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Complex Multivariate Montgomery Type Identity Leading to Complex Multivariate Ostrowski and Grüss Inequalities

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Abstract
We give a general complex multivariate Montgomery type identity which is a representation formula for a complex multivariate function. Using it we produce general tight complex multivariate high order Ostrowski and Grüss type inequalities. The estimates involve L_p norms, any $1 \leq p \leq \infty$. We include also applications.

Keywords: Multivariate complex integral, Multivariate complex continuous functions, Multivariate complex analytic functions, Multivariate complex montgomery identity, Multivariate complex Ostrowski and Grüss inequalities.
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1. Introduction

Our motivation comes from the following results:

Theorem 1.1. *(A. Ostrowski, 1938 [1]). Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a),$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

Theorem 1.2. *(G. Grüss, 1934 [2]). Let $f,g : [a,b] \to \mathbb{R}$ be Lebesgue integrable functions, and $m,M,n,N \in \mathbb{R}$ such that: $-\infty < m \leq f \leq M < \infty$, $-\infty < n \leq g \leq N < \infty$, a.e. on $[a,b]$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,g(t)\,dt - \left(\frac{1}{b-a} \int_a^b f(t)\,dt \right) \left(\frac{1}{b-a} \int_a^b g(t)\,dt \right) \right| \leq \frac{1}{4}(M-m)(N-n),$$

with the constant $\frac{1}{4}$ being the best possible.

Let $f \in C^1([a,b])$ and the kernel $p : [a,b]^2 \to \mathbb{R}$ be such that

$$p(x,t) := \begin{cases} t-a, & \text{if } t \in [a,x], \\ t-b, & \text{if } t \in (x,b]. \end{cases}$$

Then, we have the basic Montgomery integral identity [3, p. 565],

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b p(x,t) f'(t)\,dt, \ \ \forall\, x \in [a,b].$$

In order to describe complex extensions of Ostrowski and Grüss inequalities using the complex integral we need the following preparation.

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz,$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,\left|z'(t)\right| dt$$

and the length of the curve γ is then

$$l(\gamma) = \int_{\gamma_{u,w}} |dz| := \int_a^b \left|z'(t)\right| dt.$$

Let f and g be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the integration by parts formula

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz.$$

We recall also the triangle inequality for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\, l(\gamma),$$

where $\|f\|_{\gamma,\infty} := \sup_{z \in \gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz|\right)^{\frac{1}{p}}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [l(\gamma)]^{\frac{1}{q}}\,\|f\|_{\gamma,p}.$$

First, we mention a Complex extension of Ostrowski inequality to the complex integral by providing upper bounds for the quantity

$$\left|f(v)(w-u)-\int_{\gamma} f(z)\,dz\right|$$

under the assumption that γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$, $u = z(a)$, $v = z(x)$ with $x \in (a,b)$ and $w = z(b)$ while f is holomorphic in G, an open domain and $\gamma \subset G$.

Secondly, we mention a Complex extension of Grüss inequality:

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ, we consider the complex Čebyšev functional defined by

$$\mathscr{D}_{\gamma}(f,g) := \frac{1}{w-u}\int_{\gamma} f(z)\,g(z)\,dz - \frac{1}{w-u}\int_{\gamma} f(z)\,dz \frac{1}{w-u}\int_{\gamma} g(z)\,dz.$$

We display upper bounds to $\left|\mathscr{D}_{\gamma}(f,g)\right|$.

We have the following results for functions of a complex variable:

Theorem 1.3. *(S. Dragomir, 2019 [4]). Let f be holomorhic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a,b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$,*

$$\left|f(v)(w-u)-\int_{\gamma} f(z)\,dz\right| \leq \left\|f'\right\|_{\gamma_{u,v};\infty}\int_{\gamma_{u,v}}|z-u|\,|dz| + \left\|f'\right\|_{\gamma_{v,w};\infty}\int_{\gamma_{u,w}}|z-w|\,|dz| \leq$$

$$\left[\int_{\gamma_{u,v}}|z-u|\,|dz| + \int_{\gamma_{u,w}}|z-w|\,|dz|\right]\left\|f'\right\|_{\gamma_{u,w};\infty},$$

and

$$\left|f(v)(w-u)-\int_{\gamma} f(z)\,dz\right| \leq \max_{z\in\gamma_{u,v}}|z-u|\left\|f'\right\|_{\gamma_{u,v};1} + \max_{z\in\gamma_{v,w}}|z-w|\left\|f'\right\|_{\gamma_{v,w};1} \leq$$

$$\max\left\{\max_{z\in\gamma_{u,v}}|z-u|, \max_{z\in\gamma_{v,w}}|z-w|\right\}\left\|f'\right\|_{\gamma_{u,w};1}.$$

If $p,q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, then

$$\left|f(v)(w-u)-\int_{\gamma} f(z)\,dz\right| \leq \left(\int_{\gamma_{u,v}}|z-u|^{q}\,|dz|\right)^{\frac{1}{q}}\left\|f'\right\|_{\gamma_{u,v};p} + \left(\int_{\gamma_{u,w}}|z-w|^{q}\,|dz|\right)^{\frac{1}{q}}\left\|f'\right\|_{\gamma_{v,w};p} \leq$$

$$\left(\int_{\gamma_{u,v}}|z-u|^{q}\,|dz| + \int_{\gamma_{u,w}}|z-w|^{q}\,|dz|\right)^{\frac{1}{q}}\left\|f'\right\|_{\gamma_{u,w};p}.$$

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$. Now, for $\phi, \Phi \in \mathbb{C}$ define the set of complex-valued functions

$$\overline{\Delta}_{\gamma}(\phi,\Phi) := \left\{f : \gamma \to \mathbb{C} \middle| \left|f(z) - \frac{\phi+\Phi}{2}\right| \leq \frac{1}{2}|\Phi-\phi| \text{ for each } z \in \gamma\right\}.$$

We have the following complex Grüss type inequalities:

Theorem 1.4. *(S. Dragomir, 2018 [5]). Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ and there exist $\phi, \Phi, \psi, \Psi \in \mathbb{C}$, $\phi \neq \Phi$, $\psi \neq \Psi$ such that $f \in \overline{\Delta}_{\gamma}(\phi,\Phi)$ and $g \in \overline{\Delta}_{\gamma}(\psi,\Psi)$ then*

$$\left|\mathscr{D}_{\gamma}(f,g)\right| \leq \frac{1}{4}|\Phi-\phi|\,|\Psi-\psi|\,\frac{l^{2}(\gamma)}{|w-u|^{2}}.$$

If the path γ is a segment $[u,w]$ connecting two distinct points u and w in $\mathbb{C}$ then we write $\int_\gamma f(z)\,dz$ as $\int_u^w f(z)\,dz$.

