an arbitrary Kaehler manifold $\tilde{M}$. Then the second fundamental form h and the warping functions $f_2, \ldots, f_k$ satisfy*

$$||h||^2 \geq 2 \sum_{i=2}^{k} n_i ||\nabla(\ln f_i)||^2. \tag{1.5}$$

The equality sign of (1.5) *holds identically if and only if the following statements hold:*

(i) *N_T is a totally geodesic holomorphic submanifold of $\tilde{M}$;*

(ii) *For each $i \in \{2, \ldots, k\}$, N_i is a totally umbilical submanifold of $\tilde{M}$ with $-\nabla(\ln f_i)$ as its mean curvature vector;*

(iii) *${}_{f_2}N_2 \times \cdots \times_{f_k} N_k$ is immersed as mixed totally geodesic submanifold in $\tilde{M}$; and*

(iv) *For each point $p \in N$, the first normal space* $\text{Im}\, h_p$ *is a subspace of $J(T_pN_\perp)$.*

In the last section we provide examples to illustrate that inequalities (1.2) and (1.5) are both sharp.

2. Preliminaries

Let N be a Riemannian n-manifold isometrically immersed in a Riemannian m-manifold $\tilde{M}^m$. We choose a local field of orthonormal frame $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ in $\tilde{M}^m$ such that, restricted to N, the vectors $e_1, \ldots, e_n$ are tangent to N and $e_{n+1}, \ldots, e_m$ are normal to N.

Let $K(e_i, e_j), 1 \leq i < j \leq n$, denote the sectional curvature of the plane section spanned by e_i and e_j. Then the scalar curvature of N is given by

$$\tau = \sum_{i<j} K(e_i, e_j).$$

Let L be a subspace of T_pN of dimension $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of the r-plane section L is defined by $\tau(L) = \sum_{\alpha<\beta} K(e_\alpha, e_\beta), \quad 1 \leq \alpha, \beta \leq r$.

For a submanifold N in $\tilde{M}^m$ we denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of N and $\tilde{M}^m$, respectively. The Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$$

for vector fields X, Y tangent to N and ξ normal to N, where h denotes the second fundamental form, D the normal connection, and A the shape operator of the submanifold. Let $\{h^r_{ij}\}$, $i, j = 1, \ldots, n$; $r = n+1, \ldots, m$, denote the coefficients of the second fundamental form h with respect to $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$.

The mean curvature vector $\overrightarrow{H}$ is defined by

$$\overrightarrow{H} = \frac{1}{n}\,\mathrm{trace}\, h = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of the tangent bundle TN of N. The squared mean curvature is given by $H^2 = \langle \overrightarrow{H}, \overrightarrow{H} \rangle$, where $\langle\ ,\ \rangle$ denotes the inner product. A submanifold N is called minimal in $\tilde{M}^m$ if the mean curvature vector of N in $\tilde{M}^m$ vanishes identically.

Denote by R and $\tilde{R}$ the Riemann curvature tensor of N and $\tilde{M}^m$, respectively. Then the *equation of Gauss* is given by (see, for instance, [1])

$$\begin{aligned} R(X,Y;Z,W) = {} & \tilde{R}(X,Y;Z,W) + \langle h(X,W), h(Y,Z)\rangle \\ & - \langle h(X,Z), h(Y,W)\rangle, \end{aligned} \tag{2.1}$$

for vectors X, Y, Z, W tangent to N.

Let M be a Riemannian p-manifold and $\{e_1, \ldots, e_p\}$ be an orthonormal frame field on M. For a differentiable function φ on M, the Laplacian of φ is defined by

$$\Delta\varphi = \sum_{j=1}^{p} \{(\nabla_{e_j} e_j)\varphi - e_j e_j \varphi\}.$$

Recall that if M is compact, every eigenvalue of Δ is non-negative.