If f, g are continuous on $[u,w]$ and there exists $\phi,\Phi,\psi,\Psi\in\mathbb{C}$, $\phi\neq\Phi$, $\psi\neq\Psi$ such that $f\in\overline{\Delta}_{[u,w]}(\phi,\Phi)$ and $g\in\overline{\Delta}_{[u,w]}(\psi,\Psi)$ then

$$\left|\frac{1}{w-u}\int_u^w f(z)\,g(z)\,dz-\frac{1}{w-u}\int_u^w f(z)\,dz\frac{1}{w-u}\int_u^w g(z)\,dz\right|\leq\frac{1}{4}\left|\Phi-\phi\right|\left|\Psi-\psi\right|.$$

We will use the complex Montgomery identity which follows:

Theorem 1.5. *(S. Dragomir, 2018 [4]) Let f be holomorphic in G, an open domain and suppose $\gamma\subset G$ is a smooth path from $z(a)=u$ to $z(b)=w$. If $v=z(t)$ with $t\in[a,b]$, then $\gamma_{u,w}=\gamma_{u,v}\cup\gamma_{v,w}$, and*

$$f(v)=\frac{1}{w-u}\left[\int_\gamma f(z)\,dz+\int_{\gamma_{u,v}}(z-u)\,f'(z)\,dz+\int_{\gamma_{v,w}}(z-w)\,f'(z)\,dz\right].$$

Define

$$p(v,z):=\begin{cases} z-u, & \text{if } z\in\gamma_{u,v}\\ z-w, & \text{if } z\in\gamma_{v,w}.\end{cases}$$

Thus, it holds

$$f(v)=\frac{1}{w-u}\int_\gamma f(z)\,dz+\frac{1}{w-u}\int_\gamma p(v,z)\,f'(z)\,dz, \tag{1.1}$$

a form which we will use a lot in this article.

Representation formula (1.1) is the main inspiration to write this article.

We will use (1.1) to derive a multivariate Complex Montgomery type identity then based on it, we will produce Complex multivariate Ostrowski and Grüss type inequalities.

For the last we need:

Definition 1.6. *Here we extend the notion of line (curve) integral into multivariate case. Let γ_j, $j=1,\dots,m$, be a smooth path parametrized by $z_j(t_j)$, $t_j\in[a_j,b_j]$ and f is a complex valued function which is continuous on $\prod_{j=1}^m\gamma_j\subseteq\mathbb{C}^m$. Put $z_j(a_j)=u_j$ and $z_j(b_j)=w_j$, with $u_j,w_j\in\mathbb{C}$, $j=1,\dots,m$.*

We define the complex multivariate integral of f on $\prod_{j=1}^m\gamma_j:=\prod_{j=1}^m\gamma_{u_j,w_j}$ as

$$\int_{\gamma_1}\dots\int_{\gamma_m}f(z_1,\dots,z_m)\,dz_1\dots dz_m:=\int_{\prod_{j=1}^m\gamma_j}f(z_1,\dots,z_m)\,dz_1\dots dz_m:=$$

$$\int_{\gamma_{u_1,w_1}}\dots\int_{\gamma_{u_m,w_m}}f(z_1,\dots,z_m)\,dz_1\dots dz_m:=\int_{\prod_{j=1}^m\gamma_{u_j,w_j}}f(z_1,\dots,z_m)\,dz_1\dots dz_m:=$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\dots\int_{a_m}^{b_m}f(z_1(t_1),\dots,z_m(t_m))\prod_{j=1}^m z_j'(t_j)\,dt_1\dots dt_m. \tag{1.2}$$

We make

Remark 1.7. *Clearly here $z_j\in C^1([a_j,b_j],\mathbb{C})$, $j=1,\dots,m$. The integrand in (1.2) is a continuous complex valued function over $\prod_{j=1}^m[a_j,b_j]$. Therefore $|f(z_1(t_1),\dots,z_m(t_m))|\prod_{j=1}^m z_j'(t_j)$ is also continuous but from $\prod_{j=1}^m[a_j,b_j]$ into $\mathbb{R}$, hence it is bounded. Consequently it holds*

$$\int_{\prod_{j=1}^m[a_j,b_j]}|f(z_1(t_1),\dots,z_m(t_m))|\prod_{j=1}^m\left|z_j'(t_j)\right|\prod_{j=1}^m dt_j<+\infty.$$

Therefore, by Fubini's theorem, the order integration in (1.2) is immaterial.

Clearly it holds

$$\left|\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} f(z_1(t_1),\ldots,z_m(t_m))\prod_{j=1}^{m} z_j'(t_j)\,dt_1\ldots dt_m\right| \le \int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} |f(z_1(t_1),\ldots,z_m(t_m))|\prod_{j=1}^{m}\left|z_j'(t_j)\right| dt_1\ldots dt_m. \tag{1.3}$$

We also define the integral with respect to arc-lengths

$$\int_{\prod_{j=1}^{m}\gamma_{u_j,w_j}} f(z_1,\ldots,z_m)\,|dz_1|\,|dz_2|\ldots|dz_m| := \int_{\prod_{j=1}^{m}[a_j,b_j]} f(z_1(t_1),\ldots,z_m(t_m))\prod_{j=1}^{m}\left|z_j'(t_j)\right| dt_1\ldots dt_m. \tag{1.4}$$

It holds (by (1.3), (1.4))

$$\left|\int_{\prod_{j=1}^{m}\gamma_j} f(z_1,\ldots,z_m)\,dz_1\ldots dz_m\right| \le \int_{\prod_{j=1}^{m}\gamma_{u_j,w_j}} |f(z_1,\ldots,z_m)|\,|dz_1|\,|dz_2|\ldots|dz_m| \le \|f\|_{\prod_{j=1}^{m}\gamma_j,\infty}\prod_{j=1}^{m} l(\gamma_j),$$

where

$$\|f\|_{\prod_{j=1}^{m}\gamma_j,\infty} := \sup_{(z_1,\ldots,z_m)\in\prod_{j=1}^{m}\gamma_j} |f(z_1,\ldots,z_m)|,$$

and

$$l(\gamma_j) = \int_{\gamma_{u_j,w_j}} \left|dz_j\right| = \int_{a_j}^{b_j}\left|z_j'(t_j)\right| dt_j,\ \ j=1,\ldots,m.$$

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\prod_{j=1}^{m}\gamma_j,p} := \left(\int_{\prod_{j=1}^{m}\gamma_j} |f(z_1,\ldots,z_m)|^p\,|dz_1|\,|dz_2|\ldots|dz_m|\right)^{\frac{1}{p}}.$$

For $p=1$ we have

$$\|f\|_{\prod_{j=1}^{m}\gamma_j,1} := \int_{\prod_{j=1}^{m}\gamma_j} |f(z_1,\ldots,z_m)|\,|dz_1|\,|dz_2|\ldots|dz_m|.$$

If $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$\|f\|_{\prod_{j=1}^{m}\gamma_j,1} \le \left(\prod_{j=1}^{m} l(\gamma_j)\right)^{\frac{1}{q}}\|f\|_{\prod_{j=1}^{m}\gamma_j,p}.$$

2. Main results

We start by presenting a complex trivariate Montgomery type representation identity of complex functions:

Theorem 2.1. *Let $f:\prod_{j=1}^{3} D_j \subseteq \mathbb{C}^3 \to \mathbb{C}$ be a continuous function that is analytic per coordinate on the domain D_j, $j=1,2,3$, and $x=(x_1,x_2,x_3)\in\prod_{j=1}^{3} D_j$. For $j=1,2,3$, suppose $\gamma_j \subset D_j$ is a smooth path parametrized by $z_j(t_j)$, $t_j\in[a_j,b_j]$ with $z_j(a_j)=u_j$, $z_j(t_j)=x_j$ and $z_j(b_j)=w_j$, where $u_j,w_j\in D_j$, $u_j\neq w_j$. Assume also that all partial derivatives of f up to order three are continuous functions on $\prod_{j=1}^{3} D_j$.*

Here we define the kernels for $i = 1,2,3$, $p_i : \gamma_i^2 \to \mathbb{C}$

$$p_i(x_i,s_i) := \begin{cases} s_i - u_i, & \text{if } s_i \in \gamma_{u_i,x_i}, \\ s_i - w_i, & \text{if } s_i \in \gamma_{x_i,w_i}. \end{cases}$$

Then

$$f(x_1,x_2,x_3) = \frac{1}{\prod_{i=1}^{3}(w_i - u_i)} \left\{ \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3} f(s_1,s_2,s_3)\,ds_3 ds_2 ds_1 + \sum_{j=1}^{3}\left(\int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3} p_j(x_j,s_j)\frac{\partial f(s_1,s_2,s_3)}{\partial s_j} ds_3 ds_2 ds_1 \right) \right.$$

$$\left. + \sum_{\substack{l=1 \\ j<k}}^{3}\left(\int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3} p_j(x_j,s_j)\,p_k(x_k,s_k)\frac{\partial^2 f(s_1,s_2,s_3)}{\partial s_k \partial s_j} ds_3 ds_2 ds_1 \right)(l) + \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3}\left(\prod_{i=1}^{3} p_i(x_i,s_i) \right)\frac{\partial^3 f(s_1,s_2,s_3)}{\partial s_3 \partial s_2 \partial s_1} ds_3 ds_2 ds_1 \right\}. \tag{2.1}$$

Above l *counts* $(j,k) : j<k;\ j,k \in \{1,2,3\}$.

Proof. Here we apply (1.1) repeatedly.

First we see that

$$f(x_1,x_2,x_3) = A_0 + B_0,$$

where

$$A_0 := \frac{1}{w_1 - u_1}\int_{\gamma_1} f(s_1,x_2,x_3)\,ds_1,$$

and

$$B_0 := \frac{1}{w_1 - u_1}\int_{\gamma_1} p_1(x_1,s_1)\frac{\partial f(s_1,x_2,x_3)}{\partial s_1} ds_1.$$

Furthermore we have

$$f(s_1,x_2,x_3) = A_1 + B_1,$$

where

$$A_1 := \frac{1}{w_2 - u_2}\int_{\gamma_2} f(s_1,s_2,x_3)\,ds_2,$$

and

$$B_1 := \frac{1}{w_2 - u_2}\int_{\gamma_2} p_2(x_2,s_2)\frac{\partial f(s_1,s_2,x_3)}{\partial s_2} ds_2.$$

Also we find that

$$f(s_1,s_2,x_3) = \frac{1}{w_3 - u_3}\int_{\gamma_3} f(s_1,s_2,s_3)\,ds_3 +$$

$$\frac{1}{w_3 - u_3}\int_{\gamma_3} p_3(x_3,s_3)\frac{\partial f(s_1,s_2,s_3)}{\partial s_3} ds_3.$$

Next we put things together, and we derive

$$A_1 = \frac{1}{(w_2 - u_2)(w_3 - u_3)}\int_{\gamma_2}\int_{\gamma_3} f(s_1,s_2,s_3)\,ds_3 ds_2 + \frac{1}{(w_2 - u_2)(w_3 - u_3)}\int_{\gamma_2}\int_{\gamma_3} p_3(x_3,s_3)\frac{\partial f(s_1,s_2,s_3)}{\partial s_3} ds_3 ds_2.$$

And we get

$$A_0 = \frac{1}{\prod_{i=1}^{3}(w_i - u_i)} \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3} f(s_1,s_2,s_3)\,ds_3ds_2ds_1 + \frac{1}{\prod_{i=1}^{3}(w_i - u_i)} \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3} p_3(x_3,s_3)\frac{\partial f(s_1,s_2,s_3)}{\partial s_3}ds_3ds_2ds_1$$

$$+\frac{1}{(w_1-u_1)(w_2-u_2)}\int_{\gamma_1}\int_{\gamma_2} p_2(x_2,s_2)\frac{\partial f(s_1,s_2,x_3)}{\partial s_2}ds_2ds_1.$$

Also we obtain

$$\frac{\partial f(s_1,s_2,x_3)}{\partial s_2} = \frac{1}{w_3-u_3}\int_{\gamma_3}\frac{\partial f(s_1,s_2,s_3)}{\partial s_2}ds_3 + \frac{1}{w_3-u_3}\int_{\gamma_3} p_3(x_3,s_3)\frac{\partial^2 f(s_1,s_2,s_3)}{\partial s_3\partial s_2}ds_3.$$

Therefore we get

$$A_0 = \frac{1}{\prod_{i=1}^{3}(w_i - u_i)} \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3} f(s_1,s_2,s_3)\,ds_3ds_2ds_1 + \frac{1}{\prod_{i=1}^{3}(w_i - u_i)} \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3} p_3(x_3,s_3)\frac{\partial f(s_1,s_2,s_3)}{\partial s_3}ds_3ds_2ds_1$$

$$+\frac{1}{\prod_{i=1}^{3}(w_i - u_i)} \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3} p_2(x_2,s_2)\frac{\partial f(s_1,s_2,s_3)}{\partial s_2}ds_3ds_2ds_1 + \frac{1}{\prod_{i=1}^{3}(w_i - u_i)} \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3} p_2(x_2,s_2)\,p_3(x_3,s_3)\frac{\partial^2 f(s_1,s_2,s_3)}{\partial s_3\partial s_2}ds_3ds_2ds_1.$$

Similarly we obtain that

$$B_0 = \frac{1}{\prod_{i=1}^{3}(w_i - u_i)} \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3} p_1(x_1,s_1)\frac{\partial f(s_1,s_2,s_3)}{\partial s_1}ds_3ds_2ds_1 +$$

$$\frac{1}{\prod_{i=1}^{3}(w_i - u_i)} \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3} p_1(x_1,s_1)\,p_3(x_3,s_3)\frac{\partial^2 f(s_1,s_2,s_3)}{\partial s_3\partial s_1}ds_3ds_2ds_1 +$$

$$\frac{1}{\prod_{i=1}^{3}(w_i - u_i)} \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3} p_1(x_1,s_1)\,p_2(x_2,s_2)\frac{\partial^2 f(s_1,s_2,s_3)}{\partial s_2\partial s_1}ds_3ds_2ds_1 +$$

$$\frac{1}{\prod_{i=1}^{3}(w_i - u_i)} \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3} p_1(x_1,s_1)\,p_2(x_2,s_2)\,p_3(x_3,s_3)\frac{\partial^3 f(s_1,s_2,s_3)}{\partial s_3\partial s_2\partial s_1}ds_3ds_2ds_1.$$