A warped product immersion is defined as follows: Let $M_1 \times_{\rho_2} M_2 \times \cdots \times_{\rho_k} M_k$ be a warped product and let $\psi_i : N_i \to M_i$, $i = 1, \cdots, k$, be isometric immersions, and define $f_i := \rho_i \circ \psi_1 : N_1 \to \mathbf{R}_+$ for $i = 2, \cdots, k$. Then the map

$$\psi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to M_1 \times_{\rho_2} M_2 \times \cdots \times_{\rho_k} M_k$$

given by $\psi(x_1, \cdots, x_k) := (\psi_1(x_1), \cdots, \psi_k(x_k))$ is an isometric immersion, which is called a warped product immersion [6] (see, also [5]).

A multiply warped product manifold $M_1 \times_{\rho_2} M_2 \times \cdots \times_{\rho_k} M_k$ is called a *multiply warped product representation* of a real space form $R^m(c)$ of constant sectional curvature c if the multiply warped product $M_1 \times_{\rho_2} M_2 \times \cdots \times_{\rho_k} M_k$ is an open dense subset of $R^m(c)$.

Let $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$, $M = M_1 \times_{\rho_2} M_2 \times \cdots \times_{\rho_k} M_k$ and let $\psi : N \to M$ be a warped product immersion. Denote by $\pi_i : N \to N_i$ and $\bar{\pi}_i : M \to M_i, i = 1, \ldots, k$ be the canonical projections. Then the second fundamental forms for h^ψ and h^{ψ_i} ($\psi_i = \psi|_{N_i}$) are related by (cf. Lemma 12 of [6]):

$$\bar{\pi}_{1*} h^\psi(X, Y) = h^{\psi_1}(\pi_{1*}(X), \pi_{1*} Y), \tag{2.2}$$

$$\bar{\pi}_{i*} h^\psi(v, w) = h^{\psi_i 1}(\pi_{i*}(v), \pi_{i*} w), \quad i = 2, \ldots, k, \tag{2.3}$$

for $v, w \in T_p N$ and X, Y tangent to horizontal the lift of N_1 at $p \in N$.

We recall the following result of S. Nölker [6] for later use.

Nölker's Theorem. *Let $f : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to R^N(c)$ be an isometric immersion into a Riemannian manifold of constant curvature c. If f is mixed totally geodesic, then f is locally a warped product immersion.*

Let n be a natural number ≥ 2 and let $n_1, \ldots, n_k$ be k natural numbers. If $n_1 + \cdots + n_k = n$, then $(n_1, \ldots, n_k)$ is called a *partition* of n.

We also need the following general algebraic lemma from [4].

Lemma 2.1. *Let $a_1, \ldots, a_n$ be n real numbers and let k be an integer in $[2, n-1]$. Then, for any partition $(n_1, \ldots, n_k)$ of n, we have*

$$\sum_{1 \leq i_1 < j_1 \leq n_1} a_{i_1} a_{j_1} + \sum_{n_1+1 \leq i_2 < j_2 \leq n_1+n_2} a_{i_2} a_{j_2} + \cdots + \sum_{n_1 \cdots + n_{k-1}+1 \leq i_1 < j_1 \leq n} a_{i_k} a_{j_k}$$
$$\geq \frac{1}{2k} \Big\{ (a_1 + \cdots + a_n)^2 - k(a_1^2 + \cdots + a_n^2) \Big\},$$

with the equality holding if and only if

$$a_1 + \cdots + a_{n_1} = \cdots = a_{n_1 + \cdots + n_{k-1}+1} + \cdots + a_n.$$

3. Proof of Theorem 1.1

Let $\hat{\nabla}, \hat{R}, \cdots, etc.$, denote the Levi-Civita connection, the Riemann curvature tensor, $\cdots, etc.$, of the Riemannian product $N_1 \times N_2 \times \cdots \times N_k$; and by $\nabla, R, \cdots$ the corresponding quantities of the multiply warped product $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$.