We have proved (2.1). □

Next comes the general complex multivariate Montgomery type representation identity of complex functions:

Theorem 2.2. *Let $f : \prod_{j=1}^{m} D_j \subseteq \mathbb{C}^m \to \mathbb{C}$ be a continuous function that is analytic per coordinate on the domain D_j, $j = 1,...,m$, and $x = (x_1,...,x_m) \in \prod_{j=1}^{m} D_j$. For $j = 1,...,m$, suppose $\gamma_j \subset D_j$ is a smooth path parametrized by $z_j(t_j)$, $t_j \in [a_j,b_j]$ with $z_j(a_j) = u_j$, $z_j(t_j) = x_j$ and $z_j(b_j) = w_j$, where $u_j, w_j \in D_j$, $u_j \neq w_j$. Assume also that all partial derivatives of f up to order $m \in \mathbb{N}$ are continuous functions on $\prod_{j=1}^{m} D_j$.*

We define the kernels $p_i : \gamma_i^2 \to \mathbb{C}$

$$p_i(x_i,s_i) := \begin{cases} s_i - u_i, & \text{if } s_i \in \gamma_{u_i,x_i}, \\ s_i - w_i, & \text{if } s_i \in \gamma_{x_i,w_i}, \end{cases}$$

for $i = 1,2,...,m$.

Then

$$f(x_1,x_2,...,x_m) = \frac{1}{\prod_{i=1}^{m}(w_i-u_i)} \left\{ \int_{\prod_{i=1}^{m}\gamma_i} f(s_1,s_2,...,s_m)\,ds_m ds_{m-1}...ds_1 + \sum_{j=1}^{m}\left(\int_{\prod_{i=1}^{m}\gamma_i} p_j(x_j,s_j)\frac{\partial f(s_1,s_2,...,s_m)}{\partial s_j} ds_m...ds_1\right) + \right.$$

$$\sum_{\substack{l_1=1\\ j<k}}^{\binom{m}{2}} \left(\int_{\prod_{i=1}^{m}\gamma_i} p_j(x_j,s_j)\,p_k(x_k,s_k)\frac{\partial^2 f(s_1,s_2,...,s_m)}{\partial s_k\partial s_j} ds_m...ds_1\right)_{(l_1)} +$$

$$\sum_{\substack{l_2=1\\ j<k<r}}^{\binom{m}{3}} \left(\int_{\prod_{i=1}^{m}\gamma_i} p_j(x_j,s_j)\,p_k(x_k,s_k)\,p_r(x_r,s_r)\frac{\partial^3 f(s_1,...,s_m)}{\partial s_r\partial s_k\partial s_j} ds_m...ds_1\right)_{(l_2)} + ... +$$

$$\sum_{l=1}^{\binom{m}{m-1}} \left(\int_{\prod_{i=1}^{m}\gamma_i} p_1(x_1,s_1)\,...\widehat{p_l(x_l,s_l)}...p_m(x_m,s_m)\frac{\partial^{m-1} f(s_1,,...,s_m)}{\partial s_m...\widehat{\partial s_l}...\partial s_1} ds_m...\widehat{ds_l}...ds_1\right)$$

$$\left. + \int_{\prod_{i=1}^{m}\gamma_i} \left(\prod_{i=1}^{m} p_i(x_i,s_i)\right)\frac{\partial^m f(s_1,,...,s_m)}{\partial s_m...\partial s_1} ds_m...ds_1 \right\}. \tag{2.2}$$

Above l_1 counts $(j,k): j<k;\ j,k\in\{1,2,...,m\}$, also l_2 counts $(j,k,r): j<k<r;\ j,k,r\in\{1,2,...,m\}$, etc. Also $\widehat{p_l(x_l,s_l)}$ and $\widehat{\partial s_l}$ means that $p_l(x_l,s_l)$ and ∂s_l are missing, respectively.

Proof. Similar to Theorem 2.1. □

We make

Remark 2.3. *(on Theorems 2.1, 2.2)*
By (2.1) we get

$$E_f(x_1,x_2,x_3) := f(x_1,x_2,x_3) - \frac{1}{\prod_{i=1}^{3}(w_i-u_i)} \left\{ \int_{\prod_{i=1}^{3}\gamma_i} f(s_1,s_2,s_3)\,ds_3ds_2ds_1 \right.$$

$$\left. - \sum_{j=1}^{3}\left(\int_{\prod_{i=1}^{3}\gamma_i} p_j(x_j,s_j)\frac{\partial f(s_1,s_2,s_3)}{\partial s_j} ds_3ds_2ds_1\right) - \sum_{\substack{l=1\\ j<k}}^{3}\left(\int_{\prod_{i=1}^{3}\gamma_i} p_j(x_j,s_j)\,p_k(x_k,s_k)\frac{\partial^2 f(s_1,s_2,s_3)}{\partial s_k\partial s_j} ds_3ds_2ds_1\right)(l) \right\}$$

$$= \frac{1}{\prod_{i=1}^{3}(w_i-u_i)} \left(\int_{\prod_{i=1}^{3}\gamma_i} \left(\prod_{i=1}^{3} p_i(x_i,s_i)\right)\frac{\partial^3 f(s_1,s_2,s_3)}{\partial s_3\partial s_2\partial s_1} ds_1ds_2ds_3\right).$$

Above l counts $(j,k): j<k;\ j,k\in\{1,2,3\}$.
Similarly, by (2.2) we find

$$E_f(x_1,x_2,...,x_m) = f(x_1,x_2,...,x_m) -$$

$$\frac{1}{\prod_{i=1}^{m}(w_i-u_i)} \left\{ \int_{\prod_{i=1}^{m}\gamma_i} f(s_1,...,s_m)\,ds_1...ds_m - \sum_{j=1}^{m}\left(\int_{\prod_{i=1}^{m}\gamma_i} p_j(x_j,s_j)\frac{\partial f(s_1,...,s_m)}{\partial s_j} ds_1...ds_m\right) - \right.$$

$$\sum_{\substack{l_1=1\\ j<k}}^{\binom{m}{2}}\left(\int_{\prod_{i=1}^{m}\gamma_i} p_j(x_j,s_j)\,p_k(x_k,s_k)\,\frac{\partial^2 f(s_1,...,s_m)}{\partial s_k\partial s_j}ds_1...ds_m\right)_{(l_1)} -$$

$$\sum_{\substack{l_2=1\\ j<k<r}}^{\binom{m}{3}}\left(\int_{\prod_{i=1}^{m}\gamma_i} p_j(x_j,s_j)\,p_k(x_k,s_k)\,p_r(x_r,s_r)\,\frac{\partial^3 f(s_1,...,s_m)}{\partial s_r\partial s_k\partial s_j}ds_1...ds_m\right)_{(l_2)} - ... -$$

$$\left.\sum_{l=1}^{\binom{m}{m-1}}\left(\int_{\prod_{i=1}^{m}\gamma_i} p_1(x_1,s_1)\,...\widehat{p_l(x_l,s_l)}...p_m(x_m,s_m)\,\frac{\partial^{m-1} f(s_1,,...,s_m)}{\partial s_m...\widehat{\partial s_l}...\partial s_1}ds_1...\widehat{ds_l}...ds_m\right)\right\}$$

$$=\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\left(\int_{\prod_{i=1}^{m}\gamma_i}\left(\prod_{i=1}^{m}p_i(x_i,s_i)\right)\frac{\partial^m f(s_1,,...,s_m)}{\partial s_m...\partial s_1}ds_1...ds_m\right).$$