Denote by ∇f the gradient of f. If we put $H_i = -\nabla((\ln f_i) \circ \pi_1)$, then we have (cf. [6])

$$\nabla_X Y - \hat{\nabla}_X Y = \sum_{i=2}^{k} \big(\langle X^i, Y^i \rangle H_i - \langle H_i, X \rangle Y^i - \langle H_i, Y \rangle X^i \big), \tag{3.1}$$

$$R(X, Y) - \hat{R}(X, Y) = \sum_{i=2}^{k} (\nabla_{X^1} H_i - \langle H_i, X \rangle H_i) \wedge Y^i \tag{3.2}$$
$$+ \sum_{i=2}^{k} X^i \wedge (\nabla_{Y^1} H_i - \langle H_i, Y \rangle H_i) - \sum_{i,j=2}^{k} \langle H_i, H_j \rangle X^i \wedge Y^j,$$

where X^i denotes the N_i-component of X and $X \wedge Y$ is defined by

$$(X \wedge Y)Z := \langle Z, Y \rangle X - \langle Z, X \rangle Y. \tag{3.3}$$

By applying (3.1) and (3.2), we know that the sectional curvature function of the multiply warped product $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ satisfies

$$
\begin{aligned}
K(X_1, X_i) &= \frac{1}{f_i}\left(\nabla_{X_1} X_1 f_i - X_1^2 f_i\right), \\
K(X_i, X_j) &= -\frac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j}, \quad i, j = 2, \dots, k,
\end{aligned}
\tag{3.4}
$$

for each unit vector X_i tangent to N_i. In particular, (3.4) yields

$$
\Delta f_i = f_i \sum_{j=1}^{n_1} K(e_j, X_i), \quad i = 2, \dots, k, \tag{3.5}
$$

for any unit vector X_i tangent to N_i.

From the equation (2.1) of Gauss, we know that the scalar curvature τ and the squared mean curvature H^2 of N in $\tilde{M}^m$ satisfy

$$
2\tau(p) = n^2 H^2(p) - ||h||^2(p) + 2\tilde{\tau}(T_p(N)), \quad p \in N, \tag{3.6}
$$

where $n_i = \dim N_i$, $n = n_1 + \cdots + n_k$, $||h||^2$ is the squared norm of the second fundamental form h of N in $\tilde{M}^m$ and $\tilde{\tau}(T_p(N))$ the scalar curvature of the subspace $T_p(N)$ in $\tilde{M}^m$. Let us put

$$
\eta = 2\tau - n^2\Big(1 - \frac{1}{k}\Big)H^2 - 2\tilde{\tau}(T_p(N)). \tag{3.7}
$$

Then from (3.6) and (3.7) we find

$$
n^2 H^2 = k\left(\eta + ||h||^2\right). \tag{3.8}
$$

Let us also put

$$
\Delta_1 = \{1, \dots, n_1\}, \ \dots, \ \Delta_k = \{n_1 + \cdots + n_{k-1} + 1, \dots, n_1 + \cdots + n_k\}.
$$

For a given point $p \in N$ we choose an orthonormal basis $e_1, \dots, e_m$ at p such that, for each $j \in \Delta_i$, e_j is tangent to N_i for $i = 1, \dots, k$. Moreover, we choose the normal vector e_{n+1} in the direction of the mean curvature vector at p (When the mean curvature vanishes at p, e_{n+1} can be chosen to be any unit normal vector at p). Then from (3.8) we have

$$
\left(\sum_{A=1}^{n} a_A\right)^2 - k\sum_{A=1}^{n}(a_A)^2 = k\left[\eta + \sum_{A\neq B}(h_{AB}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{A,B=1}^{n}(h_{AB}^r)^2\right], \tag{3.9}
$$

where $a_A = h_{AA}^{n+1}$ and $h_{AB}^r = \langle h(e_A, e_B), e_r \rangle$ with $1 \leq A, B \leq n$ and $n+1 \leq r \leq m$.

Since $(n_1, \dots, n_k)$ is a partition of n, we may apply Lemma 2.1 to (3.9) to obtain

$$
\begin{aligned}
&\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} + \sum_{\alpha_2<\beta_2} a_{\alpha_2}a_{\beta_2} + \cdots + \sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k} \\
&\geq \frac{\eta}{2} + \sum_{A<B}(h_{AB}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{A,B=1}^{n}(h_{AB}^r)^2,
\end{aligned}
\tag{3.10}
$$

where $\alpha_i, \beta_i \in \Delta_i$, $i = 1, \dots, k$.

On the other hand, it follows from the equation of Gauss and (3.5) that

$$\begin{aligned}\sum_{i=2}^{k} n_i \frac{\Delta f_i}{f_i} &= \sum_{j\in\Delta_1}\sum_{\beta\in\Delta_2\cup\cdots\cup\Delta_k} K(e_j,e_\beta)\\ &= \tau - \sum_{1\le j_1<j_2\le n_1} K(e_{j_1},e_{j_2}) - \sum_{n_1+1\le\alpha<\beta\le n} K(e_\alpha,e_\beta)\\ &= \tau - \tilde\tau(\mathcal{D}_1) - \sum_{r=n+1}^{m}\sum_{1\le j_1<j_2\le n_1}\left(h^r_{j_1j_1}h^r_{j_2j_2}-(h^r_{j_1j_2})^2\right)\\ &\quad - \tilde\tau(\mathcal{D}_2\oplus\cdots\oplus\mathcal{D}_k) - \sum_{r=n+1}^{m}\sum_{n_1+1\le\alpha<\beta<n}\left(h^r_{\alpha\alpha}h^r_{\beta\beta}-(h^r_{\alpha\beta})^2\right).\end{aligned}\tag{3.11}$$

Therefore, by combining (3.7), (3.10) and (3.11), we obtain

$$\begin{aligned}\sum_{i=2}^{k} n_i \frac{\Delta f_i}{f_i} \;&\le\; \tau - \tilde\tau(T(N)) + \sum_{j\in\Delta_1}\sum_{\beta\in\Delta_2\cup\cdots\cup\Delta_k}\tilde K(e_j,e_\beta)\\ &\quad -\frac{1}{2}\sum_{r=n+2}^{m}\sum_{A,B=1}^{n}(h^r_{AB})^2 - \sum_{1\le j\le n_1;\, n_1+1\le\alpha\le n}(h^{n+1}_{j\alpha})^2\\ &\quad + \sum_{r=n+2}^{m}\sum_{1\le j_1<j_2\le n_1}\left((h^r_{j_1j_2})^2 - h^r_{j_1j_1}h^r_{j_1j_2}\right)\\ &\quad + \sum_{r=n+2}^{m}\sum_{n_1+1\le\alpha<\beta<n}\left((h^r_{\alpha\beta})^2 - h^r_{\alpha\alpha}h^r_{\beta\beta}\right) - \frac{\eta}{2}\\ &=\; \tau - \tilde\tau(T(N)) + \sum_{j\in\Delta_1}\sum_{\beta\in\Delta_2\cup\cdots\cup\Delta_k}\tilde K(e_j,e_\beta) - \frac{\eta}{2}\\ &\quad - \sum_{r=n+1}^{m}\sum_{1\le j\le n_1}\sum_{n_1+1\le\alpha\le n}(h^r_{j\alpha})^2\\ &\quad -\frac{1}{2}\sum_{r=n+2}^{m}\Big(\sum_{1\le j\le n_1}h^r_{jj}\Big)^2 - \frac{1}{2}\sum_{r=n+2}^{m}\Big(\sum_{n_1+1\le\alpha\le n}h^r_{\alpha\alpha}\Big)^2\\ &\le\; \tau - \tilde\tau(T(N)) + n_1(n-n_1)\max\tilde K - \frac{\eta}{2}\\ &= \frac{n^2}{4}H^2 + n_1(n-n_1)\max\tilde K.\end{aligned}\tag{3.12}$$

This proves inequality (1.2).

If the equality sign of (1.2) holds, then all of inequalities in (3.10) and (3.12) become equalities. Hence, by applying Lemma 2.1, we know that if the equality sign of (1.2) holds, then the immersion is mixed totally geodesic and also trace $h_1 = \cdots =$ trace h_k holds identically. This gives (1) in Theorem (1.1). From the equality case of the last inequality in (3.12) we have (2) as well.