Above l_1 counts $(j,k): j<k;\ j,k\in\{1,...,m\}$, l_2 counts $(j,k,r): j<k<r;\ j,k,r\in\{1,2,...,m\}$, etc. Also $\widehat{p_l(x_l,s_l)}$ and $\widehat{\partial s_l}$ means that $p_l(x_l,s_l)$ and ∂s_l are missing, respectively.

Hence it holds

$$\left|E_f(x_1,x_2,x_3)\right| \leq \frac{1}{\prod_{i=1}^{3}|w_i-u_i|}\times\left(\int_{\prod_{i=1}^{3}\gamma_i}\left(\prod_{i=1}^{3}|p_i(x_i,s_i)|\right)\left|\frac{\partial^3 f(s_1,s_2,s_3)}{\partial s_3\partial s_2\partial s_1}\right||ds_1||ds_2||ds_3|\right), \tag{2.3}$$

and

$$\left|E_f(x_1,...,x_m)\right| \leq \frac{1}{\prod_{i=1}^{m}|w_i-u_i|}\times\left(\int_{\prod_{i=1}^{m}\gamma_i}\left(\prod_{i=1}^{m}|p_i(x_i,s_i)|\right)\left|\frac{\partial^m f(s_1,,...,s_m)}{\partial s_m...\partial s_1}\right||ds_1|...|ds_m|\right). \tag{2.4}$$

We give the following complex multivariate Ostrowski type inequalities:

Theorem 2.4. *All as in Theorem 2.1. Here $r_1,r_2,r_3,r_4>0: \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}=1$. Then*

$$\left|E_f(x_1,x_2,x_3)\right| \leq \frac{1}{\prod_{i=1}^{3}|w_i-u_i|}\times\min\left\{\left(\prod_{i=1}^{3}\int_{\gamma_i}|p_i(x_i,s_i)||ds_i|\right)\left\|\frac{\partial^3 f}{\partial s_3\partial s_2\partial s_1}\right\|_{\infty,\prod_{j=1}^{3}\gamma_j},\right.$$

$$\left(\prod_{i=1}^{3}\|p_i(x_i,s_i)\|_{r_j,\gamma_j}\right)\left(\prod_{i=1}^{3}(l(\gamma_j))^{\frac{2}{r_j}}\right)\left\|\frac{\partial^3 f}{\partial s_3\partial s_2\partial s_1}\right\|_{r_4,\prod_{i=1}^{3}\gamma_j},$$

$$\left.\left(\sup_{(s_1,s_2,s_3)\in\prod_{j=1}^{3}\gamma_j}\left(\prod_{i=1}^{3}|p_i(x_i,s_i)|\right)\right)\left\|\frac{\partial^3 f}{\partial s_3\partial s_2\partial s_1}\right\|_{1,\prod_{j=1}^{3}\gamma_j}\right\},$$

$\forall\ (x_1,x_2,x_3)\in\prod_{j=1}^{3}\gamma_j$.

Proof. By (2.3) and generalized Hölder's inequality. □

Theorem 2.5. *All as in Theorem 2.2. Here* $r_1, r_2, ..., r_m, r_{m+1} > 0 : \sum_{i=1}^{m+1} \frac{1}{r_i} = 1$. *Then*

$$\left|E_f(x_1,...,x_m)\right| \leq \frac{1}{\prod_{i=1}^{m}|w_i - u_i|} \times \min\left\{\left(\prod_{i=1}^{m}\int_{\gamma_i}|p_i(x_i,s_i)||ds_i|\right)\left\|\frac{\partial^m f}{\partial s_m ... \partial s_1}\right\|_{\infty, \prod_{j=1}^{m}\gamma_j},\right.$$

$$\left(\prod_{i=1}^{m}\|p_i(x_i,s_i)\|_{r_j,\gamma_j}\right)\left(\prod_{i=1}^{m}(l(\gamma_j))^{\frac{m-1}{r_j}}\right)\left\|\frac{\partial^m f}{\partial s_m ... \partial s_1}\right\|_{r_{m+1}, \prod_{i=1}^{m}\gamma_j},$$

$$\left.\left(\sup_{(s_1,...,s_m)\in\prod_{j=1}^{m}\gamma_j}\left(\prod_{i=1}^{m}|p_i(x_i,s_i)|\right)\right)\left\|\frac{\partial^m f}{\partial s_m ... \partial s_1}\right\|_{1, \prod_{j=1}^{m}\gamma_j}\right\},$$

$\forall\ (x_1,...,x_m) \in \prod_{j=1}^{m}\gamma_j$.

Proof. By (2.4) and generalized Hölder's inequality. □

We make

Remark 2.6. *Working further on (2.1) we call:*

$$A_f^{(3)} := A_f^{(3)}(x_1,x_2,x_3) := \sum_{j=1}^{3}\left(\int_{\prod_{i=1}^{3}\gamma_i} p_j(x_j,s_j)\frac{\partial f(s_1,s_2,s_3)}{\partial s_j}ds_1 ds_2 ds_3\right)$$

$$+\sum_{\substack{l=1\\ j<k}}^{3}\left(\int_{\prod_{i=1}^{3}\gamma_i} p_j(x_j,s_j)\,p_k(x_k,s_k)\frac{\partial^2 f(s_1,s_2,s_3)}{\partial s_k \partial s_j}ds_1 ds_2 ds_3\right)(l),$$

and

$$B_f^{(3)} := B_f^{(3)}(x_1,x_2,x_3) := \int_{\prod_{i=1}^{3}\gamma_i}\left(\prod_{i=1}^{3}p_i(x_i,s_i)\right)\frac{\partial^3 f(s_1,s_2,s_3)}{\partial s_3 \partial s_2 \partial s_1}ds_1 ds_2 ds_3.$$