The converse can be easily verified. □

4. **Proof of Theorem 1.2**

For a CR-submanifold M in an arbitrary Kaehler manifold $\tilde{M}$, we denote by ν the complementary orthogonal subbundle of $J\mathcal{D}^{\perp}$ in the normal bundle $T^{\perp}N$. Hence we have the following orthogonal direct sum decomposition:

$$T^{\perp}N = J\mathcal{D}^{\perp} \oplus \nu, \quad J\mathcal{D}^{\perp} \perp \nu.$$

Assume that $N := N_T \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ is a multiply CR-warped product in a Kaehler manifold $\tilde{M}$. Denote ${}_{f_2}N_2 \times \cdots \times_{f_k} N_k$ by $N_{\perp}$. Let $\mathcal{D}_T, \mathcal{D}_2, \ldots, \mathcal{D}_k, \mathcal{D}_{\perp}$ denote the distributions obtained from vectors tangent to $N_T, N_2, \ldots, N_k, N_{\perp}$, respectively. Then we have

$$J\nabla_X Z + J\sigma(X,Z) = -A_{JZ}X + D_X JZ \tag{4.1}$$

for any vector fields X, Y in $\mathcal{D}_T$ and Z in $\mathcal{D}_{\perp}$. Thus, by taking the inner product of (4.1) with JY, we find

$$\langle \nabla_X Z, Y \rangle = -\langle A_{JZ}X, JY \rangle = -\langle \sigma(X, JY), JZ \rangle . \tag{4.2}$$

Since $N_T \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ is a multiply warped product, (3.1) implies that N_T is totally geodesic in N. Thus we have

$$\langle \nabla_X Z, Y \rangle = \langle \nabla_X Y, Z \rangle = 0 \tag{4.3}$$

for vector fields X, Y in $\mathcal{D}_T$ and Z in $\mathcal{D}_{\perp}$. By combining (4.2) and (4.3) we obtain

$$\langle \sigma(X,Y), JZ \rangle = 0. \tag{4.4}$$

for vector fields X, Y in $\mathcal{D}_T$ and Z in $\mathcal{D}_{\perp}$.

It follows from (3.1) that,

$$\nabla_X Z = \sum_{i=2}^{k} (X(\ln f_i)) Z^i \tag{4.5}$$

for vector fields X, Y in $\mathcal{D}_T$ and Z in $\mathcal{D}_{\perp}$, where Z^i is the N_i-component of Z. By applying (4.5) and Lemma 2.1 of [2] we find

$$\begin{aligned} \langle h(JX,Z), JW \rangle &= -\langle JA_{JW}Z, X \rangle = -\langle \nabla_Z W, X \rangle \\ &= \sum_{i=2}^{n} (X(\ln f_i)) \left\langle Z^i, W^i \right\rangle \end{aligned} \tag{4.6}$$

for vector fields X in $\mathcal{D}_T$ and Z, W in $\mathcal{D}_{\perp}$.

For a given point $p \in N$ we may choose an orthonormal basis $e_1, \ldots, e_n$ at p such that e_α is tangent to N_i for each $\alpha \in \Delta_i,\ i = 2, \ldots, k$. For each $i \in \{2, \ldots, k\}$, (4.6) implies that

$$\sum_{\alpha \in \Delta_i} \langle h(JX, e_\alpha), Je_\alpha \rangle = n_i \sum_{i=2}^{n} X(\ln f_i). \tag{4.7}$$

Now, inequality (1.5) follows from (4.4) and (4.7).

It follows from (4.7) that the equality sign of (1.5) holds identically if and only if we have

$$h(\mathcal{D}_T, \mathcal{D}_T) = \{0\}, \ \ h(\mathcal{D}_{\perp}, \mathcal{D}_{\perp}) = \{0\}, \ \ h(\mathcal{D}_T, \mathcal{D}_{\perp}) \subset J\mathcal{D}_{\perp}. \tag{4.8}$$

Because N_T is totally geodesic in $N_T \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$, the first condition in (4.8) implies that N_T is totally geodesic in $\tilde{M}$. This gives statement (i).