Set also

$$T_f^{(3)} := T_f^{(3)}(x_1,x_2,x_3) := A_f^{(3)} + B_f^{(3)}.$$

Thus, we have ($x = (x_1,x_2,x_3)$)

$$f(x) = f(x_1,x_2,x_3) = \frac{1}{\prod_{i=1}^{3}(w_i - u_i)}\int_{\prod_{i=1}^{3}\gamma_i} f(s_1,s_2,s_3)\,ds_1 ds_2 ds_3 + \frac{1}{\prod_{i=1}^{3}(w_i - u_i)}\left(A_f^{(3)} + B_f^{(3)}\right) =$$

$$\frac{1}{\prod_{i=1}^{3}(w_i - u_i)}\int_{\prod_{i=1}^{3}\gamma_i} f(s_1,s_2,s_3)\,ds_1 ds_2 ds_3 + \frac{1}{\prod_{i=1}^{3}(w_i - u_i)}T_f^{(3)}.$$

Working further on (2.2) we call:

$$A_f^{(m)} := A_f^{(m)}(x_1,...,x_m) := \sum_{j=1}^{m}\left(\int_{\prod_{i=1}^{m}\gamma_i} p_j(x_j,s_j)\frac{\partial f(s_1,...,s_m)}{\partial s_j}ds_1 ... ds_m\right) +$$

$$\sum_{\substack{l_1=1\\ j<k}}^{\binom{m}{2}}\left(\int_{\prod_{i=1}^{m}\gamma_i} p_j(x_j,s_j)\,p_k(x_k,s_k)\,\frac{\partial^2 f(s_1,\dots,s_m)}{\partial s_k\partial s_j}ds_1\dots ds_m\right)_{(l_1)}+$$

$$\sum_{\substack{l_2=1\\ j<k<r}}^{\binom{m}{3}}\left(\int_{\prod_{i=1}^{m}\gamma_i} p_j(x_j,s_j)\,p_k(x_k,s_k)\,p_r(x_r,s_r)\,\frac{\partial^3 f(s_1,\dots,s_m)}{\partial s_r\partial s_k\partial s_j}ds_1\dots ds_m\right)(l_2)+\dots+$$

$$\sum_{l=1}^{\binom{m}{m-1}}\left(\int_{\prod_{i=1}^{m}\gamma_i} p_1(x_1,s_1)\dots\widehat{p_l(x_l,s_l)}\dots p_m(x_m,s_m)\,\frac{\partial^{m-1} f(s_1,,\dots,s_m)}{\partial s_m\dots\widehat{\partial s_l}\dots\partial s_1}ds_1\dots\widehat{ds_l}\dots ds_m\right),$$

and

$$B_f^{(m)}:=B_f^{(m)}(x_1,\dots,x_m):=\int_{\prod_{i=1}^{m}\gamma_i}\left(\prod_{i=1}^{m}p_i(x_i,s_i)\right)\frac{\partial^m f(s_1,,\dots,s_m)}{\partial s_m\dots\partial s_1}ds_1\dots ds_m.$$

Set also

$$T_f^{(m)}:=T_f^{(m)}(x_1,\dots,x_m):=A_f^{(m)}+B_f^{(m)}.$$

Thus, we have ($x=(x_1,\dots,x_m)$)

$$f(x)=f(x_1,\dots,x_m)=\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\int_{\prod_{i=1}^{m}\gamma_i} f(s_1,\dots,s_m)\,ds_1\dots ds_m+\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\left(A_f^{(m)}+B_f^{(m)}\right)=$$

$$\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\int_{\prod_{i=1}^{m}\gamma_i} f(s_1,\dots,s_m)\,ds_1\dots ds_m+\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}T_f^{(m)}. \tag{2.5}$$

Let function g as in Theorem 2.2. Then as in (2.5) we obtain

$$g(x)=g(x_1,\dots,x_m)=\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\int_{\prod_{i=1}^{m}\gamma_i} g(s_1,\dots,s_m)\,ds_1\dots ds_m+\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\left(A_g^{(m)}+B_g^{(m)}\right)= \tag{2.6}$$

$$\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\int_{\prod_{i=1}^{m}\gamma_i} g(s_1,\dots,s_m)\,ds_1\dots ds_m+\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}T_g^{(m)}.$$

Above $A_g^{(m)},B_g^{(m)},T_g^{(m)}$ *have the obvious meaning.*

By (2.5) we get

$$f(x)g(x)=\frac{g(x)}{\prod_{i=1}^{m}(w_i-u_i)}\int_{\prod_{i=1}^{m}\gamma_i} f(s_1,\dots,s_m)\prod_{i=1}^{m}ds_i+\frac{g(x)}{\prod_{i=1}^{m}(w_i-u_i)}T_f^{(m)},$$

and by (2.6) we get

$$g(x)f(x)=\frac{f(x)}{\prod_{i=1}^{m}(w_i-u_i)}\int_{\prod_{i=1}^{m}\gamma_i} g(s_1,\dots,s_m)\prod_{i=1}^{m}ds_i+\frac{f(x)}{\prod_{i=1}^{m}(w_i-u_i)}T_g^{(m)}.$$

Consequently after integration we get:

(set $s := (s_1, ..., s_m)$*)*

$$\int_{\prod_{i=1}^{m}\gamma_i} f(s)\, g(s) \prod_{i=1}^{m} ds_i = \frac{\int_{\prod_{i=1}^{m}\gamma_i} g(s) \prod_{i=1}^{m} ds_i}{\prod_{i=1}^{m}(w_i - u_i)} \int_{\prod_{i=1}^{m}\gamma_i} f(s) \prod_{i=1}^{m} ds_i + \frac{1}{\prod_{i=1}^{m}(w_i - u_i)} \int_{\prod_{i=1}^{m}\gamma_i} g(s)\, T_f^{(m)}(s) \prod_{i=1}^{m} ds_i, \tag{2.7}$$

and

$$\int_{\prod_{i=1}^{m}\gamma_i} f(s)\, g(s) \prod_{i=1}^{m} ds_i = \frac{\int_{\prod_{i=1}^{m}\gamma_i} f(s) \prod_{i=1}^{m} ds_i}{\prod_{i=1}^{m}(w_i - u_i)} \int_{\prod_{i=1}^{m}\gamma_i} g(s) \prod_{i=1}^{m} ds_i + \frac{1}{\prod_{i=1}^{m}(w_i - u_i)} \int_{\prod_{i=1}^{m}\gamma_i} f(s)\, T_g^{(m)}(s) \prod_{i=1}^{m} ds_i. \tag{2.8}$$

By (2.7) and (2.8) we obtain

$$\int_{\prod_{i=1}^{m}\gamma_i} f(s)\, g(s) \prod_{i=1}^{m} ds_i - \frac{1}{\prod_{i=1}^{m}(w_i - u_i)} \left(\int_{\prod_{i=1}^{m}\gamma_i} f(s) \prod_{i=1}^{m} ds_i \right) \left(\int_{\prod_{i=1}^{m}\gamma_i} g(s) \prod_{i=1}^{m} ds_i \right) = \frac{1}{\prod_{i=1}^{m}(w_i - u_i)} \int_{\prod_{i=1}^{m}\gamma_i} f(s)\, T_g^{(m)}(s) \prod_{i=1}^{m} ds_i =$$

$$\frac{1}{\prod_{i=1}^{m}(w_i - u_i)} \int_{\prod_{i=1}^{m}\gamma_i} g(s)\, T_f^{(m)}(s) \prod_{i=1}^{m} ds_i.$$