From (3.1) we know that, for any $2 \leq i \neq j \leq k$, and any vector field Z_i in $\mathcal{D}_i$ and Z_j in $\mathcal{Z}_j$, we have $\nabla_{Z_i} Z_j = 0$. This yields

$$\langle \nabla_{Z_i} W_i, Z_j \rangle = 0$$

Thus, if $\hat{h}_i$ denotes the second fundamental form of N_i in N, we have

$$\hat{h}(\mathcal{D}_i, \mathcal{D}_i) \subset \mathcal{D}_T. \tag{4.9}$$

From (4.6) and (4.9) we find

$$\hat{h}(Z_i, W_i) = -(X(\ln f_i)) \langle Z_i, W_i \rangle \tag{4.10}$$

for Z_i, W_i tangent to N_i. Therefore, by combining the first condition in (4.8) and (4.10), we obtain statement (ii).

Statement (iii) follows immediately from (3.1) and the second condition in (4.8). The last statement follows from (4.8).

The converse is easy to verify. □

5. **Remarks and applications**

Combining Theorem 1.1. and Nölker's theorem gives immediately the following.

Corollary 5.1. *Let $\phi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to R^m(c)$ be an isometric immersion of the multiply warped product $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into a Riemannian m-manifold of constant curvature c. If we have*

$$\sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j} = \frac{n^2}{4} H^2 + n_1(n - n_1)c,$$

then ϕ is a warped product immersion.

By applying Theorem 1.1 we have the following.

Corollary 5.2. *If $f_2, \ldots, f_k$ are harmonic functions on N_1 or eigenfunctions of the Laplacian Δ on N_1 with positive eigenvalues, then the multiply warped product manifold $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ cannot be isometrically immersed into any Riemannian manifold of negative sectional curvature as a minimal submanifold.*

Corollary 5.3. *If $f_2, \ldots, f_k$ are eigenfunctions of the Laplacian Δ on N_1 with nonnegative eigenvalues and at least one of $f_2, \ldots, f_k$ is non-harmonic, then the multiply warped product manifold $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ cannot be isometrically immersed into any Riemannian manifold of non-positive sectional curvature as a minimal submanifold.*

By applying Theorem 1.1 we also have the following.

Corollary 5.4. *If $f_2, \ldots, f_k$ are harmonic functions on N_1, then every isometric minimal immersion of the multiply warped product manifold $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into a Euclidean space is a warped product immersion.*

Proof. Assume that $\phi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to \mathbb{E}^m$ is an isometric immersion of a multiply warped product $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into Euclidean m-space. If $f_2, \ldots, f_k$ are harmonic functions on N_1, then the minimality of $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ in the Euclidean space and the harmonicity of $f_2, \ldots, f_k$ imply that the equality sign of (1.2) holds identically. Thus, the immersion is mixed totally geodesic according to Theorem 1.1. Hence, by applying Nölker's theorem, we know that ϕ is locally a warped product immersion. □

Since the proof of Theorem 1.1 bases only on the equation of Gauss, the same proof as Theorem 1.1 yields the following.

Corollary 5.5. *Let* $\phi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to \tilde{M}^m(4c)$ *be a totally real isometric immersion of the multiply warped product* $N_1 \times_{f_2} N_2 \times \cdots \times_{\sigma_k} N_k$ *into a complex space form of constant holomorphic sectional curvature* $4c$ (*or a quaternionic space form of constant quaternionic sectional curvature* $4c$). *Then we have*

$$\sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j} \le \frac{n^2}{4} H^2 + n_1(n - n_1)c, \quad n = \sum_{j=0}^{k} n_j.$$

6. Examples and remark

The following two examples show that inequalities (1.2) and (1.5) are both sharp.

Example 6.1. Let $M_1 \times_{\rho_2} M_2 \times \cdots \times_{\rho_k} M_k$ be a *multiply warped product representation* of a real space form $R^m(c)$. Assume that $\psi_1 : N_1 \to M_1$ is a minimal immersion of N_1 into M_1 and let $f_2, \ldots, f_k$ be the restrictions of $\rho_2, \ldots, \rho_k$ on N_1. Then the following warped product immersion:

$$\psi = (\psi_1, id, \ldots, id) : N_1 \times_{f_2} M_2 \times \cdots \times_{f_k} M_k \to M_1 \times_{\rho_2} M_2 \times \cdots \times_{\rho_k} M_k \subset R^m(c)$$

is a mixed totally geodesic warped product submanifold of $R^m(c)$ which satisfies the condition:

$$\operatorname{trace} h_1 = \cdots = \operatorname{trace} h_k = 0.$$

Thus, the immersion ψ satisfies the equality case of (1.2) according to Theorem 1.1. Therefore, inequality (1.2) is optimal.

Example 6.2. Assume that h and k are natural numbers with $h \ge k$. Let $N_T = \mathbf{C}^h := \{(z_1, \ldots, z_h) : z_1, \ldots, z_h \in \mathbf{C}\}$ and let $N_i = S^{n_i}$ denote the unit n_i-spheres for $i = 2, \ldots, k$.

Consider the immersion ψ of $N_T \times S^{n_2} \times \cdots \times S^{n_k}$ into $\mathbf{C}^{h+n_2+\cdots+n_k}$ defined by

$$\psi = (z_1 w_{2,0}, \ldots, z_1 w_{2,n_2}, \ldots, z_k w_{k,0}, \ldots, z_k w_{k,n_k}, z_{k+1}, \ldots, z_h),$$

where $(w_{i,0}, \ldots, w_{i,n_i}) \in \mathbf{R}^{n_i+1}$ satisfy $\sum_{\alpha=0}^{n_i} w_{i,\alpha}^2 = 1$ for $i = 2, \ldots, k$.

It is easy to see that the product manifold $\mathbf{C}^h \times S^{n_2} \times \cdots \times S^{n_k}$ endowed with the induced metric via ψ is the multiply warped product manifold $\mathbf{C}^h \times_{f_2} S^{n_2} \times \cdots \times_{f_k} S^{n_k}$ with $f_i = |z_i|$. Moreover, with respect to the canonical complex structure of $\mathbf{C}^{h+n_2+\cdots+n_k}$, the immersion ψ is a multiply CR-warped product submanifold.

A straightforward computation shows that this example of multiply CR-warped product submanifold satisfies the equality case of (1.5). This examples shows that inequality (1.5) is also optimal.

Example 6.3. Let $M_1 \times_{f_2} M_2 \times \cdots \times_{f_k} M_k$ be a *multiply warped product representation* of a Riemannian m-manifold $R^m(c)$ of constant curvature c. Assume that $\psi^i : N_i \to M_i$, $i = 2, \ldots, k$, are minimal immersions. Then it follows from (2.2) and (2.3) that the immersion:

$$\psi : M_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to M_1 \times_{f_2} M_2 \times \cdots \times_{f_k} M_k$$

defined by $\psi = (id, \psi_2, \ldots, \psi_k)$ is a minimal isometric immersion of the multiply warped product manifold $M_1 \times_{\rho_2} N_2 \times \cdots \times_{\rho_k} N_k$ into $R^m(c)$.

On the other hand, since $M_1 \times_{f_2} M_2 \times \cdots \times_{f_k} M_k$ is of constant curvature c, it follows from (3.4) that the warping functions $f_2, \ldots, f_k$ are eigenfunctions of the Laplacian Δ of M_1 with eigenvalues given by $n_2 c, \ldots, n_k c$, respectively. In particular, if $c = 0$ the warping functions $f_2, \ldots, f_k$ are harmonic functions.

Example 6.3 illustrates that the warping functions $f_2, \ldots, f_k$ in Corollary 5.5 cannot be replaced by eigenfunctions with negative eigenvalue. Moreover, the target space in Corollary 5.5 cannot be replaced either by Euclidean space or by spheres. Therefore Corollary 5.5 is sharp.

Example 6.3 with $c = 0$ implies that Corollary 5.3 is sharp as well.

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