We conclude that (set $d\overrightarrow{s} := \prod_{i=1}^{m} ds_i$*)*

$$\int_{\prod_{i=1}^{m}\gamma_i} f(s)\, g(s)\, d\overrightarrow{s} - \frac{1}{\prod_{i=1}^{m}(w_i - u_i)} \left(\int_{\prod_{i=1}^{m}\gamma_i} f(s)\, d\overrightarrow{s} \right) \left(\int_{\prod_{i=1}^{m}\gamma_i} g(s)\, d\overrightarrow{s} \right) = \frac{1}{2\left(\prod_{i=1}^{m}(w_i - u_i)\right)} \left[\int_{\prod_{i=1}^{m}\gamma_i} \left(f(s)\, T_g^{(m)}(s) + g(s)\, T_f^{(m)}(s) \right) d\overrightarrow{s} \right].$$

Therefore we have

$$\frac{1}{\prod_{i=1}^{m}(w_i - u_i)} \int_{\prod_{i=1}^{m}\gamma_i} f(s)\, g(s)\, d\overrightarrow{s} - \frac{1}{\prod_{i=1}^{m}(w_i - u_i)} \left(\int_{\prod_{i=1}^{m}\gamma_i} f(s)\, d\overrightarrow{s} \right) \frac{1}{\prod_{i=1}^{m}(w_i - u_i)} \left(\int_{\prod_{i=1}^{m}\gamma_i} g(s)\, d\overrightarrow{s} \right) = \frac{1}{2\left(\prod_{i=1}^{m}(w_i - u_i)\right)^2}$$

$$\left[\int_{\prod_{i=1}^{m}\gamma_i} \left\{ f(s) \left(A_g^{(m)}(s) + B_g^{(m)}(s) \right) + g(s) \left(A_f^{(m)}(s) + B_f^{(m)}(s) \right) \right\} d\overrightarrow{s} \right].$$

Hence it holds

$$\Delta(f,g) := \frac{1}{\prod_{i=1}^{m}(w_i - u_i)} \int_{\prod_{i=1}^{m}\gamma_i} f(s)\, g(s)\, d\overrightarrow{s} - \frac{1}{\prod_{i=1}^{m}(w_i - u_i)} \left(\int_{\prod_{i=1}^{m}\gamma_i} f(s)\, d\overrightarrow{s} \right) \frac{1}{\prod_{i=1}^{m}(w_i - u_i)} \left(\int_{\prod_{i=1}^{m}\gamma_i} g(s)\, d\overrightarrow{s} \right) -$$

$$\frac{1}{2\left(\prod_{i=1}^{m}(w_i - u_i)\right)^2} \left[\int_{\prod_{i=1}^{m}\gamma_i} \left\{ f(s) A_g^{(m)}(s) + g(s) A_f^{(m)}(s) \right\} d\overrightarrow{s} \right] = \frac{1}{2\left(\prod_{i=1}^{m}(w_i - u_i)\right)^2} \left[\int_{\prod_{i=1}^{m}\gamma_i} \left\{ f(s) B_g^{(m)}(s) + g(s) B_f^{(m)}(s) \right\} d\overrightarrow{s} \right].$$

Clearly we derive that ($\left|d\overrightarrow{s}\right| := \prod_{i=1}^{m} |ds_i|$*)*

$$|\Delta(f,g)| \leq \frac{1}{2\left(\prod_{i=1}^{m}|w_i - u_i|\right)^2} \left[\int_{\prod_{i=1}^{m}\gamma_i} \left\{ |f(s)| \left| B_g^{(m)}(s) \right| + |g(s)| \left| B_f^{(m)}(s) \right| \right\} \left|d\overrightarrow{s}\right| \right] = \tag{2.9}$$

$$\frac{1}{2\left(\prod_{i=1}^{m}|w_i-u_i|\right)^2}\left[\int_{\prod_{i=1}^{m}\gamma_i}|f(s)|\left|B_g^{(m)}(s)\right|\left|d\overrightarrow{s}\right|+\int_{\prod_{i=1}^{m}\gamma_i}|g(s)|\left|B_f^{(m)}(s)\right|\left|d\overrightarrow{s}\right|\right]\leq$$

$$\frac{1}{2\left(\prod_{i=1}^{m}|w_i-u_i|\right)^2}\left[\|f\|_{\infty,\prod_{i=1}^{m}\gamma_i}\left(\int_{\prod_{i=1}^{m}\gamma_i}\left|B_g^{(m)}(s)\right|\left|d\overrightarrow{s}\right|\right)+\|g\|_{\infty,\prod_{i=1}^{m}\gamma_i}\left(\int_{\prod_{i=1}^{m}\gamma_i}\left|B_f^{(m)}(s)\right|\left|d\overrightarrow{s}\right|\right)\right].$$

We have established the following complex multivariate Grüss type inequality:

Theorem 2.7. *Let f,g and all as in Theorem 2.2. Then*

$$\left|\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\int_{\prod_{i=1}^{m}\gamma_i}f(s)g(s)d\overrightarrow{s}-\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\left(\int_{\prod_{i=1}^{m}\gamma_i}f(s)d\overrightarrow{s}\right)\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\left(\int_{\prod_{i=1}^{m}\gamma_i}g(s)d\overrightarrow{s}_i\right)-\right.$$

$$\left.\frac{1}{2\left(\prod_{i=1}^{m}(w_i-u_i)\right)^2}\left[\int_{\prod_{i=1}^{m}\gamma_i}\left(f(s)A_g^{(m)}(s)+g(s)A_f^{(m)}(s)\right)d\overrightarrow{s}\right]\right|\leq$$

$$\frac{1}{2\left(\prod_{i=1}^{m}|w_i-u_i|\right)^2}\left[\|f\|_{\infty,\prod_{i=1}^{m}\gamma_i}\left(\int_{\prod_{i=1}^{m}\gamma_i}\left|B_g^{(m)}(s)\right|\left|d\overrightarrow{s}\right|\right)+\|g\|_{\infty,\prod_{i=1}^{m}\gamma_i}\left(\int_{\prod_{i=1}^{m}\gamma_i}\left|B_f^{(m)}(s)\right|\left|d\overrightarrow{s}\right|\right)\right].$$

The corresponding L_p Grüss inequality follows:

Theorem 2.8. *Let f,g and all as in Theorem 2.2 and $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\left|\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\int_{\prod_{i=1}^{m}\gamma_i}f(s)g(s)d\overrightarrow{s}-\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\left(\int_{\prod_{i=1}^{m}\gamma_i}f(s)d\overrightarrow{s}\right)\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\left(\int_{\prod_{i=1}^{m}\gamma_i}g(s)d\overrightarrow{s}\right)-\right.$$

$$\left.\frac{1}{2\left(\prod_{i=1}^{m}(w_i-u_i)\right)^2}\left[\int_{\prod_{i=1}^{m}\gamma_i}\left(f(s)A_g^{(m)}(s)+g(s)A_f^{(m)}(s)\right)d\overrightarrow{s}\right]\right|\leq$$

$$\frac{1}{2\left(\prod_{i=1}^{m}|w_i-u_i|\right)^2}\left[\|f\|_{p,\prod_{i=1}^{m}\gamma_i}\left\|B_g^{(m)}\right\|_{q,\prod_{i=1}^{m}\gamma_i}+\|g\|_{p,\prod_{i=1}^{m}\gamma_i}\left\|B_f^{(m)}\right\|_{q,\prod_{i=1}^{m}\gamma_i}\right].$$

Proof. Use of (2.9) and Hölder inequality. □

The corrsponding L_1 Grüss inequality follows:

Theorem 2.9. *Let f,g and all as in Theorem 2.2. Then*

$$\left|\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\int_{\prod_{i=1}^{m}\gamma_i}f(s)g(s)d\overrightarrow{s}-\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\left(\int_{\prod_{i=1}^{m}\gamma_i}f(s)d\overrightarrow{s}\right)\frac{1}{\prod_{i=1}^{m}(w_i-u_i)}\left(\int_{\prod_{i=1}^{m}\gamma_i}g(s)d\overrightarrow{s}\right)-\right.$$

$$\left. \frac{1}{2\left(\prod_{i=1}^{m}(w_i-u_i)\right)^2}\left[\int_{\prod_{i=1}^{m}\gamma_i}\left(f(s)A_g^{(m)}(s)+g(s)A_f^{(m)}(s)\right)d\overrightarrow{s}\right]\right| \le$$

$$\frac{1}{2\left(\prod_{i=1}^{m}|w_i-u_i|\right)^2}\left[\|f\|_{1,\prod_{i=1}^{m}\gamma_i}\left\|B_g^{(m)}\right\|_{\infty,\prod_{i=1}^{m}\gamma_i}+\|g\|_{1,\prod_{i=1}^{m}\gamma_i}\left\|B_f^{(m)}\right\|_{\infty,\prod_{i=1}^{m}\gamma_i}\right].$$

Proof. By (2.9) □

Corollary 2.10. *Let f,g and all as in Theorem 2.1. Then*

$$\left|\frac{1}{\prod_{i=1}^{3}(w_i-u_i)}\int_{\prod_{i=1}^{3}\gamma_i}f(s)g(s)d\overrightarrow{s}-\frac{1}{\prod_{i=1}^{3}(w_i-u_i)}\left(\int_{\prod_{i=1}^{3}\gamma_i}f(s)d\overrightarrow{s}\right)\frac{1}{\prod_{i=1}^{3}(w_i-u_i)}\left(\int_{\prod_{i=1}^{3}\gamma_i}g(s)d\overrightarrow{s}\right)-\right.$$

$$\left. \frac{1}{2\left(\prod_{i=1}^{3}(w_i-u_i)\right)^2}\left[\int_{\prod_{i=1}^{3}\gamma_i}\left(f(s)A_g^{(3)}(s)+g(s)A_f^{(3)}(s)\right)d\overrightarrow{s}\right]\right| \le$$

$$\frac{1}{2\left(\prod_{i=1}^{m}|w_i-u_i|\right)^2}\left[\|f\|_{\infty,\prod_{i=1}^{3}\gamma_i}\left(\int_{\prod_{i=1}^{3}\gamma_i}\left|B_g^{(3)}(s)\right|\left|d\overrightarrow{s}\right|\right)+\|g\|_{\infty,\prod_{i=1}^{3}\gamma_i}\left(\int_{\prod_{i=1}^{3}\gamma_i}\left|B_f^{(3)}(s)\right|\left|d\overrightarrow{s}\right|\right)\right].$$

Proof. By Theorem 2.7 for $m=3$. □

Corollary 2.11. *Let f,g and all as in Theorem 2.1 and $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\left|\frac{1}{\prod_{i=1}^{3}(w_i-u_i)}\int_{\prod_{i=1}^{3}\gamma_i}f(s)g(s)d\overrightarrow{s}-\frac{1}{\prod_{i=1}^{3}(w_i-u_i)}\left(\int_{\prod_{i=1}^{3}\gamma_i}f(s)d\overrightarrow{s}\right)\frac{1}{\prod_{i=1}^{3}(w_i-u_i)}\left(\int_{\prod_{i=1}^{3}\gamma_i}g(s)d\overrightarrow{s}\right)-\right.$$

$$\left. \frac{1}{2\left(\prod_{i=1}^{3}(w_i-u_i)\right)^2}\left[\int_{\prod_{i=1}^{3}\gamma_i}\left(f(s)A_g^{(3)}(s)+g(s)A_f^{(3)}(s)\right)d\overrightarrow{s}\right]\right| \le$$

$$\frac{1}{2\left(\prod_{i=1}^{3}|w_i-u_i|\right)^2}\left[\|f\|_{p,\prod_{i=1}^{3}\gamma_i}\left\|B_g^{(3)}\right\|_{q,\prod_{i=1}^{3}\gamma_i}+\|g\|_{p,\prod_{i=1}^{3}\gamma_i}\left\|B_f^{(3)}\right\|_{q,\prod_{i=1}^{3}\gamma_i}\right].$$

Proof. By Theorem 2.8 for $m=3$. □

Corollary 2.12. *Let f,g and all as in Theorem 2.1. Then*

$$\left|\frac{1}{\prod_{i=1}^{3}(w_i-u_i)}\int_{\prod_{i=1}^{3}\gamma_i}f(s)g(s)d\overrightarrow{s}-\frac{1}{\prod_{i=1}^{3}(w_i-u_i)}\left(\int_{\prod_{i=1}^{3}\gamma_i}f(s)d\overrightarrow{s}\right)\frac{1}{\prod_{i=1}^{3}(w_i-u_i)}\left(\int_{\prod_{i=1}^{3}\gamma_i}g(s)d\overrightarrow{s}\right)-\right.$$

$$\frac{1}{2\left(\prod_{i=1}^{3}(w_i-u_i)\right)^2}\left[\int_{\prod_{i=1}^{3}\gamma_i}\left(f(s)A_g^{(3)}(s)+g(s)A_f^{(3)}(s)\right)d\overrightarrow{s}\right]\Bigg| \leq$$

$$\frac{1}{2\left(\prod_{i=1}^{3}|w_i-u_i|\right)^2}\left[\|f\|_{1,\prod_{i=1}^{3}\gamma_i}\left\|B_g^{(3)}\right\|_{\infty,\prod_{i=1}^{3}\gamma_i}+\|g\|_{1,\prod_{i=1}^{3}\gamma_i}\left\|B_f^{(3)}\right\|_{\infty,\prod_{i=1}^{3}\gamma_i}\right].$$

Proof. By Theorem 2.9 for $m = 3$. □

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